Proofs and Types
Natural Deduction

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Deduction

- A *deduction* of A is a finite tree with root A.

$$\begin{array}{c}\vdots\\ A\end{array}$$

- The tree will have leaves labelled by sentences.
  - Two types of leaves: *dead* or *alive*.

Leaves

- An alive leaf is a *hypothesis*.
- Here is a deduction of A with a hypothesis A:

$$A$$

- A dead leaf does not play an active role in the deduction.
- Here is a deduction of $A \Rightarrow B$ with a dead leaf A:

$$\dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \Rightarrow B} \Rightarrow \mathcal{I}$$

- Note that a number of leaves with the same label can be *discharged* while other leaves with the same label remain alive.

$$\dfrac{\begin{array}{c}A \quad [A] \\ \vdots \\ B\end{array}}{A \Rightarrow B} \Rightarrow \mathcal{I}$$

Rules

- *Hypothesis*: A
- *Introductions*:

$$\frac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}\ \wedge\mathcal{I} \qquad \frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B}\ \Rightarrow\mathcal{I} \qquad \frac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall\xi.A}\ \forall\mathcal{I}$$

- *Eliminations*:

$$\frac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\ \wedge 1\mathcal{I} \qquad \frac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\ \wedge 2\mathcal{E} \qquad \frac{\begin{matrix}\vdots & \vdots \\ A & A \Rightarrow B\end{matrix}}{B}\ \Rightarrow\mathcal{E}$$

$$\frac{\begin{matrix}\vdots \\ \forall\xi.A\end{matrix}}{A[a/\xi]}\ \forall\mathcal{E}$$

- In $\forall\mathcal{I}$, ξ is not free in any hypothesis.

Examples

- A deduction of $A \Rightarrow (B \Rightarrow A)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A] \quad [B]}{A \wedge B}\wedge\mathcal{I}}{A}\wedge 1\mathcal{E}}{B \Rightarrow A}\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\Rightarrow\mathcal{I}$$

- A deduction of $A \Rightarrow (B \wedge C)$ from hypotheses $(A \Rightarrow B) \wedge (A \Rightarrow C)$:

$$\dfrac{\dfrac{\dfrac{[A] \quad \dfrac{(A \Rightarrow B) \wedge (A \Rightarrow C)}{A \Rightarrow B}\wedge 1\mathcal{E}}{B}\Rightarrow\mathcal{E} \quad \dfrac{[A] \quad \dfrac{(A \Rightarrow B) \wedge (A \Rightarrow C)}{A \Rightarrow C}\wedge 2\mathcal{E}}{C}\Rightarrow\mathcal{E}}{B \wedge C}\wedge\mathcal{I}}{A \Rightarrow (B \wedge C)}\Rightarrow\mathcal{I}$$

More Examples

- A deduction of $(\forall x P x) \Rightarrow (\forall y P y)$.

$$\dfrac{\dfrac{\dfrac{[\forall x P x]}{P y}\ \forall\mathcal{E}}{\forall y P y}\ \forall\mathcal{I}}{\forall x P x \Rightarrow \forall y P y}\ \Rightarrow\mathcal{I}$$

- Find the problem in the "deduction" of $Px \Rightarrow Py$.

$$\dfrac{\dfrac{\dfrac{[P x]}{\forall x P x}\ \forall\mathcal{I}}{P y}\ \forall\mathcal{E}}{P x \Rightarrow P y}\ \Rightarrow\mathcal{I}$$

Interpretation a la Heyting

- A formula A is seen as the set of its possible deductions.
  - If δ is a deduction of A (δ "proves" A), we write $\delta \in A$.
  - A is a theorem if and only if $A \neq \emptyset$.
- A deduction of A on the hypothesis $B_1, \ldots, B_n$ is a function $t[x_1, \ldots, x_n]$ with parameters $x_1, \ldots, x_n$ such that $t[b_1, \ldots, b_n] \in A$ if $b_i \in B_i$ for all $1 \leq i \leq n$.
  - Recall that a hypothesis B_i may have several occurrences. The set of occurrences corresponding to B_i is called a *parcel*.

$$\begin{array}{c} {\color{green}B_1} \quad {\color{brown}B_3} \quad {\color{blue}B_2} \quad {\color{red}B_1} \quad {\color{red}B_1} \quad {\color{blue}B_2} \\ \vdots \\ A \end{array}$$

In this example, a deduction of A on B_1, B_1, B_2, B_3 is $t[x_1, x_2, x_3, x_4]$. The parcels of x_1, x_2, x_3, and x_4 have 1, 2, 2, and 1 occurrences respectively.

Interpretation of the Rules

- A deduction of a single hypothesis A is represented by a variable x for an element of A.
- A deduction ending in $\wedge\mathcal{I}$ is represented by $\langle u[x_1, \ldots, x_n], v[x_1, \ldots, x_n]\rangle$ where $u[x_1, \ldots, x_n]$ and $v[x_1, \ldots, x_n]$ are deductions of the two children.
- A deduction ending in $\wedge 1\mathcal{E}$ is represented by $\pi^1 t[x_1, \ldots, x_n]$ where $t[x_1, \ldots, x_n]$ is a deduction of the child and π^1 the *first projection*. A deduction ending in $\wedge 2\mathcal{E}$ is represented by $\pi^2 t[x_1, \ldots, x_n]$ similarly. We will use the following equations:

$$\pi^1\langle u, v\rangle = u \qquad \pi^2\langle u, v\rangle = v \qquad \langle \pi^1 t, \pi^2 t\rangle = t$$

