Special Topics on Applied Mathematical Logic

Spring 2012

Lecture 05

Jie-Hong Roland Jiang

National Taiwan University

March 17, 2012

Outline

- First-Order Logic
  - Deductive Calculi (Proof Systems)
    - Deductions
    - Logical Axioms
  - Deductions and Metatheorems
  - Deduction Strategy

A Deductive Calculus (A Proof System)

We want to prove $\Sigma \models \tau$

A satisfactory proof system should be

1. finitely long
   - ensured by Compactness Theorem
2. checkable mechanically (e.g., enumerating provable sentences) and effectively
   - ensured by Enumerability Theorem

Compactness and Enumerability Theorems

Theorem (Compactness Theorem (CT))

If $\Sigma \models \tau$, then there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$

Theorem (Enumerability Theorem (ET))

For a reasonable language, the set of valid wffs can be effectively enumerated

CT and ET $\Longleftarrow$ satisfactory proofs exist (necessary)
CT and ET $\Longrightarrow$ satisfactory proofs exist (sufficient)

CT: There exists $\Sigma_0 = \{\sigma_0, \ldots, \sigma_n\} \subseteq \Sigma$ such that $\Sigma_0 \models \tau$. So $\sigma_0 \Rightarrow \cdots \Rightarrow \sigma_n \Rightarrow \tau$ is valid.
ET: $(\sigma_0 \Rightarrow \cdots \Rightarrow \sigma_n \Rightarrow \tau)$ is a proof that can be generated by enumerating the validities

Formal Deductions

- Let Λ be an infinite set of wffs, called **logical axioms**. The **theorems** of a set Γ of wffs are the wffs that can be obtained from $\Gamma \cup \Lambda$ by using the rule of *inference* some finite number of times.
- $\Gamma \vdash \varphi$ denotes that φ is a theorem of Γ, or φ is deducible/provable from Γ
- For $\Gamma \vdash \varphi$, a **deduction** of φ from Γ is a sequence of wffs that records how φ is obtained from $\Gamma \cup \Lambda$ with the rule of inference

Deductions

- The choices of Λ and the rule(s) of inference are not unique
- We use *modus ponens*:

$$\frac{\alpha, \qquad \alpha \Rightarrow \beta}{\beta}$$

  as our only one rule of inference (at the expanse of infinite Λ)
  - This is a *Hilbert-style deduction system* (with a large set of axioms and a small set of inference rules)
    - Approach of the textbook
  - On the contrary, a *Gentzen-style deduction system* (*natural deduction*) includes many deduction rules but very few or no axioms at all
    - Approach of theorem provers
- The theorems of Γ are the wffs obtained from $\Gamma \cup \Lambda$ by applying modus ponens some finite number of times

Deductions

A **deduction of φ from Γ** is a finite sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ of wffs such that $\alpha_n = \varphi$ and, for each $k \leq n$, either

1. $\alpha_k \in \Gamma \cup \Lambda$, or
2. α_k is obtained by modus ponens from α_i and $\alpha_j = (\alpha_i \Rightarrow \alpha_k)$ for some $i, j < k$

Deductions

A set S of wffs is *closed* under modus ponens, if $\alpha \in S$ and $(\alpha \Rightarrow \beta) \in S$, then $\beta \in S$

- By induction principle, for S that includes $\Gamma \cup \Lambda$ and is closed under modus ponens, then S contains every theorem of Γ
    - E.g., if $\{\alpha, \beta, \alpha \Rightarrow \beta \Rightarrow \gamma\} \subseteq \Gamma \cup \Lambda$ (not closed), then $\Gamma \vdash \gamma$

$$\dfrac{\beta, \qquad \dfrac{\alpha, \quad \alpha \Rightarrow \beta \Rightarrow \gamma}{\beta \Rightarrow \gamma}}{\gamma}$$

Logical Axioms

What is the set Λ of logical axioms?

