Proofs and Types
Sequent Calculus

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Sequents

- A sequent is of the form $A \vdash B$ where $A$ and $B$ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$.
- Informally, $A \vdash B$ means the conjunction of $A$ implies the disjunction of $B$. Particularly,
  - $\vdash B$ asserts $\bigvee_j B_j$.
  - $A \vdash$ asserts $\neg \bigwedge_i A_i$.
  - $\vdash$ asserts contradiction.
Structural Rules

- **The exchange rules:**
  \[
  \frac{A, C, D, A' \vdash B}{A, D, C, A' \vdash B} \quad LX
  \]
  \[
  \frac{A \vdash B, C, D, B'}{A \vdash B, D, C, B'} \quad RX
  \]

- **The weakening rules:**
  \[
  \frac{A \vdash B}{A, C \vdash B} \quad LW
  \]
  \[
  \frac{A \vdash B}{A \vdash C, B} \quad RW
  \]

- **The contraction rules:**
  \[
  \frac{A, C, C \vdash B}{A, C \vdash B} \quad LC
  \]
  \[
  \frac{A \vdash C, C, B}{A \vdash C, B} \quad RC
  \]

- The structural rules essentially say that \( A \) and \( B \) in the sequence \( A \vdash B \) are multisets.
- Observe the (beautiful) symmetry in sequent calculus.
Intuitionistic Structural Rules

- An *intuitionistic sequent* is a sequent $\Gamma \vdash B$ where $B$ has at most one formula.
- The *exchange* and *contraction* rules:
  \[
  \frac{A, C, D, A' \vdash B}{A, D, C, A' \vdash B} \quad \text{LX} \quad \frac{A, C, C \vdash B}{A, C \vdash B} \quad \text{LC}
  \]
- The *weakening* rules:
  \[
  \frac{\Gamma \vdash B}{\Gamma, C \vdash B} \quad \text{LW} \quad \frac{A \vdash \cdot}{A \vdash C} \quad \text{RW}
  \]
- Note that $\text{RX}$ and $\text{RC}$ rules are not possible.
- And the symmetry is broken...
“Identity” Rules

- For every formula $C$, we have the identity axiom.
  \[
  \frac{}{C \vdash C}
  \]
- The cut rule:
  \[
  \frac{A \vdash C, B \quad A', C \vdash B'}{A, A' \vdash B, B'} \quad \text{Cut}
  \]
- The cut rule can be seen as the symmetric rule to identity axiom.
  - The identity axiom states $C$-left is stronger than $C$-right.
  - The cut rule states $C$-right is stronger than $C$-left.
- The cut rule is not welcome in proof search.
  - How can an algorithm guess $C$ to prove $A, A' \vdash B, B'$?
- Surprisingly, the cut rule is not necessary.
  - For every proof for a sequent, there is a cut-free proof for the same sequent.
Intuitionistic “Identity” Rules

- For every formula $C$, we have the *identity axiom*.
  \[
  \frac{}{C \vdash C}
  \]

- The *cut* rule:
  \[
  \frac{A \vdash C, A', C \vdash B'}{A, A' \vdash B'} \quad \text{Cut}
  \]

- Intuitionistic identity rules are as expected.
Logic Rules – I

▫ **Negation.**

\[
\frac{A \vdash C, B}{A, \neg C \vdash B} \quad \mathcal{L}\neg
\]

\[
\frac{A, C \vdash B}{A, \neg C \vdash B} \quad \mathcal{R}\neg
\]

▫ **Conjunction.**

\[
\frac{A, C \vdash B}{A, C \land D \vdash B} \quad \mathcal{L}1\land
\]

\[
\frac{A, D \vdash B}{A, C \land D \vdash B} \quad \mathcal{L}2\land
\]

\[
\frac{A \vdash C, B}{A, A' \vdash C \land D, B, B'} \quad \mathcal{R}\land
\]

▫ **Disjunction.**

\[
\frac{A, C \vdash B}{A, A', C \lor D \vdash B, B'} \quad \mathcal{L}\lor
\]

\[
\frac{A \vdash D, B}{A \vdash C \lor D, B} \quad \mathcal{R}1\lor
\]

\[
\frac{A \vdash D, B}{A \vdash C \lor D, B} \quad \mathcal{R}2\lor
\]
Logical Rules – II

- **Implication.**

\[
\begin{align*}
\frac{A \vdash C, B}{A, A', D \vdash B'} & \quad \mathcal{L} \Rightarrow \\
\frac{A, A', C \Rightarrow D \vdash B, B'}{A \vdash C \Rightarrow D, B} & \quad \mathcal{R} \Rightarrow
\end{align*}
\]

