Proofs and Types
Strong Normalisation Theorem

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Strong Normalisation Theorem

- There is a strategy of reduction that finds the normal form of any typed λ-term.
  - The weak normalisation theorem.
- All strategies of reduction in fact find the normal form of any typed λ-term.
  - The strong normalisation theorem.
- We will demonstrate a technique for proving the strong normalisation theorem.
- The technique can be generalized to other systems. Particularly,
  - Gödel's system **T**. Since Peano Arithmetic can be encoded in system **T** and system **T** is strongly normalising, Peano Arithmetic is consistent.
  - Girard's system **F**.

Reducibility

Definition 1

The set RED_T (*reducible* terms of type T) is defined as follows.

- For t of atomic type T, $t \in \mathsf{RED}_T$ if t is strongly normalisable.
- For t of type $U \times V$, $t \in \mathsf{RED}_{U \times V}$ if $\pi^1 t \in \mathsf{RED}_U$ and $\pi^2 \in \mathsf{RED}_V$.
- For t of type $U \to V$, $t \in \mathsf{RED}_{U \to V}$ if $tu \in \mathsf{RED}_V$ for all $u \in \mathsf{RED}_U$.

- A term is *neutral* if it is one of the following forms:

$$x \qquad \pi^1 t \qquad \pi^2 t \qquad tu$$

Properties of Reducibility

- We will prove the following properties by induction on the T:

  CR 1 If $t \in \mathsf{RED}_T$, then t is strongly normalisable.

  CR 2 If $t \in \mathsf{RED}_T$ and $t \rightsquigarrow t'$, then $t' \in \mathsf{RED}_T$.

  CR 3 If t is neutral and $t' \in \mathsf{RED}_T$ for every t' obtained by converting a redex in t, then $t \in \mathsf{RED}_T$.

- Particularly, we have

  CR 4 If t is neutral and normal, then $t \in \mathsf{RED}_T$.

- We now proceed to prove **CR 1** to **3** simultaneously by induction on T.

Length of Normalisation

Lemma 2 (König)

A finitely branching tree with no infinite branch is finite.

Lemma 3

t is strongly normalisable iff there is a number $\nu(t)$ which bounds the length of every normalisation sequence from t.

Proof.

If there is $\nu(t)$, t is clearly strongly normalisable.
Conversely, suppose t is strongly normalisable. Note that t has fintely many redexes. Hence all strategies of normalisation form a finitely branching tree. Moreover, every branch of the tree is finite becuase t is strongly normalisable. By König's lemma, the tree is finite. The height of the tree is $\nu(t)$. □

T is an Atomic Type

CR 1 If $t \in \mathsf{RED}_T$, then t is strongly normalisable.

Proof.

Since $t \in \mathsf{RED}_T$, t is strongly normalisable by the definition of RED_T. □

CR 2 If $t \in \mathsf{RED}_T$ and $t \rightsquigarrow t'$, then $t' \in \mathsf{RED}_T$.

Proof.

Let $t \rightsquigarrow t'$. Clearly, t' is strongly normalisable and hence $t' \in \mathsf{RED}_T$. □

CR 3 If t is neutral and $t' \in \mathsf{RED}_T$ for every t' obtained by converting a redex in t, then $t \in \mathsf{RED}_T$.

Proof.

Let t be neutral and $t' \in \mathsf{RED}_T$ for every t' obtained by converting a redex in t. We have $\nu(t) = 1 + \max_{t'} \nu(t')$. Hence t is strongly normalisable and then $t \in \mathsf{RED}_T$. □

$T = U \times V$ is a Product Type

CR 1 If $t \in \mathsf{RED}_T$, then t is strongly normalisable.

Proof.

Since $t \in \mathsf{RED}_{U \times V}$, $\pi^1 t \in \mathsf{RED}_U$ and $\pi^2 t \in \mathsf{RED}_V$. By IH (**CR 1**), $\pi^1 t$ and $\pi^2 t$ are strongly normalisable. Observe that $\nu(t) \leq \nu(\pi^1 t)$. t is strongly normalisable. $\qquad \square$

CR 2 If $t \in \mathsf{RED}_T$ and $t \rightsquigarrow t'$, then $t' \in \mathsf{RED}_T$.