- A wff φ is a **generalization** of ψ iff $\varphi = \forall x_1 \ldots \forall x_n \psi$ for some variables $x_1, \ldots, x_n$ and $n \geq 0$
- Λ is the set of all generalizations of wffs of the following forms:
  1. Tautologies
  2. $\forall x \alpha \Rightarrow \alpha_t^x$, where α_t^x is obtained from α by replacing x (whenever free in α) by term t
  3. $\forall x(\alpha \Rightarrow \beta) \Rightarrow (\forall x \alpha \Rightarrow \forall x \beta)$
  4. $\alpha \Rightarrow \forall x \alpha$, where x does not occur free in α
  5. $x = x$
  6. $x = y \Rightarrow (\alpha \Rightarrow \alpha')$, where α is atomic and α' is obtained from α by replacing x in some places by y
     - Axiom-groups 3 and 4 will be useful in proving Generalization Theorem
     - Axiom-groups 5 and 6 are for languages with equality

Substitution (Axiom-group 2)

In Axiom-group 2, α_t^x can be obtained by recursion:

case 1 atomic formula

by replacing variable x by t in α

case 2 $\neg\alpha$

$$(\neg\alpha)_t^x = \neg(\alpha)_t^x$$

case 3 $\alpha \Rightarrow \beta$

$$(\alpha \Rightarrow \beta)_t^x = \alpha_t^x \Rightarrow \beta_t^x$$

case 4 $\forall y \alpha$

$$(\forall y \alpha)_t^x = \begin{cases} \forall y \alpha & \text{if } x = y \\ \forall y (\alpha)_t^x & \text{if } x \neq y \end{cases}$$

Substitution

E.g.,

- $\varphi^x_x = \varphi$
- $(Qx \Rightarrow \forall x Px)^x_y = (Qy \Rightarrow \forall x Px)$
- $(\neg \forall y x = y)^x_z = \neg \forall y z = y$
- $(\neg \forall y x = y)^x_y$?

A term t is not substitutable for x in α if there is some variable y in t that is captured by $\forall y$ in α^x_t

Substitutability

Recursive definition of substitutability φ^x_t:
t is **substitutable** for x in φ if

case 1 φ being atomic formula
always substitutable

case 2 φ being $\neg\alpha$
t is substitutable in α

case 3 φ being $\alpha \Rightarrow \beta$
t is substitutable in both α and β

case 4 φ being $\forall y \alpha$
either (a) x does not occur free in $\forall y \alpha$, or (b) y does not appear in t and t is substitutable for x in α

- In (a), we do not need to perform substitution, e.g., $(Qx \Rightarrow \forall x Px)^x_y = (Qy \Rightarrow \forall x Px)$

Is $(\forall x x = t)^x_t$ substitutable ?

Tautologies (Axiom-group 1)

Tautologies are wffs obtainable from tautologies of sentential logic by replacing each sentence symbol by a wff of the first-order language
E.g.,
$\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$ with $A = \forall x \neg Px$ and $B = Qy$ is a tautology

Tautologies

Another view of tautologies

wffs

prime | nonprime

atomic / $\forall x \alpha$ | $\neg\alpha$ / $\alpha \Rightarrow \beta$

- Follow sentential logic, but take sentence symbols to be prime formulas of our first-order language
  - Any formula can be built up from prime formulas by operations $\mathcal{E}_\neg$ and $\mathcal{E}_\Rightarrow$
    - $\forall x(Px \Rightarrow Px)$ is not a tautology
    - $\forall x Px \Rightarrow Px$ is not a tautology
- If Γ tautologically implies φ, then Γ logically implies φ
  - The converse is not true, e.g., $\Gamma = \forall x Px$ and $\varphi = Pc$

Tautologies

- Note that here we have no assumption that our first-order language has only countably many formulas
- We are speaking of sentential logic with potentially uncountably many sentence symbols

Tautologies

Theorem (24B)

$\Gamma \vdash \varphi$ iff $\Gamma \cup \Lambda$ tautologically implies φ

Proof.