- **Universal quantification.**

\[
\begin{align*}
\frac{A, C[a/\xi] \vdash B}{A, \forall \xi. C \vdash B} & \quad \mathcal{L}\forall \\
\frac{A \vdash C, B}{A \vdash \forall \xi. C, B} & \quad \mathcal{R}\forall
\end{align*}
\]

- **Existential quantification.**

\[
\begin{align*}
\frac{A, C \vdash B}{A, \exists \xi. C \vdash B} & \quad \mathcal{L}\exists \\
\frac{A \vdash C[a/\xi], B}{A \vdash \exists \xi. C, B} & \quad \mathcal{R}\exists
\end{align*}
\]

- Observe again the symmetry in these rules.
Intuitionistic Logical Rules

- **Negation.**
  \[
  \frac{\Gamma \vdash C}{\Gamma, \neg C \vdash} \quad \frac{\Gamma, C \vdash}{\Gamma \vdash \neg C}
  \]

- **Conjunction.**
  \[
  \frac{\Gamma, C \vdash B}{\Gamma, C \land D \vdash B} \quad \frac{\Gamma, D \vdash B}{\Gamma, C \land D \vdash B}
  \]

- **Disjunction.**
  \[
  \frac{\Gamma \vdash C}{\Gamma, A' \vdash C \land D} \quad \frac{\Gamma, D \vdash B}{\Gamma, A' \vdash C \land D}
  \]

- **All rules except \( \mathcal{L} \lor \) are as expected.**
Intuitionistic Logical Rules – II

- **Implication.**
  \[
  \frac{A \vdash C \quad A', D \vdash B'}{A, A', C \Rightarrow D \vdash B'} \quad \mathcal{L} \Rightarrow \quad \frac{A, C \vdash D}{A \vdash C \Rightarrow D} \quad \mathcal{R} \Rightarrow
  \]

- **Universal quantification.**
  \[
  \frac{A, C[a/\xi] \vdash B}{A, \forall \xi. C \vdash B} \quad \mathcal{L}\forall \quad \frac{A \vdash C}{A \vdash \forall \xi. C} \quad \mathcal{R}\forall
  \]

- **Existential quantification.**
  \[
  \frac{A, C \vdash B}{A, \exists \xi. C \vdash B} \quad \mathcal{L}\exists \quad \frac{A \vdash C[a/\xi]}{A \vdash \exists \xi. C} \quad \mathcal{R}\exists
  \]

- All rules are as expected.
Examples

- Consider \( \vdash A \implies (B \implies A) \).

  \[
  \begin{array}{c}
  \frac{A \vdash A}{A \land B \vdash A} \quad \frac{A \vdash A \quad B \vdash B}{A, B \vdash A \land B} \quad \frac{A, B \vdash A}{A, B \vdash A} \quad \frac{A \vdash A}{\vdash A \implies A} \quad \frac{A \vdash B \implies A}{\vdash A \implies (B \implies A)}
  \end{array}
  \]

- Consider \( \vdash \forall x. Px \implies \forall y. Py \).

  \[
  \begin{array}{c}
  \frac{Py \vdash Py}{\forall x. Px \vdash Py} \quad \frac{Py \vdash Py}{\forall x. Px \vdash \forall y. Py} \quad \frac{\forall x. Px \vdash \forall y. Py}{\vdash \forall x. Px \implies \forall y. Py}
  \end{array}
  \]
Consider a proof of \( \vdash A \) without cut.
What could be the last rule?
- Structural rules cannot give us \( \vdash A \).
- The identity axiom does not give us \( \vdash A \).
- Left logical rules cannot do.

The last rule must be a right logical rule.
- If \( A = A' \lor A'' \), the last rule must be \( R1\lor \) or \( R2\lor \). That is, we have \( \vdash A' \) or \( \vdash A'' \). If \( \vdash A' \lor A'' \), then \( \vdash A' \) or \( \vdash A'' \). This is called the Disjunction Property.
- If \( A = \exists \xi. A' \), the last rule must be \( R\exists \). That is, we have \( \vdash A'[a/\xi] \). If \( \vdash \exists \xi. A' \) is provable, then \( \vdash A'[a/\xi] \) for some term \( a \). This is called the Existence Property.
Subformula Property