Proof.

Let $t \rightsquigarrow t'$. Then $\pi^1 t \rightsquigarrow \pi^1 t'$ and $\pi^2 t \rightsquigarrow \pi^2 t'$. Since $\pi^1 t \in \mathsf{RED}_U$ and $\pi^2 \in \mathsf{RED}_V$, $\pi^1 t' \in \mathsf{RED}_U$ and $\pi^2 t' \in \mathsf{RED}_V$ by IH (**CR 2**). Thus $t' \in \mathsf{RED}_{U \times V}$. $\qquad \square$

CR 3 If t is neutral and $t' \in \mathsf{RED}_T$ for every t' obtained by converting a redex in t, then $t \in \mathsf{RED}_T$.

Proof.

Let t be neutral. Since $t \neq \langle u, v \rangle$, we obtain $\pi^1 t'$ after converting a redex in $\pi^1 t$, where t' is obtained by converting a redex in t. Hence $\pi^1 t' \in \mathsf{RED}_U$ by the assumption and definition of $\mathsf{RED}_{U \times V}$. For any $\pi^1 t'$ obtained by converting a redex in $\pi^1 t$, we have $\pi^1 t' \in \mathsf{RED}_U$. By IH (**CR 3**), $\pi^1 t \in \mathsf{RED}_U$. Similarly, $\pi^2 t \in \mathsf{RED}_V$. $\qquad \square$

$T = U \to V$ is an Arrow Type

CR 1 If $t \in \mathsf{RED}_T$, then t is strongly normalisable.

Proof.

Let x be a variable of type U. Since x is neutral and normal, $x \in \mathsf{RED}_U$. Thus $tx \in \mathsf{RED}_V$. By IH (**CR 1**), tx is strongly normalisable. Observe that $\nu(t) \leq \nu(tx)$. $\qquad\square$

CR 2 If $t \in \mathsf{RED}_T$ and $t \rightsquigarrow t'$, then $t' \in \mathsf{RED}_T$.

Proof.

Let $u \in \mathsf{RED}_U$ and $t \rightsquigarrow t'$. $tu \in \mathsf{RED}_V$ and $tu \rightsquigarrow t'u$. By IH (**CR 2**), $t'u \in \mathsf{RED}_V$. $\qquad\square$

CR 3 If t is neutral and $t' \in \mathsf{RED}_T$ for every t' obtained by converting a redex in t, then $t \in \mathsf{RED}_T$.

Proof.

Let $u \in \mathsf{RED}_U$. By IH (**CR 1**), u is strongly normalisable. In one step, tu converts to
    (1) $t'u$ with t' one step from t. $t'u \in \mathsf{RED}_V$ for $t' \in \mathsf{RED}_{U \to V}$ by assumption.
    (2) tu' with u' one step from u. By IH (**CR 2**), $u' \in \mathsf{RED}_U$ and $\nu(u') < \nu(u)$. Hence $tu' \in \mathsf{RED}_V$ by IH ($\nu(u)$).
By IH (**CR 3**), $tu \in \mathsf{RED}_V$. $\qquad\square$

Reducibility Theorem

Lemma 4

If $u \in \mathsf{RED}_U$ and $v \in \mathsf{RED}_v$, $\langle u, v \rangle \in \mathsf{RED}_{U \times V}$.

Proof.

By **CR 1**, u and v are strongly normalisable. $\pi^1 \langle u, v \rangle$ converts to

- u. $u \in \mathsf{RED}_U$.
- $\pi^1 \langle u', v \rangle$ with u' one step from u. By **CR 2**, $u' \in \mathsf{RED}_U$ and $\nu(u') < \nu(u)$. By IH ($\nu(u) + \nu(v)$), $\pi^1 \langle u', v \rangle \in \mathsf{RED}_U$.
- $\pi^1 \langle u, v' \rangle$ with v' one step from v. By IH ($\nu(u) + \nu(v)$), $\pi^1 \langle u, v' \rangle \in \mathsf{RED}_U$.