($\Longrightarrow$) Note that $\{\alpha, \alpha \Rightarrow \beta\}$ tautologically implies β. $\Gamma \vdash \varphi$ indicates there is a sequence of modus ponens from $\Gamma \cup \Lambda$ leading to φ. By induction, it can be shown $\Gamma \cup \Lambda$ tautologically implies φ.
($\Longleftarrow$) By the corollary of Compactness Theorem of sentential logic (p.60), there is a finite subset

$$\{\gamma_1, \ldots, \gamma_m, \lambda_1, \ldots, \lambda_n\} \subseteq \Gamma \cup \Lambda$$

that tautologically implies φ. Hence

$$\gamma_1 \Rightarrow \cdots \Rightarrow \gamma_m \Rightarrow \lambda_1 \Rightarrow \cdots \Rightarrow \lambda_n \Rightarrow \varphi$$

is a tautology and is in Λ. So φ can be derived by applying modus ponens $m + n$ times. $\square$

Deductions and Metatheorems

E.g., $\vdash Px \Rightarrow \exists y Py$

In modus ponens,

$$\frac{\forall y \neg Py \Rightarrow \neg Px \text{ [AG2]}, \quad (\forall y \neg Py \Rightarrow \neg Px) \Rightarrow (Px \Rightarrow \neg \forall y \neg Py) \text{ [AG1]}}{Px \Rightarrow \neg \forall y \neg Py}$$

In pedigree tree,

$\lambda_3 : Px \Rightarrow \neg \forall y \neg Py$

$\lambda_2 : \forall y \neg Py \Rightarrow \neg Px$ $\qquad$ $\lambda_1 : (\forall y \neg Py \Rightarrow \neg Px) \Rightarrow (Px \Rightarrow \neg \forall y \neg Py)$

Notice that $(\lambda_1 \Rightarrow \lambda_2 \Rightarrow \lambda_3)$ is a deduction of $Px \Rightarrow \exists y Py$

Deductions and Metatheorems

Theorem (Generalization Theorem)

If $\Gamma \vdash \varphi$ and x do not occur free in any formula in Γ, then $\Gamma \vdash \forall x \varphi$

(x can occur free in φ.)

Proof of Generalization Theorem

By induction, we show that $\{\varphi \mid \Gamma \vdash \forall x\varphi\}$ contains $\Gamma \cup \Lambda$ and is closed under modus ponens (because this set contains every theorem by the induction principle).

case 1 $\varphi \in \Lambda$

$\forall x\varphi \in \Lambda$ (check the 6 AGs)

case 2 $\varphi \in \Gamma$

$\because x$ does not occur free in φ $\therefore \varphi \Rightarrow \forall x\varphi$ is in AG 4

$$\frac{\varphi \text{ in } \Gamma, \quad \varphi \Rightarrow \forall x\varphi \text{ in AG 4}}{\forall x\varphi}$$

case 3 $\frac{\psi, \quad \psi \Rightarrow \varphi}{\varphi}$ with $\Gamma \vdash \forall x\psi$ and $\Gamma \vdash \forall x(\psi \Rightarrow \varphi)$

$$\frac{\forall x\psi, \qquad \dfrac{\forall x(\psi \Rightarrow \varphi), \quad \forall x(\psi \Rightarrow \varphi) \Rightarrow (\forall x\psi \Rightarrow \forall x\varphi)}{\forall x\psi \Rightarrow \forall x\varphi}}{\forall x\varphi}$$

Q.E.D.

(AG 3 and AG 4 are needed due to this proof.)

Deductions and Metatheorems

Lemma (Rule T)

If $\Gamma \vdash \alpha_1, \ldots, \Gamma \vdash \alpha_n$ *and* $\{\alpha_1, \ldots, \alpha_n\}$ *tautologically implies* β, *then* $\Gamma \vdash \beta$

Proof.

$\alpha_1 \Rightarrow \cdots \Rightarrow \alpha_n \Rightarrow \beta$ (i.e., $(\alpha_1 \wedge \cdots \wedge \alpha_n) \Rightarrow \beta$) is a tautology, and thus in Λ. By modus ponens n times, we have $\Gamma \vdash \beta$ $\square$

Deductions and Metatheorems

Theorem (Deduction Theorem)

If $\Gamma; \gamma \vdash \varphi$, *then* $\Gamma \vdash \gamma \Rightarrow \varphi$

Proof.

