Proofs and Types
Sequent Calculus

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Sequents

- A *sequent* is of the form $\underline{A} \vdash \underline{B}$ where $\underline{A}$ and $\underline{B}$ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$.
- Informally, $\underline{A} \vdash \underline{B}$ means the conjunction of $\underline{A}$ implies the disjunction of $\underline{B}$. Particularly,
  - $\vdash \underline{B}$ asserts $\bigvee_j B_j$.
  - $\underline{A} \vdash$ asserts $\neg \bigwedge_i A_i$.
  - $\vdash$ asserts contradiction.

Structural Rules

- The *exchange* rules:

$$\frac{\underline{A}, C, D, \underline{A'} \vdash \underline{B}}{\underline{A}, D, C, \underline{A'} \vdash \underline{B}}\ \mathcal{L}X \qquad \frac{\underline{A} \vdash \underline{B}, C, D, \underline{B'}}{\underline{A} \vdash \underline{B}, D, C, \underline{B'}}\ \mathcal{R}X$$

- The *weakening* rules:

$$\frac{\underline{A} \vdash \underline{B}}{\underline{A}, C \vdash \underline{B}}\ \mathcal{L}W \qquad \frac{\underline{A} \vdash \underline{B}}{\underline{A} \vdash C, \underline{B}}\ \mathcal{R}W$$

- The *contraction* rules:

$$\frac{\underline{A}, C, C \vdash \underline{B}}{\underline{A}, C \vdash \underline{B}}\ \mathcal{L}C \qquad \frac{\underline{A} \vdash C, C, \underline{B}}{\underline{A} \vdash C, \underline{B}}\ \mathcal{R}C$$

- The structural rules essentially say that $\underline{A}$ and $\underline{B}$ in the sequence $\underline{A} \vdash \underline{B}$ are multisets.
- Observe the (beautiful) symmetry in sequent calculus.

Intuitionistic Structural Rules

- An *intuitionistic sequent* is a sequent $\underline{A} \vdash \underline{B}$ where $\underline{B}$ has at most one formula.
- The *exchange* and *contraction* rules:

$$\frac{\underline{A}, C, D, \underline{A'} \vdash \underline{B}}{\underline{A}, D, C, \underline{A'} \vdash \underline{B}}\ \mathcal{L}X \qquad \frac{\underline{A}, C, C \vdash \underline{B}}{\underline{A}, C \vdash \underline{B}}\ \mathcal{L}C$$

- The *weakening* rules:

$$\frac{\underline{A} \vdash \underline{B}}{\underline{A}, C \vdash \underline{B}}\ \mathcal{L}W \qquad \frac{\underline{A} \vdash}{\underline{A} \vdash C}\ \mathcal{R}W$$

- Note that $\mathcal{R}X$ and $\mathcal{R}C$ rules are not possible.
- And the symmetry is broken...

"Identity" Rules

- For every formula C, we have the *identity axiom*.

$$\frac{}{C \vdash C}$$

- The *cut* rule:

$$\frac{\underline{A} \vdash C, \underline{B} \qquad \underline{A'}, C \vdash \underline{B'}}{\underline{A}, \underline{A'} \vdash \underline{B}, \underline{B'}}\ Cut$$

- The cut rule can be seen as the symmetric rule to identity axiom.
  - The identity axiom states C-left is stronger than C-right.
  - The cut rule states C-right is stronger than C-left.
- The cut rule is not welcome in proof search.
  - How can an algorithm guess C to prove $\underline{A}, \underline{A'} \vdash \underline{B}, \underline{B'}$?
- Surprisingly, the cut rule is not necessary.
  - For every proof for a sequent, there is a cut-free proof for the same sequent.

Intuitionistic "Identity" Rules

- For every formula C, we have the *identity axiom*.

$$\frac{}{C \vdash C}$$

- The *cut* rule:

$$\frac{\underline{A} \vdash C \qquad \underline{A}', C \vdash \underline{B}'}{\underline{A}, \underline{A}' \vdash \underline{B}'}\ Cut$$

- Intuitionistic identity rules are as expected.