Since $\pi^1 \langle u, v \rangle$ is neutral, $\pi^1 \langle u, v \rangle \in \mathsf{RED}_U$ by **CR 3**. Similarly, $\pi^2 \langle u, v \rangle \in \mathsf{RED}_V$. $\qquad\square$

Reducibility Theorem

Lemma 5

If $v[u/x] \in \mathsf{RED}_V$ for all $u \in \mathsf{RED}_U$, then $\lambda x^U.v \in \mathsf{RED}_{U \to V}$.

Proof.

Recall $x \in \mathsf{RED}_U$ and $v[x/x] = v \in \mathsf{RED}_V$. Let $u \in \mathsf{RED}_U$. $(\lambda x^U.v)u$ converts to

- $v[u/x]$. $v[u/x] \in \mathsf{RED}_V$ by assumption.
- $(\lambda x^U.v)u'$ with u' one step from u. By **CR 2**, $u' \in \mathsf{RED}_U$ and $\nu(u') < \nu(u)$. By IH $(\nu(u) + \nu(v))$, $(\lambda x^U.v)u' \in \mathsf{RED}_V$.
- $(\lambda x^U.v')u$ with v' one step from v. By **CR 2**, $v' \in \mathsf{RED}_V$ and $\nu(v') < \nu(v)$. By IH $(\nu(u) + \nu(v))$, $(\lambda x^U.v')u \in \mathsf{RED}_V$.

By **CR 3**, $(\lambda x^U.v)u \in \mathsf{RED}_V$. $\qquad\square$

The Strong Normalisation Theorem

Lemma 6

Let t be a term of type T with free variables $x_1, \ldots, x_n$ of types $U_1, \ldots, U_n$. If $u_1 \in \mathsf{RED}_{U_1}, \ldots, u_n \in \mathsf{RED}_{U_n}$, then $t[u_1/x_1, \ldots, u_n/x_n] \in \mathsf{RED}_T$.

Proof.

Induction on t. We write $t[\underline{u}/\underline{x}]$ for $t[u_1/x_1, \ldots, u_n/x_n]$.

- t is x_i. Trivial.
- t is $\pi^1 w$. By IH (t), $w[\underline{u}/\underline{x}]$ is reducible for any sequence $\underline{u}$ of reducible terms. By the definition of $\mathsf{RED}_{U \times V}$, $t[\underline{u}/\underline{x}] = \pi^1 w[\underline{u}/\underline{x}]$ is reducible.
- t is $\pi^2 w$. Similar.
- t is $\langle v, w \rangle$. By IH (t), $v[\underline{u}/\underline{x}]$ and $w[\underline{u}/\underline{x}]$ are reducible. By Lemma 4, $t[\underline{u}/\underline{x}] = \langle v[\underline{u}/\underline{x}], w[\underline{u}/\underline{x}] \rangle$ is reducible.
- t is vw. By IH (t), $v[\underline{u}/\underline{x}]$ and $w[\underline{u}/\underline{x}]$ are reducible. By the definition of $\mathsf{RED}_{W \to V}$, $t[\underline{u}/\underline{x}] = (v[\underline{u}/\underline{x}])(w[\underline{u}/\underline{x}])$ is reducible.
- t is $\lambda y^V.w$. By IH (t), $w[\underline{u}/\underline{x}, v/y]$ is reducible for all reducible term v. By Lemma 5, $t[\underline{u}/\underline{x}] = \lambda y^V.(w[\underline{u}/\underline{x}])$ is reducible. $\qquad\square$

The Strong Normalisation Theorem

Theorem 7

All terms are reducible.

Proof.

Let t be a term of free variables $x_1, \ldots, x_n$ of types $U_1, \ldots, U_n$. Recall $x_1 \in \mathsf{RED}_{U_1}, \ldots, x_n \in \mathsf{RED}_{U_n}$ (**CR 3**). By Lemma 6, $t = t[\underline{x}/\underline{x}]$ is reducible. $\qquad\square$

Theorem 8

All terms are strongly normalisable.

Proof.

By **CR 1**. $\qquad\square$