Special Topics on Applied Mathematical Logic

Spring 2012

Lecture 04

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March 20, 2012

Outline

First-Order Logic
- Truth and Models (Semantics)
  - Logical Implication
  - Definability
  - Homomorphisms

Truth and Models

- Truth assignments are to sentential logic what **structures** (or **interpretations**) are to first-order logic
- A structure for a first-order language tells
  1. what the universe (the set of objects that $\forall$ refers to) is, and
  2. what the parameters (predicate, constant, function symbols) mean

Truth and Models

Formally, a **structure** $\mathfrak{A}$ is a function whose domain is the set of parameters and

1. $\mathfrak{A}$ assigns to the symbol $\forall$ a *nonempty set* $|\mathfrak{A}|$ called the **universe** (or **domain**) of $\mathfrak{A}$
   $\forall$ — for everything in $|\mathfrak{A}|$
2. $\mathfrak{A}$ assigns to each n-place predicate symbol P an n-ary relation $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$
   $Pt_1, \ldots, t_n$ — $t_1, \ldots, t_n \in |\mathfrak{A}|$ is in $P^{\mathfrak{A}}$
3. $\mathfrak{A}$ assigns to each constant symbol c a member $c^{\mathfrak{A}}$ of $|\mathfrak{A}|$
   c — $c^{\mathfrak{A}}$
4. $\mathfrak{A}$ assigns to each n-place function symbol f an n-ary operation $f^{\mathfrak{A}} : |\mathfrak{A}|^n \rightarrow |\mathfrak{A}|$ (the mapping must be total)

Truth and Models

Example

- Language of set theory
  $\exists x \forall y \neg y \in x$
  - There exists a set s.t. every set is not its member

  $\forall x \forall y \exists z \forall t (t \in z \Leftrightarrow (t = x \vee t = y))$
  - For every two sets x and y, there exists a set z such that for every set t, $t \in z$ iff $t = x$ or $t = y$ (pair-set axiom)
- Language of number theory
  Let $\mathfrak{A}$ be such that $|\mathfrak{A}| = \mathbb{N}$ and $\in^{\mathfrak{A}}$ is the set of pairs $\langle m, n \rangle$ with $m < n$
  $\exists x \forall y \neg y \in x$
  - There exists a natural number that is the smallest
  - We say $\exists x \forall y \neg y \in x$ is **true** in $\mathfrak{A}$, or $\mathfrak{A}$ is a **model** of $\exists x \forall y \neg y \in x$

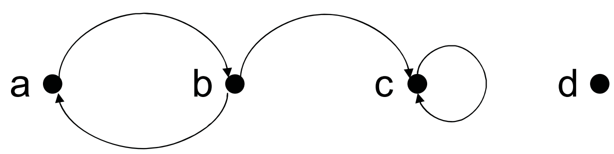
  $\forall x \forall y \exists z \forall t (t \in z \Leftrightarrow (t = x \vee t = y))$
  - The formula is not true (i.e., false) in $\mathfrak{A}$

Truth and Models

Example

Consider a language with $\forall$ and 2-place predicate symbol E
Let structure $\mathfrak{B}$ have

- $|\mathfrak{B}| = \{a, b, c, d\}$ (vertex set)
- $E^{\mathfrak{B}} = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, c \rangle\}$ (edge set)



$\exists x \forall y \neg Eyx$

- The formula is true in $\mathfrak{B}$ (there is a vertex not pointed to from any vertex)

Truth and Models

- A sentence σ *is true in* $\mathfrak{A}$, denoted $\models_{\mathfrak{A}} \sigma$
- To formally define $\models_{\mathfrak{A}} \varphi$, let
  φ be a wff of our language,
  $\mathfrak{A}$ be a structure for the language, and
  $s : V \to |\mathfrak{A}|$ for V being the set of all variables.

  Then $\models_{\mathfrak{A}} \varphi[s]$ (meaning $\mathfrak{A}$ satisfies φ with s) iff the translation of φ determined by $\mathfrak{A}$ is true, where variable x is translated as $s(x)$ wherever it occurs free.