Logic Rules – I

- *Negation.*

$$\frac{\underline{A} \vdash C, \underline{B}}{\underline{A}, \neg C \vdash \underline{B}}\ \mathcal{L}\neg \qquad \frac{\underline{A}, C \vdash \underline{B}}{\underline{A} \vdash \neg C, \underline{B}}\ \mathcal{R}\neg$$

- *Conjunction.*

$$\frac{\underline{A}, C \vdash \underline{B}}{\underline{A}, C \wedge D \vdash \underline{B}}\ \mathcal{L}1\wedge \qquad \frac{\underline{A}, D \vdash \underline{B}}{\underline{A}, C \wedge D \vdash \underline{B}}\ \mathcal{L}2\wedge$$

$$\frac{\underline{A} \vdash C, \underline{B} \qquad \underline{A}' \vdash D, \underline{B}'}{\underline{A}, \underline{A}' \vdash C \wedge D, \underline{B}, \underline{B}'}\ \mathcal{R}\wedge$$

- *Disjunction.*

$$\frac{\underline{A}, C \vdash \underline{B} \qquad \underline{A}', D \vdash \underline{B}'}{\underline{A}, \underline{A}', C \vee D \vdash \underline{B}, \underline{B}'}\ \mathcal{L}\vee$$

$$\frac{\underline{A} \vdash C, \underline{B}}{\underline{A} \vdash C \vee D, \underline{B}}\ \mathcal{R}1\vee \qquad \frac{\underline{A} \vdash D, \underline{B}}{\underline{A} \vdash C \vee D, \underline{B}}\ \mathcal{R}2\vee$$

Logical Rules – II

- *Implication.*

$$\frac{\underline{A} \vdash C, \underline{B} \qquad \underline{A}', D \vdash \underline{B}'}{\underline{A}, \underline{A}', C \Rightarrow D \vdash \underline{B}, \underline{B}'}\ \mathcal{L}\Rightarrow \qquad \frac{\underline{A}, C \vdash D, \underline{B}}{\underline{A} \vdash C \Rightarrow D, \underline{B}}\ \mathcal{R}\Rightarrow$$

- *Universal quantification.*

$$\frac{\underline{A}, C[a/\xi] \vdash \underline{B}}{\underline{A}, \forall\xi.C \vdash \underline{B}}\ \mathcal{L}\forall \qquad \frac{\underline{A} \vdash C, \underline{B}}{\underline{A} \vdash \forall\xi.C, \underline{B}}\ \mathcal{R}\forall$$

- *Existential quantification.*

$$\frac{\underline{A}, C \vdash \underline{B}}{\underline{A}, \exists\xi.C \vdash \underline{B}}\ \mathcal{L}\exists \qquad \frac{\underline{A} \vdash C[a/\xi], \underline{B}}{\underline{A} \vdash \exists\xi.C, \underline{B}}\ \mathcal{R}\exists$$

- Observe again the symmetry in these rules.

Intuitionistic Logical Rules

- *Negation.*

$$\frac{\underline{A} \vdash C}{\underline{A}, \neg C \vdash} \mathcal{L}\neg \qquad \frac{\underline{A}, C \vdash}{\underline{A} \vdash \neg C} \mathcal{R}\neg$$

- *Conjunction.*

$$\frac{\underline{A}, C \vdash \underline{B}}{\underline{A}, C \wedge D \vdash \underline{B}} \mathcal{L}1\wedge \qquad \frac{\underline{A}, D \vdash \underline{B}}{\underline{A}, C \wedge D \vdash \underline{B}} \mathcal{L}2\wedge$$

$$\frac{\underline{A} \vdash C \qquad \underline{A'} \vdash D}{\underline{A}, \underline{A'} \vdash C \wedge D} \mathcal{R}\wedge$$

- *Disjunction.*

$$\frac{\underline{A}, C \vdash \underline{B} \qquad \underline{A'}, D \vdash \underline{B}}{\underline{A}, \underline{A'}, C \vee D \vdash \underline{B}} \mathcal{L}\vee$$

$$\frac{\underline{A} \vdash C}{\underline{A} \vdash C \vee D} \mathcal{R}1\vee \qquad \frac{\underline{A} \vdash D}{\underline{A} \vdash C \vee D} \mathcal{R}2\vee$$

- All rules except $\mathcal{L}\vee$ are as expected.

Intuitionistic Logical Rules – II

- *Implication.*

$$\frac{\underline{A} \vdash C \qquad \underline{A}', D \vdash \underline{B}'}{\underline{A}, \underline{A}', C \Rightarrow D \vdash \underline{B}'} \mathcal{L} \Rightarrow \qquad \frac{\underline{A}, C \vdash D}{\underline{A} \vdash C \Rightarrow D} \mathcal{R} \Rightarrow$$

- *Universal quantification.*

$$\frac{\underline{A}, C[a/\xi] \vdash \underline{B}}{\underline{A}, \forall\xi.C \vdash \underline{B}} \mathcal{L}\forall \qquad \frac{\underline{A} \vdash C}{\underline{A} \vdash \forall\xi.C} \mathcal{R}\forall$$

- *Existential quantification.*

$$\frac{\underline{A}, C \vdash \underline{B}}{\underline{A}, \exists\xi.C \vdash \underline{B}} \mathcal{L}\exists \qquad \frac{\underline{A} \vdash C[a/\xi]}{\underline{A} \vdash \exists\xi.C} \mathcal{R}\exists$$

- All rules are as expected.