- Can we predict premises of the last rule in a proof?
  - The cut rule is unpredictable.
    - There is no way to guess the cut formula $C$.
- Define
  - The immediate subformulae of $A \land B$, $A \lor B$, and $A \Rightarrow B$ are $A$ and $B$;
  - The immediate subformula of $\neg A$ is $A$;
  - The immediate subformulae of $\forall \xi.A$ and $\exists \xi.A$ are $A[a/\xi]$ with any term $a$.
- Except the cut rule, all rules preserve “contexts” (written $A, A', B, B'$) and change only one formula; moreover, the premises are immediate subformulae of the conclusion.
  - This is called Subformula Property.
  - Subformula property is very useful in automated deduction.
    - We only consider subformulae in proof search.
Consider the fragment with $\land$, $\Rightarrow$, and $\forall$.

A proof of $\vdash A \vdash B$ corresponds to a deduction of $B$ under parcels of hypotheses $A$.

$$A_1 \ A_2 \ \cdots \ A_n$$

$A \vdash B \quad \leftrightarrow \quad B$

Conversely, a deduction of $B$ under parcels of hypotheses $A$ can be represented by a proof of $\vdash A \vdash B$.

Why not consider $A \vdash B$?

A deduction is for a formula, not formulae.
The identity group gives basic deductions.

For the identity axiom,
\[ A \vdash A \implies A. \]

For the cut rule,
\[
\frac{A \vdash B \quad A', B \vdash C}{A, A' \vdash C} \text{ Cut} \quad \implies \quad A' \\
\vdots \\
\vdots \\
B \vdots \\
C
\]
From Sequent Calculus to Natural Deduction

- Structural rules manage parcels.
- For rule $\mathcal{L}X$,
  \[
  \frac{A, C, D, A' \vdash B}{A, D, C, A' \vdash B} \quad \mathcal{L}X
  \]
  \[
  \frac{A \vdash B}{A, C \vdash B} \quad \mathcal{L}W
  \]
  \[
  \frac{A, C, C \vdash B}{A, C \vdash B} \quad \mathcal{L}C
  \]
  
  For rule $\mathcal{L}W$, add a new parcel.
  For rule $\mathcal{L}C$, merge two parcels.
From Sequent Calculus to Natural Deduction

- Right logical rules correspond to introduction.
  - For rule $\mathcal{R}\land$,
    \[
    \frac{A \vdash B \quad A' \vdash C}{A, A' \vdash B \land C} \quad \mathcal{R}\land
    \quad \mapsto
    \begin{array}{l}
    A \\
    A'
    \end{array}
    
    \begin{array}{l}
    B \\
    C
    \end{array}
    \begin{array}{l}
    B \land C
    \end{array}
    \quad \land\mathcal{I}
    \]
  
  - For rule $\mathcal{R}\Rightarrow$,
    \[
    \frac{A, B \vdash C}{A \vdash B \Rightarrow C} \quad \mathcal{R}\Rightarrow
    \quad \mapsto
    \begin{array}{l}
    A \\
    [B]
    \end{array}
    \begin{array}{l}
    B \Rightarrow C
    \end{array}
    \Rightarrow\mathcal{I}
    \]
  
  - For rule $\mathcal{R}\forall$,
    \[
    \frac{A \vdash B}{A \vdash \forall\xi.B} \quad \mathcal{R}\forall
    \quad \mapsto
    \begin{array}{l}
    A
    \end{array}
    \begin{array}{l}
    B
    \end{array}
    \begin{array}{l}
    B
    \end{array}
    \begin{array}{l}
    \forall\xi.B
    \end{array}
    \]
From Sequent Calculus to Natural Deduction

- Left logical rules correspond to elimination.

- For rule $\mathcal{L}1\land$,

  \[
  \frac{A, B \vdash D}{A, B \land C \vdash D} \quad \mathcal{L}1\land \quad \frac{B \land C}{B} \quad \land 1\varepsilon
  \]

- For rule $\mathcal{L} \Rightarrow$,

  \[
  \frac{A \vdash B}{A, A', C \vdash D} \quad \frac{A', C \vdash D}{A, A', B \Rightarrow C \vdash D} \quad \mathcal{L} \Rightarrow \quad \frac{B \Rightarrow C \vdash \varepsilon}{\Rightarrow \varepsilon}
  \]

- For rule $\mathcal{L}\forall$,

  \[
  \frac{A, B[a/\xi] \vdash C}{\forall \xi. B \vdash C} \quad \mathcal{L}\forall \quad \frac{\forall \xi. B}{B[a/\xi]} \quad \forall \varepsilon
  \]
Example