Truth and Models

Extend s to $\bar{s}$

case i (terms)

$\bar{s} : T \to |\mathfrak{A}|$ for T the set of all terms

$\bar{s}$ is defined recursively by

1. $\bar{s}(x) = s(x)$ (x: variable)
2. $\bar{s}(c) = c^{\mathfrak{A}}$ (c: constant)
3. $\bar{s}(ft_1, \ldots, t_n) = f^{\mathfrak{A}}(\bar{s}(t_1), \ldots, \bar{s}(t_n))$ (f: function)

   commutative diagram of $\bar{s}(ft) = f^{\mathfrak{A}}(\bar{s}(t))$

$$\begin{tikzcd}
T \arrow[r, "\bar{s}"] \arrow[d, "f"'] & {|\mathfrak{A}|} \arrow[d, "f^{\mathfrak{A}}"] \\
T \arrow[r, "\bar{s}"] & {|\mathfrak{A}|}
\end{tikzcd}$$

- $\bar{s}$ is unique
- $\bar{s}$ depends on both s and $\mathfrak{A}$

Truth and Models

Extend s to $\bar{s}$ (cont'd)

case ii (atomic formulas)
explicit (not recursive) definition with

1. $\models_{\mathfrak{A}} = t_1 t_2[s]$ iff $\bar{s}(t_1) = \bar{s}(t_2)$
2. $\models_{\mathfrak{A}} Pt_1 \cdots t_n[s]$ iff sequence $\langle \bar{s}(t_1), \ldots, \bar{s}(t_n) \rangle \in P^{\mathfrak{A}}$

Truth and Models

Extend s to $\bar{s}$ (cont'd)

case iii (other wffs)
recursive definition with

1. For atomic formulas, see case ii
2. $\models_{\mathfrak{A}} \neg\varphi[s]$ iff $\not\models_{\mathfrak{A}} \varphi[s]$
3. $\models_{\mathfrak{A}} (\varphi \Rightarrow \psi)[s]$ iff either $\not\models_{\mathfrak{A}} \varphi[s]$ or $\models_{\mathfrak{A}} \psi[s]$ or both
4. $\models_{\mathfrak{A}} \forall x\varphi[s]$ iff for every $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \varphi[s(x|d)]$, where $s(x|d)(y) = \begin{cases} s(y) & \text{if } y \neq x \\ d & \text{if } y = x \end{cases}$

- $\models_{\mathfrak{A}} (\alpha \wedge \beta)[s]$ iff $\models_{\mathfrak{A}} \alpha[s]$ and $\models_{\mathfrak{A}} \beta[s]$ (similarly for $\vee$ and $\Leftrightarrow$)
- $\models_{\mathfrak{A}} \exists x\alpha[s]$ iff there is some $d \in |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \alpha[s(x|d)]$

Truth and Models

- $\models_{\mathfrak{A}} \varphi[s]$ iff the translation of φ determined by $\mathfrak{A}$ is true for free variables translated as $s(x)$
- s only matters for free variables
  - If φ is a sentence, then s does not matter

Truth and Models

Example

Consider a language with $\forall$, P (two-place predicate), f (one-place function), and c (constant); let $\mathfrak{A} = (\mathbb{N}; \leq, S, 0)$, i.e.,

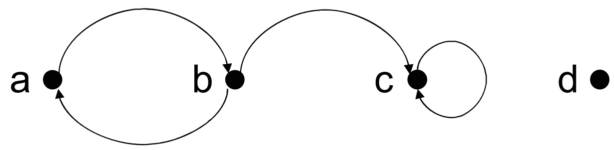
- $|\mathfrak{A}| = \mathbb{N}$
- $P^{\mathfrak{A}} = \{\langle m, n\rangle \mid m \leq n\}$
- $f^{\mathfrak{A}} = S$, i.e., $f^{\mathfrak{A}}(n) = n + 1$
- $c^{\mathfrak{A}} = 0$