Examples

- Consider $\vdash A \Rightarrow (B \Rightarrow A)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A}\,\mathcal{L}1\wedge \qquad \dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\,\mathcal{R}\wedge}{A, B \vdash A}\,\text{Cut}}{A \vdash B \Rightarrow A}\,\mathcal{R}\Rightarrow}{\vdash A \Rightarrow (B \Rightarrow A)}\,\mathcal{R}\Rightarrow$$

- Consider $\vdash \forall x.Px \Rightarrow \forall y.Py$.

$$\dfrac{\dfrac{\dfrac{Py \vdash Py}{\forall x.Px \vdash Py}\,\mathcal{L}\forall}{\forall x.Px \vdash \forall y.Py}\,\mathcal{R}\forall}{\vdash \forall x.Px \Rightarrow \forall y.Py}\,\mathcal{R}\Rightarrow$$

Properties of Intuitionistic Sequent Calculus

- Consider a proof of $\vdash A$ without cut.
- What could be the last rule?
  - Structural rules cannot give us $\vdash A$.
  - The identity axiom does not give us $\vdash A$.
  - Left logical rules cannot do.
- The last rule must be a right logical rule.
- If $A = A' \vee A''$, the last rule must be $\mathcal{R}1\vee$ or $\mathcal{R}2\vee$. That is, we have $\vdash A'$ or $\vdash A''$. If $\vdash A' \vee A''$, then $\vdash A'$ or $\vdash A''$. This is called the *Disjunction Property*.
- If $A = \exists\xi.A'$, the last rule must be $\mathcal{R}\exists$. That is, we have $\vdash A'[a/\xi]$. If $\vdash \exists\xi.A'$ is provable, then $\vdash A'[a/\xi]$ for some term a. This is called the *Existence Property*.

Subformula Property

- Can we predict premises of the last rule in a proof?
- The cut rule is unpredictable.
  - There is no way to guess the cut formula C.
- Define
  - The *immediate subformulae* of $A \wedge B$, $A \vee B$, and $A \Rightarrow B$ are A and B;
  - The *immediate subformula* of $\neg A$ is A;
  - The *immediate subformulae* of $\forall \xi . A$ and $\exists \xi . A$ are $A[a/\xi]$ with any term a.
- Except the cut rule, all rules preserve "contexts" (written $\underline{A}, \underline{A}', \underline{B}, \underline{B}'$) and change only one formula; moreover, the premises are immediate subformulae of the conclusion.
- This is called *Subformula Property*.
- Subformula property is very useful in automated deduction.
  - We only consider subformulae in proof search.

From Sequent Calculus to Natural Deduction

- Consider the fragment with $\wedge$, $\Rightarrow$, and $\forall$.
- A proof of $\underline{A} \vdash B$ corresponds to a deduction of B under parcels of hypotheses $\underline{A}$.

$$\underline{A} \vdash B \quad \longmapsto \quad \begin{array}{c} A_1 \;\; A_2 \;\; \cdots \;\; A_n \\ \vdots \\ B \end{array}$$

- Conversely, a deduction of B under parcels of hypotheses $\underline{A}$ can be represented by a proof of $\underline{A} \vdash B$.
- Why not consider $\underline{A} \vdash \underline{B}$?
  - A deduction is for a formula, not formulae.

From Sequent Calculus to Natural Deduction

- The identity group gives basic deductions.
- For the identity axiom,

$$A \vdash A \quad \longmapsto \quad A.$$

- For the cut rule,

$$\dfrac{\underline{A} \vdash B \qquad \underline{A'}, B \vdash C}{\underline{A}, \underline{A'} \vdash C}\,\text{Cut} \quad \longmapsto \quad \begin{array}{cc} & \underline{A} \\ & \vdots \\ \underline{A'} & B \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{C} \end{array}$$

From Sequent Calculus to Natural Deduction

- Structural rules manage parcels.
- For rule $\mathcal{L}X$,

$$\frac{\underline{A}, C, D, \underline{A'} \vdash B}{\underline{A}, D, C, \underline{A'} \vdash B}\,\mathcal{L}X \quad \longmapsto \quad \begin{array}{c} \underline{A}\;\; C\;\; D\;\; \underline{A'} \\ \vdots \\ B \end{array}$$

