Proofs and Types
The Normalisation Theorem

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Normalisation Theorem

- Given a typed λ-term, how to find its normal form?
  - Consider $\pi^1\langle x_1^U, \pi^2\langle y_1^U, z_1^U\rangle\rangle$. Which redex should we convert first?
  - Do all strategies give the same normal form?
    - For instance, can we have both $\lceil + \rceil \lceil i \rceil \lceil j \rceil \rightsquigarrow \lceil i + j \rceil$ and $\lceil + \rceil \lceil i \rceil \lceil j \rceil \rightsquigarrow \lceil i \times j \rceil$?
  - Do all strategies terminate?
    - Recall $(\lambda x.xx)(\lambda x.xx)$.
- The uniqueness of normal form follows from Church-Rosser property.
- The normalisation theorem has two forms:
  - The weak normalisation theorem states that there is a terminating strategy for normalisation.
  - The strong normalisation theorem states that all strategies for normalisation terminate.

Church-Rosser Property

Theorem 1 (Church-Rosser)

If $t \rightsquigarrow u$ and $t \rightsquigarrow v$, then there is w such that $u \rightsquigarrow w$ and $v \rightsquigarrow w$.

$$\begin{tikzcd} & t \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ u \arrow[dr, squiggly] & & v \arrow[dl, squiggly] \\ & w & \end{tikzcd}$$

Corollary 2

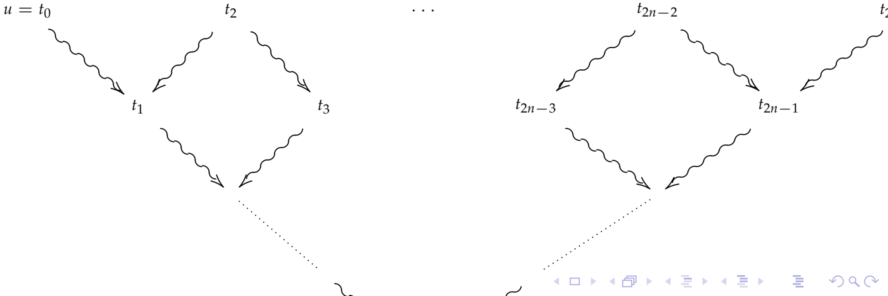
A term t has at most one normal form.

Proof.

Let $t \rightsquigarrow u$ and $t \rightsquigarrow v$ where u, v are normal. Then $u \rightsquigarrow w$ and $v \rightsquigarrow w$ for some w. Since u, v are normal, $u = w$ and $v = w$.

Consistency of Typed λ-Calculus

- Recall consistency means that $u = v$ is not deducible by equations $\pi^1\langle u, v\rangle = u$, $\pi^2\langle u, v\rangle = v$, and $(\lambda x^U.u)v = u[v/x]$ for some u, v.
  - Note that $u \rightsquigarrow v$ implies $u = v$.
- Suppose $u = v$. There are terms $u = t_0, t_1, \ldots, t_{2n-1}, t_{2n} = v$ such that $t_{2i}, t_{2i+2} \rightsquigarrow t_{2i+1}$ for $0 \leq i < n$. Hence u, v have the same normal form. Particularly, $x^U = y^U$ is not deducible.



Degree of Type, Redex, and Term

- The *degree* $\partial(T)$ of a type T is defined by
  - $\partial(T_i) = 1$ if T_i is atomic;
  - $\partial(U \times V) = \partial(U \to V) = \max(\partial(U), \partial(V)) + 1$.
- The *degree* $\partial(r)$ of a redex r is defined by
  - $\partial(\pi^1\langle u, v\rangle) = \partial(\pi^2\langle u, v\rangle) = \partial(U \times V)$ where $U \times V$ is the type of $\langle u, v\rangle$.
  - $\partial((\lambda x^U, v)u) = \partial(U \to V)$ where $U \to V$ is the type of $\lambda x^U.v$.
- The *degree* $d(t)$ of a term t is the sup of the degrees of the redexes it has. When t has no redex (that is, t is normal), $d(t) = 0$.
- Note that for any redex r of type T, $\partial(T) < \partial(r)$.
  - The types of $\pi^1\langle u, v\rangle, \pi^2\langle u, v\rangle$, and $(\lambda x^U, v)u$ are U, V, and V respectively.

Degree and Substitution

Lemma 3

If x^U is of type U, then $d(t[u/x]) \leq \max(d(t), d(u), \partial(U))$.

Proof.