Recall the proof of the sequent \( \vdash A \Rightarrow (B \Rightarrow A) \).

\[
\frac{A \vdash A}{A \land B \vdash A} \quad \frac{A \vdash A \quad B \vdash B}{A, B \vdash A \land B} \quad \frac{}{\text{Cut}}
\]

\[
\frac{A, B \vdash A}{A \vdash B \Rightarrow A} \quad \frac{}{\Rightarrow}
\]

\[
\frac{}{\Rightarrow}
\]

\[
\frac{}{\Rightarrow}
\]

\[
\frac{}{\Rightarrow}
\]

\[
\frac{}{\Rightarrow}
\]

\[
\frac{}{\Rightarrow}
\]

\[
\frac{}{\Rightarrow}
\]
Different Proofs Correspond to a Deduction

▷ Consider

\[
\frac{A \vdash A}{A, B \vdash A \land B} \quad \text{R}\land \\
\frac{B \vdash B}{A, B \vdash A \land B} \\
\frac{A \land A', B \vdash A \land B}{A \land A', B \land B' \vdash A \land B} \quad \text{L}\,1\land \\
\frac{A \land A', B \vdash A \land B}{A \land A', B \land B' \vdash A \land B}
\]

and

\[
\frac{A \vdash A}{A, B \vdash A \land B} \quad \text{R}\land \\
\frac{B \vdash B}{A, B \vdash A \land B} \\
\frac{A \land A', B \vdash A \land B}{A \land A', B \land B' \vdash A \land B} \quad \text{L}\,1\land \\
\frac{A \land A', B \vdash A \land B}{A \land A', B \land B' \vdash A \land B}
\]

▷ Both correspond to the same deduction

\[
\frac{A \land A'}{A} \quad \land\,1E \\
\frac{B \land B'}{B} \quad \land\,1E \\
\frac{A \land B}{A \land B}
\]

▷ Natural deductions reflect to our informal notion of “proofs” more closely.

▷ Sequent calculus on the other hand manipulates such “proofs.”

▷ \( A \vdash B \) means a “proof” of \( B \) from \( A \).
Direction of Expansion

- Right logical rules in sequent calculus correspond to introduction rules in natural deduction.
  - The translation expands the deduction downwards (to the root).
- Left logical rules in sequent calculus correspond to elimination rules in natural deduction.
  - The translation expands the deduction upwards (to leaves).
- We can make the translation expand downwards by the cut rule.

\[
\begin{align*}
\frac{A' \vdash A}{A', A \Rightarrow B \vdash B} & \quad \frac{B \vdash B}{A', A \Rightarrow B} \\
\frac{A', A \Rightarrow B \vdash B}{A', B' \vdash B} & \quad \frac{B' \vdash A \Rightarrow B}{A' \Rightarrow B}
\end{align*}
\]
Normal Deductions and Cut-Free Proofs

- A non-normal deduction results from an introduction followed by an elimination.

\[
\begin{align*}
\frac{[A] \quad [B]}{A \land B} \quad \land \text{I} \\
\frac{A \land B}{A} \quad \land \text{E} \\
\frac{A}{B \Rightarrow A} \quad \Rightarrow \text{I} \\
\frac{B \Rightarrow A}{A \Rightarrow (B \Rightarrow A)} \quad \Rightarrow \text{I}
\end{align*}
\]

- The cut rule can stack introduction on elimination and thus yield non-normal deduction.

\[
\begin{align*}
\frac{A \land B}{A} \quad \land \text{I} & \quad \frac{A \land B}{A} \quad \land \text{E} \\
\frac{A}{A \Rightarrow (B \Rightarrow A)} \quad \Rightarrow \text{I} & \quad \frac{A}{A \Rightarrow (B \Rightarrow A)} \quad \Rightarrow \text{I}
\end{align*}
\]

- Thus,
Normal Form, Normal Deduction, Cut-Free Proof

- A deduction corresponds to a typed $\lambda$-term.
  - Curry-Howard isomorphism.
- Any typed $\lambda$-term has a normal form.
  - The weak normalisation theorem and Church-Rosser property.
- Any deduction can be normalised.
  - Curry-Howard isomorphism.
- A sequent proof corresponds to a deduction.
- A sequent proof has a cut-free form.
  - The cut-elimination theorem (Hauptsatz).
- A cut-free sequent proof corresponds to a normal deduction.
- The cut-elimination theorem corresponds to the normalisation theorem.