Let $s(v_i) = i - 1$. Then

$$\begin{aligned}
&\bar{s}(ffv_3) = 4 \\
&\bar{s}(c) = 0 \\
&\bar{s}(fffc) = 3 \\
&\models_{\mathfrak{A}} Pcfv_1[s] \quad (\because 0 \leq 1) \\
&\models_{\mathfrak{A}} \forall v_1 Pcv_1 \\
&\not\models_{\mathfrak{A}} \forall v_1 Pv_2 v_1[s] \\
&\models_{\mathfrak{A}} \forall v_1 \exists v_2 Pv_2 v_1[s]
\end{aligned}$$

Truth and Models

Example



$$\models_{\mathfrak{B}} \forall v_2 \neg E v_1 v_2 [s] \text{ iff } s(v_1) = d$$
$$\models_{\mathfrak{B}} \forall v_2 \neg E v_2 v_1 [s] \text{ iff } s(v_1) = d$$
$$\models_{\mathfrak{B}} \exists v_2 E v_1 v_2 [s] \text{ iff } s(v_1) = a, b, c$$

Truth and Models

Theorem

Assume functions s_1 and $s_2 : V \to |\mathfrak{A}|$ agree at all free variables of φ. Then $\models_{\mathfrak{A}} \varphi[s_1]$ iff $\models_{\mathfrak{A}} \varphi[s_2]$

(prove by induction)

If $\mathfrak{A}$ and $\mathfrak{B}$ agree at all parameters that occur in φ, then $\models_{\mathfrak{A}} \varphi[s]$ iff $\models_{\mathfrak{B}} \varphi[s]$

$\mathfrak{A}$ and $\mathfrak{B}$ are **elementarily equivalent** (denoted $\mathfrak{A} \equiv \mathfrak{B}$) iff for any sentence σ, $\models_{\mathfrak{A}} \sigma$ iff $\models_{\mathfrak{B}} \sigma$

Truth and Models

Notation $\models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_k]\!]$ denotes $\mathfrak{A}$ satisfies φ with $s(v_i) = a_i$, where v_i is the ith free variable in φ

E.g., $\mathfrak{A} = (\mathbb{N}; \leq, S, 0)$
$\models_{\mathfrak{A}} \forall v_2 P v_1 v_2 [\![0]\!]$
$\not\models_{\mathfrak{A}} \forall v_2 P v_1 v_2 [\![1]\!]$

Truth and Models

Corollary

For a sentence σ, either

(a) *$\mathfrak{A}$ satisfies σ with every $s : V \to |\mathfrak{A}|$, or*

(b) *$\mathfrak{A}$ does not satisfy σ with any $s : V \to |\mathfrak{A}|$*

For case (a), we say σ *is true in* $\mathfrak{A}$ or $\mathfrak{A}$ *is a model of* σ (i.e., $\models_{\mathfrak{A}} \sigma$)
For case (b), we say σ *is false in* $\mathfrak{A}$

Truth and Models

$\mathfrak{A}$ is a model of a set Σ of sentences iff it is a model of every member of Σ

Examples

- $\mathfrak{R} = (\mathbb{R}; 0, 1, +, \times)$; $\mathfrak{Q} = (\mathbb{Q}; 0, 1, +, \times)$
  $\models_{\mathfrak{R}} \exists x(x \times x = 1 + 1)$
  $\not\models_{\mathfrak{Q}} \exists x(x \times x = 1 + 1)$
- Consider a language has only the parameters $\forall$ and a 2-place predicate P
  $\forall x \forall y x = y$
    - $\mathfrak{A} = (A; R)$ is a model iff ?
  $\forall x \forall y Pxy$
    - $\mathfrak{A} = (A; R)$ is a model iff ?
  $\forall x \forall y \neg Pxy$
    - $\mathfrak{A} = (A; R)$ is a model iff ?
  $\forall x \exists y Pxy$
    - $\mathfrak{A} = (A; R)$ is a model iff ?