- For rule $\mathcal{L}W$, add a new parcel.

$$\frac{\underline{A} \vdash B}{\underline{A}, C \vdash B}\,\mathcal{L}W \quad \longmapsto \quad \begin{array}{c} \underline{A}\;\; C \\ \vdots \\ B \end{array}$$

- For rule $\mathcal{L}C$, merge two parcels.

$$\frac{\underline{A}, C, C \vdash B}{\underline{A}, C \vdash B}\,\mathcal{L}C \quad \longmapsto \quad \begin{array}{c} \underline{A}\;\; \boxed{C\;\; C} \\ \vdots \\ B \end{array}$$

From Sequent Calculus to Natural Deduction

- Right logical rules correspond to introduction.
- For rule $\mathcal{R}\wedge$,

$$\frac{\underline{A} \vdash B \qquad \underline{A}' \vdash C}{\underline{A}, \underline{A}' \vdash B \wedge C}\,\mathcal{R}\wedge \quad \longmapsto \quad \begin{array}{c} \underline{A} \qquad \underline{A}' \\ \vdots \qquad\quad \vdots \\ \dfrac{B \qquad\quad C}{B \wedge C}\,\wedge\mathcal{I} \end{array}$$

- For rule $\mathcal{R} \Rightarrow$,

$$\frac{\underline{A}, B \vdash C}{\underline{A} \vdash B \Rightarrow C}\,\mathcal{R} \Rightarrow \quad \longmapsto \quad \begin{array}{c} \underline{A} \;\; [B] \\ \vdots \\ \dfrac{C}{B \Rightarrow C} \Rightarrow \mathcal{I} \end{array}$$

- For rule $\mathcal{R}\forall$,

$$\frac{\underline{A} \vdash B}{\underline{A} \vdash \forall\xi.B}\,\mathcal{R}\forall \quad \longmapsto \quad \begin{array}{c} \underline{A} \\ \vdots \\ B \end{array}$$

From Sequent Calculus to Natural Deduction

- Left logical rules correspond to elimination.
- For rule $\mathcal{L}1\wedge$,

$$\frac{\underline{A}, B \vdash D}{\underline{A}, B \wedge C \vdash D}\ \mathcal{L}1\wedge \qquad \longmapsto \qquad \begin{array}{c} \underline{A} \qquad \dfrac{B \wedge C}{B}\ \wedge 1\mathcal{E} \\ \vdots \\ D \end{array}$$

- For rule $\mathcal{L} \Rightarrow$,

$$\frac{\underline{A} \vdash B \qquad \underline{A'}, C \vdash D}{\underline{A}, \underline{A'}, B \Rightarrow C \vdash D}\ \mathcal{L} \Rightarrow \qquad \longmapsto \qquad \begin{array}{c} \underline{A'} \qquad \dfrac{\begin{array}{c} \underline{A} \\ \vdots \\ B \end{array} \qquad B \Rightarrow C}{C}\ \Rightarrow \mathcal{E} \\ \vdots \\ D \end{array}$$

- For rule $\mathcal{L}\forall$,

$$\frac{\underline{A}, B[a/\xi] \vdash C}{\quad}\ \mathcal{L}\forall \qquad \longmapsto \qquad \underline{A} \qquad \frac{\forall \xi . B}{B[a/\xi]}\ \forall\mathcal{E}$$

Example

- Recall the proof of the sequent $\vdash A \Rightarrow (B \Rightarrow A)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A}\,\mathcal{L}1\wedge \qquad \dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\,\mathcal{R}\wedge}{A, B \vdash A}\,\text{Cut}}{A \vdash B \Rightarrow A}\,\mathcal{R}\Rightarrow}{\vdash A \Rightarrow (B \Rightarrow A)}\,\mathcal{R}\Rightarrow$$

$$\longmapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{A \wedge B}{A}\,\wedge 1\mathcal{E} \qquad \dfrac{A \qquad B}{A \wedge B}\,\wedge\mathcal{I}}{A, B \vdash A}\,\text{Cut}}{A \vdash B \Rightarrow A}\,\mathcal{R}\Rightarrow}{\vdash A \Rightarrow (B \Rightarrow A)}\,\mathcal{R}\Rightarrow$$

$$\longmapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{A \qquad B}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge\mathcal{E}}{A \vdash B \Rightarrow A}\,\mathcal{R}\Rightarrow}{\vdash A \Rightarrow (B \Rightarrow A)}\,\mathcal{R}\Rightarrow$$

$$\longmapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{A \qquad [B]}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge\mathcal{E}}{B \Rightarrow A}\,\Rightarrow\mathcal{I}}{\vdash A \Rightarrow (B \Rightarrow A)}\,\mathcal{R}\Rightarrow$$

$$\longmapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{[A] \qquad [B]}{A \wedge B}\,\wedge\mathcal{I}}{A}\,\wedge\mathcal{E}}{B \Rightarrow A}\,\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\,\Rightarrow\mathcal{I}$$