Interpretation of the Rules

- A deduction ending in $\Rightarrow \mathcal{I}$ is represented by $t[x_1, \ldots, x_n] = \lambda x.v[x, x_1, \ldots, x_n]$ where $v[x, x_1, \ldots, x_n]$ is a deduction of the child.
    - $\lambda x.v[x, x_1, \ldots, x_n]$ is a function which maps a to $v[a, x_1, \ldots, x_n]$.
- A deduction ending in $\Rightarrow \mathcal{E}$ is represented by $t[x_1, \ldots, x_n]u[x_1, \ldots, x_n]$ where $t[x_1, \ldots, x_n]$ and $u[x_1, \ldots, x_n]$ are deductions of the children $A \Rightarrow B$ and A respectively.
    - $t[x_1, \ldots, x_n]u[x_1, \ldots, x_n]$ means applying the argument $u[x_1, \ldots, x_n]$ to the function $t[x_1, \ldots, x_n]$.
      We will need the following equations:

$$
\begin{array}{rcll}
(\lambda x.v)u & = & v[u/x] & \\
\lambda x.tx & = & t & \quad \text{when } x \text{ is not free in } t
\end{array}
$$

- Interpretations of $\forall\mathcal{I}$ and $\forall\mathcal{E}$ will be discussed later.

Examples

- Find a representation of the following deduction:

$$\cfrac{\cfrac{\cfrac{\cfrac{[A] \quad [B]}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge 1\mathcal{E}}{B \Rightarrow A}\,\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\,\Rightarrow\mathcal{I}$$

- Find a representation of the following deduction:

$$\cfrac{\cfrac{\cfrac{[A] \quad \cfrac{(A \Rightarrow B) \wedge (A \Rightarrow C)}{A \Rightarrow B}\,\wedge 1\mathcal{E}}{B}\,\Rightarrow\mathcal{E} \quad \cfrac{[A] \quad \cfrac{(A \Rightarrow B) \wedge (A \Rightarrow C)}{A \Rightarrow C}\,\wedge 2\mathcal{E}}{C}\,\Rightarrow\mathcal{E}}{B \wedge C}\,\wedge\mathcal{I}}{A \Rightarrow (B \wedge C)}\,\Rightarrow\mathcal{I}$$

Examples

- Find a representation of the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A] \quad [B]}{A \wedge B}\wedge\mathcal{I}}{A}\wedge 1\mathcal{E}}{B \Rightarrow A}\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\Rightarrow\mathcal{I}$$

$$\lambda x.\lambda y.\pi_1\langle x, y\rangle$$

- Find a representation of the following deduction:

$$\dfrac{\dfrac{\dfrac{[A] \quad \dfrac{(A \Rightarrow B) \wedge (A \Rightarrow C)}{A \Rightarrow B}\wedge 1\mathcal{E}}{B}\Rightarrow\mathcal{E} \qquad \dfrac{[A] \quad \dfrac{(A \Rightarrow B) \wedge (A \Rightarrow C)}{A \Rightarrow C}\wedge 2\mathcal{E}}{C}\Rightarrow\mathcal{E}}{B \wedge C}\wedge\mathcal{I}}{A \Rightarrow (B \wedge C)}\Rightarrow\mathcal{I}$$

$$\lambda x.\langle(\pi_1 y)x, (\pi_2 y)x\rangle$$

Untyped λ-Calculus

- *λ-terms* Λ is defined as follows.
  - $x \in \Lambda$;
  - $M \in \Lambda$ implies $\lambda x.M \in \Lambda$; and
  - $M, N \in \Lambda$ implies $MN \in \Lambda$.
- Consider the β-conversion:

$$(\lambda x.M)N \to M[N/x]$$

- A variable x is *bound* if it is in the scope of λx; otherwise x is *free*.
  - x is bound and y is free in $\lambda x.yx$;
  - x and y are bound in $\lambda y.\lambda x.yx$.

Examples

- Church numbers: $\lceil i \rceil = \lambda f.\lambda x.f^i x$.
- Addition: $\lceil + \rceil = \lambda m.\lambda n.\lambda f.\lambda x.mf(nfx)$.
- $\lceil + \rceil \lceil i \rceil \lceil j \rceil = [\lambda m.\lambda n.\lambda f.\lambda x.mf(nfx)][\lambda f.\lambda x.f^i x][\lambda f.\lambda x.f^j x] \rightarrow \lambda f.\lambda x.(\lambda f.\lambda x.f^i x)f((\lambda f.\lambda x.f^j x)fx) \rightarrow \lambda f.\lambda x.(\lambda f.\lambda x.f^i x)f(f^j x) \rightarrow \lambda f.\lambda x.f^i(f^j x) = \lambda f.\lambda x.f^{i+j} x = \lceil i + j \rceil$.
- $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.
- For any $F \in \Lambda$, $YF = (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))F \rightarrow (\lambda x.F(xx))(\lambda x.F(xx)) \rightarrow F[(\lambda x.F(xx))(\lambda x.F(xx))] = F[YF]$.
- Church-Turing Thesis:

Effectively computable functions are λ-definable.

Equivalent Deductions

- Deductions may be simplified. For instance,

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \land B}\land\mathcal{I}}{A}\land 1\mathcal{E} \qquad \text{“equals”} \qquad \begin{array}{c}\vdots \\ A\end{array}$$

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \land B}\land\mathcal{I}}{B}\land 2\mathcal{E} \qquad \text{“equals”} \qquad \begin{array}{c}\vdots \\ B\end{array}$$

$$\dfrac{\begin{array}{c}\vdots \\ A\end{array} \quad \dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I}}{B}\Rightarrow\mathcal{E} \qquad \text{“equals”} \qquad \begin{array}{c}\vdots \\ A \\ \vdots \\ B\end{array}$$