Logical Implication

A set Σ of wffs **logically implies** a wff φ, denoted $\Sigma \models \varphi$, iff for *every structure* $\mathfrak{A}$ for the language and *every function* $s : V \to |\mathfrak{A}|$ such that if $\mathfrak{A}$ satisfies every member of Σ with s, the $\mathfrak{A}$ also satisfies φ with s

- Entailment "$\models$" is a *semantical* relation
- Recall "$\models$" denotes *tautological implication* in sentential logic
- $\{\gamma\} \models \varphi$ will be written as $\gamma \models \varphi$
- φ and ψ are **logically equivalent**, denoted $\varphi \models\dashv \psi$, iff $\varphi \models \psi$ and $\psi \models \varphi$

Logical Implication

sentential logic	first-order logic
$\models \tau$ τ is a tautology	$\models \varphi$ φ is a **valid wff**, i.e., for every $\mathfrak{A}$ and every $s : V \to \lvert\mathfrak{A}\rvert$, $\mathfrak{A}$ satisfies φ with s

Logical Implication

Corollary

For a set $\Sigma; \tau$ of sentences, $\Sigma \models \tau$ iff every model of Σ is also a model of τ. A sentence is valid iff it is true in every structure.

Examples

- $\forall v_1 Q v_1 \models Q v_2$
- $Q v_1 \not\models \forall v_1 Q v_1$
- $\models \neg\neg\sigma \Rightarrow \sigma$
- $\forall v_1 Q v_1 \models \exists v_2 Q v_2$
- $\exists x \forall y Pxy \models \forall y \exists x Pxy$
- $\models \exists x (Qx \Rightarrow \forall x Qx)$

Logical Implication

- Checking *tautology* (of sentential logic) is a finite process
- Checking *validity* (of first-order logic) is an infinite process
  - Must consider every structure

Logical Implication

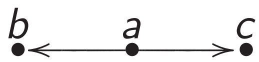
- The set of valid formulas is not decidable, but semi-decidable (i.e., effectively enumerable)
  - wffs of sentential logic vs. wffs of first-order logic
- Later we will show that *validity* ($\models$) and *deducibility* ($\vdash$) are equivalent in first order logic

Definiability in a Structure

- $\{\langle a_1, \ldots, a_k \rangle \mid \models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_k]\!]\}$ is a relation that φ **defines** in $\mathfrak{A}$
- A k-ary relation on $|\mathfrak{A}|$ is **definiable** in $\mathfrak{A}$ iff there is a formula (with free variables v_1, ..., v_k) that defines the relation in $\mathfrak{A}$

  E.g.,
  $\mathfrak{R} = (\mathbb{R}; 0, 1, +, \times)$
  - $\models_{\mathfrak{R}} \exists v_2 v_1 = v_2 \times v_2[\![a]\!] \Leftrightarrow a \geq 0$
    - $\therefore [0, \infty)$ is definiable in $\mathfrak{R}$
  - $\models_{\mathfrak{R}} \exists v_3 v_1 = v_2 + v_3 \times v_3[\![a, b]\!] \Leftrightarrow a \geq b$
    - $\therefore \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R} \mid a \geq b\}$ is definiable in $\mathfrak{R}$

  $\mathfrak{A} = (\{a, b, c\}; E = \{\langle a, b \rangle, \langle a, c \rangle\})$ where the language has parameters $\forall$ and $\exists$

  - $\{b, c\}$ is defined by $\exists v_2 E v_2 v_1$
  - $\{b\}$ is not definable in $\mathfrak{A}$

Definiability in a Structure

E.g.,
$\mathfrak{N} = (\mathbb{N}; 0, S, +, \cdot)$ under the language for number theory

- $\{\langle m, n \rangle \mid m < n\}$ is defined in $\mathfrak{N}$ by $\exists v_3 v_1 + S v_3 = v_2$
- $\{2\}$ is defined in $\mathfrak{N}$ by $v_1 = SS0$
- The set of primes is defined in $\mathfrak{N}$ by $S0 < v_1 \wedge \forall v_2 \forall v_3 (v_1 = v_2 \cdot v_3 \Rightarrow v_2 = 1 \vee v_3 = 1)$
- Exponentiation $\{\langle m, n, p \rangle \mid p = m^n\}$ is definable in $\mathfrak{N}$

Definiability in a Structure

Let L be a language that does not include an n-place predicate symbol P, L^+ be the language that extends L and includes P, and τ be a theory in L^+.