Different Proofs Correspond to a Deduction

- Consider

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\,\mathcal{R}\wedge}{\overline{\overline{A \wedge A', B \vdash A \wedge B}}}\,\mathcal{L}1\wedge}{A \wedge A', B \wedge B' \vdash A \wedge B}\,\mathcal{L}1\wedge \quad \text{and} \quad \dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\,\mathcal{R}\wedge}{A, B \wedge B' \vdash A \wedge B}\,\mathcal{L}1\wedge}{\overline{\overline{A \wedge A', B \wedge B' \vdash A \wedge B}}}\,\mathcal{L}1\wedge$$

- Both correspond to the same deduction

$$\dfrac{\dfrac{A \wedge A'}{A}\,\wedge 1\mathcal{E} \qquad \dfrac{B \wedge B'}{B}\,\wedge 1\mathcal{E}}{A \wedge B}\,\wedge\mathcal{I}$$

- Natural deductions reflect to our informal notion of "proofs" more closely.
- Sequent calculus on the other hand manipulates such "proofs."
  - $\underline{A} \vdash B$ means a "proof" of B from $\underline{A}$.

Direction of Expansion

- Right logical rules in sequent calculus correpond to introduction rules in natural deduction.
  - The translation expands the deduction downwards (to the root).
- Left logical rules in sequent calculus correspond to elimination rules in natural deduction.
  - The translation expands the deduction upwards (to leaves).
- We can make the translation expand downwards by the cut rule.

$$\dfrac{\dfrac{\underline{A'} \vdash A \qquad B \vdash B}{\underline{A'}, A \Rightarrow B \vdash B}\,\mathcal{L}\Rightarrow \qquad \underline{B'} \vdash A \Rightarrow B}{\underline{A'}, \underline{B'} \vdash B}\,\text{Cut} \qquad \longmapsto \qquad \dfrac{\begin{array}{c}\underline{A'} \\ \vdots \\ A\end{array} \qquad \begin{array}{c}\underline{B'} \\ \vdots \\ A \Rightarrow B\end{array}}{B}\Rightarrow\mathcal{E}$$

Normal Deductions and Cut-Free Proofs

- A non-normal deduction results from an introduction followed by an elimination.

$$\dfrac{\dfrac{\dfrac{\dfrac{[A] \qquad [B]}{A \wedge B}\wedge\mathcal{I}}{A}\wedge\mathcal{E}}{B \Rightarrow A}\Rightarrow\mathcal{I}}{A \Rightarrow (B \Rightarrow A)}\Rightarrow\mathcal{I}$$

- The cut rule can stack introduction on elimination and thus yield non-normal deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \wedge B}{A}\wedge 1\mathcal{E} \qquad \dfrac{A \qquad B}{A \wedge B}\wedge\mathcal{I}}{A, B \vdash A}\text{Cut}}{A \vdash B \Rightarrow A}\mathcal{R}\Rightarrow}{\vdash A \Rightarrow (B \Rightarrow A)}\mathcal{R}\Rightarrow \quad \longmapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{A \qquad B}{A \wedge B}\wedge\mathcal{I}}{A}\wedge\mathcal{E}}{A \vdash B \Rightarrow A}\mathcal{R}\Rightarrow}{\vdash A \Rightarrow (B \Rightarrow A)}\mathcal{R}\Rightarrow$$

- Thus,

Normal Form, Normal Deduction, Cut-Free Proof

- A deduction corresponds to a typed λ-term.
  - Curry-Howard isomorphism.
- Any typed λ-term has a normal form.
  - The weak normalisation theorem and Church-Rosser property.
- Any deduction can be normalised.
  - Curry-Howard isomorphism.
- A sequent proof corresponds to a deduction.
- A sequent proof has a cut-free form.
  - The cut-elimination theorem (Hauptsatz).
- A cut-free sequent proof corresponds to a normal deduction.
- The cut-elimination theorem corresponds to the normalisation theorem.