Proofs and Types
The Curry-Howard Isomorphism

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Spring 2012

Dichotomy between Sense and Denotation

- Recall

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I}}{B}\Rightarrow\mathcal{E} \qquad \text{“equals”} \qquad \begin{array}{c}\vdots\\ A\\ \vdots\\ B\end{array}$$

$$(\lambda x.t_B)t_A \quad = \quad t_B[t_A/x]$$

$$(\lambda x.t_B)t_A \quad \rightarrow \quad t_B[t_A/x]$$

- Equations define the equality of terms (the static view).
- Rewrite rules calculate terms by reduction (the dynamic view).

Typed λ-Calculus

- Formulae are *types*.
  - $T_1, \ldots, T_n$ are types; and
  - If U and V are types, $U \times V$ and $U \to V$ are types.
- Proofs are *terms*.
  - The variables $x_0^T, \ldots, x_n^T, \ldots$ are terms of type T;
  - If u and v are terms of types U and V respectively, $\langle u, v \rangle$ is a term of type $U \times V$;
  - If t is a term of type $U \times V$, $\pi^1 t$ and $\pi^2 t$ are of types U and V respectively;
  - If v is a term of type V and x_n^U is a variable of type U, $\lambda x_n^U.v$ is a term of type $U \to V$;
  - If t and u are terms of types $U \to V$ and U respectively, tu is a term of type V.

- Consider the following (*primary*) equations

$$\pi^1\langle u, v\rangle = u \qquad \pi^2\langle u, v\rangle = v \qquad (\lambda x^U.v)u = v[u/x]$$

- And the *secondary* equations

$$\langle \pi^1 t, \pi^2 t\rangle = t \qquad \lambda x^U.tx = t$$

- A system is *consistent* if the equality $x = y$ for distinct x and y cannot be proved.

Theorem 1

The system of typed λ-calculus with the primary equations is consistent and decidable.

Dynamic View

- Terms represent programs; and programs compute.
- To give a dynamic view, we consider rewrite rules derived by the primary equations.
- A term t (called *redex*) *converts* to a term t' (called *contractum*) when

$$
\begin{array}{c|ccc}
t & \pi^1\langle u, v\rangle & \pi^2\langle u, v\rangle & (\lambda x^U.v)u \\
 & \downarrow & \downarrow & \downarrow \\
t' & u & v & v[u/x]
\end{array}
$$

- A term u *reduces* to a term v (written $u \rightsquigarrow v$) if there is a sequence $u = t_0, t_1, \ldots, t_n = v$ such that t_{i+1} is obtained by replacing a redex with its contractum.
- Recall $\lceil i \rceil = \lambda f^{U \to U}.\lambda x^U.f^i x$ and $\lceil + \rceil = \lambda m^{(U \to U) \to U \to U}.\lambda n^{(U \to U) \to U \to U}.\lambda f^{U \to U}.\lambda x^U.mf(nfx)$. We have $\lceil + \rceil \lceil i \rceil \lceil j \rceil \rightsquigarrow \lceil i + j \rceil$.

Normal Form

- A term is *normal* if none of its subterms is of the form

$$\pi^1\langle u, v\rangle \qquad \pi^2\langle u, v\rangle \qquad (\lambda x^U, v)u$$

- A *normal form* for t is a term u such that $t \rightsquigarrow u$ and u is normal.
- $\lceil i \rceil$ and $\lceil + \rceil$ are normal terms.
- The normal form for $\lceil + \rceil \lceil i \rceil \lceil j \rceil$ is $\lceil i + j \rceil$.
- An untyped term may not have a normal form. Let $\omega = \lambda x.xx$. Then $\omega\omega$ has no normal form.

Head Normal Form

- The following lemma for untyped λ-calculus will be useful.

Lemma 2

A term t is normal if and only if it is in head normal form:

$$\lambda x_1.\lambda x_2.\cdots\lambda x_n.yu_1u_2\cdots u_m$$

and u_j are normal for $1 \leq j \leq m$.

Proof.

By induction on t. If t is a variable x or an abstraction $\lambda x.u$, we are done. If t is uv, then u must be normal. By IH, u is in head normal form. But t is normal, u can only be $yu_1u_2\cdots u_m$. Thus uv is in hnf. □

Corollary 3

If the types of the free variables of a normal term t are strictly simpler than the type of t, then t is an abstraction.

Curry-Howard Isomorphism

- We now give a precise description of the isomorphism.
  - The deduction A (A in parcel i) corresponds to the variable x_i^A.
  - The deduction $\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B} \wedge\mathcal{I}$ corresponds to $\langle u, v \rangle$ where u and v correspond to the deduction of A and B respectively.
  - The deductions $\dfrac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{A} \wedge 1\mathcal{E}$ and $\dfrac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{B} \wedge 2\mathcal{E}$ correspond to $\pi^1 t$ and $\pi^2 t$ respectively, where t corresponds to the deductionof $A \wedge B$.
  - The deduction $\dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \Rightarrow B} \Rightarrow \mathcal{I}$ corresponds to $\lambda x_i^A . v$ where the discharged hypotheses form parcel i and v corresponds to the deduction of B.
  - The deduction $\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & A \Rightarrow B\end{array}}{B} \Rightarrow \mathcal{E}$ corresponds to tu where t and u correspond to the deductions of $A \Rightarrow B$ and A respectively.

Examples (revised)

- Find a representation of the following deduction:

$$\cfrac{\cfrac{\cfrac{\cfrac{[A] \quad [B]}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge 1\mathcal{E}}{B \Rightarrow A}\,\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\,\Rightarrow\mathcal{I}$$

$$\lambda x_1^A.\lambda x_1^B.\pi_1\langle x_1^A, x_1^B\rangle$$

- Find a representation of the following deduction:

$$\cfrac{\cfrac{\cfrac{[A] \quad \cfrac{\color{green}{(A \Rightarrow B) \wedge (A \Rightarrow C)}}{A \Rightarrow B}\,\wedge 1\mathcal{E}}{B}\,\Rightarrow\mathcal{E} \qquad \cfrac{[A] \quad \cfrac{\color{red}{(A \Rightarrow B) \wedge (A \Rightarrow C)}}{A \Rightarrow C}\,\wedge 2\mathcal{E}}{C}\,\Rightarrow\mathcal{E}}{B \wedge C}\,\wedge\mathcal{I}}{A \Rightarrow (B \wedge C)}\,\Rightarrow\mathcal{I}$$

$$\lambda x_1^A.\langle(\pi_1 \color{green}{x_1^{(A\Rightarrow B)\wedge(A\Rightarrow C)}})x_1^A, (\pi_2 \color{red}{x_2^{(A\Rightarrow B)\wedge(A\Rightarrow C)}})x_1^A\rangle$$

Normal Proofs

- A proof is *normal* if it does not contain any sequence of an introduction followed by an elimination rule:

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge 1\mathcal{E} \qquad \dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I}}{B}\,\wedge 2\mathcal{E} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \quad \dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B}\,\Rightarrow\mathcal{I}}{B}\,\Rightarrow\mathcal{E}$$

- Recall a λ-term is normal if it does not contain subterm of the form $\pi^1\langle u, v\rangle$, $\pi^2\langle u, v\rangle$, and $(\lambda x^U.v)u$.
- It is possible to define proof conversion as well.
- In fact, the notions of conversion, normality, and reduction exist independently in natural deduction.
- In other words, proofs have not only static interpretations (A has a deduction) but also dynamic operations (normalizing the deduction of A).