- P is *explicitly definable* if there is an L formula ϕ with free variables $x_1, \ldots, x_n$ such that $\tau \models Px_1 \ldots x_n \Leftrightarrow \phi$.
- P is *implicitly definable* if for any structure $\mathfrak{A}$ and any $R_1, R_2 \subseteq |\mathfrak{A}|^n$ if both $(|\mathfrak{A}|, R_1)$ and $(|\mathfrak{A}|, R_2)$ are models of τ, then $R_1 = R_2$.

Theorem (Beth's Definiability Theorem)

P is explicitly definable if, and only if, P is implicitly definable.

Definiability in a Structure

- There are uncountably many relations on $\mathbb{N}$, but only countably many possible defining formulas
- Any decidable relation on $\mathbb{N}$ is definable in $\mathfrak{N}$ (§3.5)

Definability of a Class of Structures

- Let $Mod\Sigma$ be the *class* of all models of a set Σ of sentences. (That is, the class of all *structures* for the language in which every member of Σ is true.)
  - A set is a class
  - Classes are beyond sets (e.g., the class of all sets)
- A class $\mathcal{K}$ of structures for the language is an **elementary class** (EC) iff $\mathcal{K} = Mod\tau$ (i.e., $Mod\{\tau\}$) for some sentence τ
  - where "elementary" means "first order"
- A class $\mathcal{K}$ is an elementary class in a wider sense iff $\mathcal{K} = Mod\Sigma$

Definability of a Class of Structures

E.g., consider the language $\mathcal{L}$ with $=$, $\forall$, and a 2-place predicate E

- A graph is a structure $\mathfrak{A} = (V; E^{\mathfrak{A}})$ for $\mathcal{L}$, where $|\mathfrak{A}| = V$ is the (nonempty) set of vertices and $E^{\mathfrak{A}}$ is an edge relation that is *symmetric* and *irreflexive*
  - with axiom: $\forall x(\forall y(Exy \Leftrightarrow Eyx) \wedge \neg Exx)$
- The class of graphs is an elementary class

Homomorphisms

A **homomorphism** h of $\mathfrak{A}$ into $\mathfrak{B}$ is a function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ such that

1. for predicate symbol P
   $\langle a_1, \ldots, a_n \rangle \in P^{\mathfrak{A}}$ iff $\langle h(a_1), \ldots, h(a_n) \rangle \in P^{\mathfrak{B}}$
2. for function symbol f
   $h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$
   $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$

E.g., $\mathfrak{A} = (\mathbb{N}; +, \cdot)$, $\mathfrak{B} = (\{e, o\}; +^{\mathfrak{B}}, \cdot^{\mathfrak{B}})$

$+^{\mathfrak{B}}$	e	o
e	e	o
o	o	e

$\cdot^{\mathfrak{B}}$	e	o
e	e	e
o	e	o

$$h(n) = \begin{cases} e & \text{if } n \text{ is even} \\ o & \text{if } n \text{ is odd} \end{cases}$$

h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$

Isomorphism

- A homomorphism h is called an **isomorphism** (or **isomorphic embedding**) of $\mathfrak{A}$ into $\mathfrak{B}$ if h is *one-to-one*
- Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are **isomorphic** (denoted $\mathfrak{A} \cong \mathfrak{B}$) if there is an isomorphism of $\mathfrak{A}$ *onto* $\mathfrak{B}$
- Two isomorphic structures satisfy exactly the same sentences

Substructures

$\mathfrak{A}$ is a **substructure** of $\mathfrak{B}$, or $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$, if $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and the identity map $Id : |\mathfrak{A}| \to |\mathfrak{B}|$ is an *isomorphism* of $\mathfrak{A}$ into $\mathfrak{B}$, equivalently