We examine redexes in $t[u/x]$. In $t[u/x]$, we have

- the redexes of t modified by the substitution;
  - for instance, $t = (\lambda y^U.y)(\lambda z^U.zx)$.
- the redexes of u proliferated by occurrences of x;
  - for instance, $t = \langle x, x \rangle$ and $u = \pi^1 \langle u', u'' \rangle$.
- new redexes from substitution when $\pi^1 x, \pi^2 x, xv$ are subterms of t and $u = \langle u', u'' \rangle, \langle u', u'' \rangle, \lambda y^{U'}.u'$ respectively. These redexes have degrees equal to $\partial(U)$. □

Degree and Conversion

Lemma 4

If $t \rightsquigarrow u$, $d(u) \leq d(t)$.

Proof.

It suffices to consider one conversion where u is obtained by replacing a redex r with its contractum c in t. In u, we have

- redexes in t but not in r. Their degrees are unchanged.
  - for instance, $t = (\lambda y^U.y)(\lambda z^U.\pi^1\langle y^U, z^U\rangle)$.
- redexes in c. But c is obtained by simplification ($\pi^1\langle r', r''\rangle$ or $\pi^2\langle r', r''\rangle$), or substitution ($(\lambda x^U.r')c'$). For simplication, $d(c) \leq d(r)$. For substitution, $d(c) = d(r'[c'/x]) \leq \max(d(r'), d(c'), \partial(U))$. But $d(r'), d(c') \leq d(r)$ and $\partial(U) < d(r)$, $d(c) \leq d(r)$.
- redexes from replacing r with c ($\pi^1 c$, $\pi^2 c$, or cv). They have degrees equal to $\partial(T)$ where T is the type of r. But

Conversion of Maximal Degree

Lemma 5

Let r be a redex in t with maximal degree n. Suppose all proper sub-redexes of r have degrees less than n. If u is obtained from t by converting r to c, then u has strictly fewer redexes of degree n.

Proof.

After conversion, observe that

- redexes outside r remain unchanged.
- redexes strictly inside r are proliferated. But they all have degrees less than n.
  - For instance, $(\lambda x^U.\langle x^U, x^U\rangle)u$.
- the redex r is destroyed and possibly replaced by redexes with degrees less than n ($\pi^1 c$, $\pi^2 c$, or cv). Recall $\partial(\pi^1 c) = \partial(\pi^2 c) = \partial(cv) = \partial(T) < \partial(r)$ where T is the type of c and r.

Weak Normalisation Theorem

Theorem 6

For any term t, there is a strategy to reduce t to its normal form.

Proof.

For a term t, consider $\mu(t) = (n, m)$ where $n = d(t)$ and $m =$ the number of redexes of degree n. We obtain t' by converting the redex of degree n whose strict sub-redexes all have degrees less than n. Then $\mu(t') < \mu(t)$ in lexicographical order (Lemma 5). The result follows by double induction. □

- Recall that $(\lambda x.xx)(\lambda x.xx)$ does not have a normal form.
  - Can you give a type to $(\lambda x.xx)(\lambda x.xx)$?
  - The weak normalisation theorem holds only for typed λ-calculus.

Weak Normalisation Theorem

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Example

- Recall the term $t = \lambda x_1^A.\lambda x_1^B.\pi_1\langle x_1^A, x_1^B\rangle$ for the proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A] \qquad [B]}{A \wedge B}\wedge\mathcal{I}}{A}\wedge 1\mathcal{E}}{B \Rightarrow A}\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\Rightarrow\mathcal{I}$$

- t has 1 redex $r = \pi_1\langle x_1^A, x_1^B\rangle$.
  - $\partial(r) = \partial(A \times B) = \max(\partial(A), \partial(B)) + 1 = 1$.
  - Hence $d(t) = 1$.
- We convert t by converting r and obtain $t' = \lambda x_1^A.\lambda_1^B.x_1^A$.
  - t' has no redex and hence $d(t') = 0$.
- Here is the proof tree corresponding to t':

$$\dfrac{\dfrac{[A] \qquad [B]}{B \Rightarrow A}\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\Rightarrow\mathcal{I}$$

Decidability of Equality

- Given terms u and v, is $u \stackrel{?}{=} v$ decidable?
- Recall that if $u = v$, then u and v have the same normal form.
  - See the proof of consistency.
- By the proof of the weak normalisation theorem, we can compute the normal forms of u and v effectively.
- Return YES if and only if their normal forms coincide.