1. $P^{\mathfrak{A}}$ is the restriction of $P^{\mathfrak{B}}$ to $|\mathfrak{A}|$
2. $f^{\mathfrak{A}}$ is the restriction of $f^{\mathfrak{B}}$ to $|\mathfrak{A}|$, and $c^{\mathfrak{A}} = c^{\mathfrak{B}}$

E.g., $\mathfrak{A} = (\mathbb{P}; <)$ for $\mathbb{P}$: positive integers, $\mathfrak{B} = (\mathbb{N}; <)$

- $Id(n) = n$ is an isomorphism
  - $\mathfrak{A}$ is a substructure of $\mathfrak{B}$
- $h(n) = n - 1$ is an isomorphism

Homomorphism Theorem

Theorem (Homomorphism Theorem)

Let h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and $s : V \to |\mathfrak{A}|$, where V is the set of variables. Then

(a) *for any term t, $h(\overline{s}(t)) = \overline{h \circ s}(t)$*

(b) *for any quantifier-free formula α not containing the equality symbol, $\models_{\mathfrak{A}} \alpha[s]$ iff $\models_{\mathfrak{B}} \alpha[h \circ s]$*

  - *The "quantifier-free" criterion is due to the fact that h may not be onto*
  - *The exclusion of the equality symbol is due to the fact that h may not be one-to-one*

Homomorphism

E.g., $\mathfrak{A} = (\mathbb{P}; <)$, $\mathfrak{B} = (\mathbb{N}; <)$

$\models_{\mathfrak{A}} \forall v_2(v_1 \neq v_2 \rightarrow v_1 < v_2)[\![1]\!]$
$\not\models_{\mathfrak{B}} \forall v_2(v_1 \neq v_2 \rightarrow v_1 < v_2)[\![1]\!]$

- $h = Id : |\mathfrak{A}| \rightarrow |\mathfrak{B}|$ is not onto

$\not\models_{\mathfrak{A}} v_1 = v_2[\![1, 2]\!]$
$\models_{\mathfrak{B}} v_1 = v_2[\![0, 0]\!]$

- $h(n) = \begin{cases} 0 & \text{if } n = 1 \\ n - 2 & \text{if } n \geq 2 \end{cases}$ is not one-to-one

Elementary Equivalence

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the language are **elementarily equivalent** (denoted $\mathfrak{A} \equiv \mathfrak{B}$) iff for every sentence σ, $\models_{\mathfrak{A}} \sigma$ iff $\models_{\mathfrak{B}} \sigma$

Corollary

If $\mathfrak{A} \cong \mathfrak{B}$, *then* $\mathfrak{A} \equiv \mathfrak{B}$

- The converse is not true
  E.g., $(\mathbb{R}; <) \equiv (\mathbb{Q}; <)$, but $(\mathbb{R}; <) \not\cong (\mathbb{Q}; <)$

Automorphism

An **automorphism** of the structure $\mathfrak{A}$ is an *isomorphism* (namely, one-to-one homomorphism) of $\mathfrak{A}$ *onto* $\mathfrak{A}$

Corollary

Let h be an automorphism of $\mathfrak{A}$, and R be an n-ary relation on $\mathfrak{A}$ definable in $\mathfrak{A}$. Then $\langle a_1, \ldots, a_n \rangle \in R$ iff $\langle h(a_1), \ldots, h(a_n) \rangle \in R$.

Proof.

Let φ defines R in $\mathfrak{A}$. By Homomorphism Theorem,
$\models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_n]\!]$ iff $\models_{\mathfrak{A}} \varphi[\![h(a_1), \ldots, h(a_n)]\!]$ $\square$

Therefore, *automorphism preserves definable relations* and is useful in showing some relation is *not* definable.
E.g., $\mathbb{N}$ is not definable in $(\mathbb{R}; <)$

- By automorphism $h(a) = a^3$ ($\sqrt[3]{2} \notin \mathbb{N}$, but $2 \in \mathbb{N}$)