$\Gamma; \gamma \vdash \varphi$

iff $\{\Gamma; \gamma\} \cup \Lambda$ tautologically implies φ (by Thm 24B)

iff $\Gamma \cup \Lambda$ tautologically implies $\gamma \Rightarrow \varphi$ (by Compactness Thm of Sentential Logic; either $\Gamma \cup \Lambda$ tautologically implies φ, or $\Gamma \cup \Lambda$ does not tautologically imply γ)

iff $\Gamma \vdash \gamma \Rightarrow \varphi$ (by Thm 24B) $\square$

The converse of the theorem is true as well, in essence, the rule of modus ponens. $(\Gamma; \gamma \vdash \gamma)$

Deductions and Metatheorems

Corollary (Contraposition)

$\Gamma; \varphi \vdash \neg\psi$ *iff* $\Gamma; \psi \vdash \neg\varphi$

Proof.

$\Gamma; \varphi \vdash \neg\psi$

implies $\Gamma \vdash \varphi \Rightarrow \neg\psi$

implies $\Gamma \vdash \psi \Rightarrow \neg\varphi$

implies $\Gamma; \psi \vdash \neg\varphi$ $\square$

Deductions and Metatheorems

Corollary (Reductio ad Absurdum)

If $\Gamma; \varphi$ *is inconsistent, the* $\Gamma \vdash \neg\varphi$

Proof.

We have $\Gamma; \varphi \vdash \alpha$ and $\Gamma; \varphi \vdash \neg\alpha$.
$\because \{\varphi \Rightarrow \alpha, \varphi \Rightarrow \neg\alpha\}$ tautologically implies $\neg\varphi$
$\therefore \Gamma \vdash \neg\varphi$ $\square$

A set of formulas is inconsistent iff for some α, both α and $\neg\alpha$ are theorems of the set

Deduction Strategy

1 Show $\Gamma \vdash \psi \Rightarrow \theta$ by $\Gamma; \psi \vdash \theta$

2 Show $\Gamma \vdash \forall x\psi$

1. if x is not free in Γ, prove $\Gamma \vdash \psi$
2. if x is free in Γ, prove $\Gamma \vdash \forall y(\psi)^x_y$ and $\forall y(\psi)^x_y \vdash \forall x\psi$ with some variable y

3a Show $\Gamma \vdash \neg(\psi \Rightarrow \theta)$ by $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\theta$

3b Show $\Gamma \vdash \neg\neg\psi$ by $\Gamma \vdash \psi$

3c Show $\Gamma \vdash \neg\forall x\psi$ by $\Gamma \vdash \neg\psi^x_t$ (for t is substitutable for x in ψ)

- Note that this is useful but not always possible
  - E.g., when $\Gamma = \emptyset$ and $\psi = \neg(Px \Rightarrow \forall yPy)$, $\Gamma \vdash \neg\forall x\psi$ and yet, for every t, $\Gamma \nvdash \neg\psi^x_t$
- $\Gamma; \alpha \vdash \neg\forall x\psi$ iff $\Gamma; \forall x\psi \vdash \neg\alpha$
- If $\Gamma; \forall x\psi \vdash \neg\alpha$, then $\Gamma; \forall y\alpha \vdash \neg\forall x\psi$

4 Try reductio ad absurdum if above fail

Deduction Strategy

E.g., $\forall x \forall y(x = y \Rightarrow y = x)$

Proof.

1. $\vdash x = y \Rightarrow x = x \Rightarrow y = x$ (Ax6)
2. $\vdash x = x$ (Ax5)
3. $\vdash x = y \Rightarrow y = x$ (1,2;T)
4. $\vdash \forall x \forall y(x = y \Rightarrow y = x)$ (3;gen^2)

$\square$

Deduction Strategy

Theorem (Generalization on Constants)

Assume that $\Gamma \vdash \varphi$ and that constant symbol c does not occur in Γ. Then there is a variable y (not occur in φ) such that $\Gamma \vdash \forall y(\varphi)^c_y$. Further, there is a deduction of $\forall y(\varphi)^c_y$ from Γ in which c does not occur.

Deduction Strategy

Corollary

Assume that $\Gamma \vdash \varphi_c^x$*, where the constant symbol* c *does not occur in* Γ *and in* φ*. Then* $\Gamma \vdash \forall x \varphi$*, and there is a deduction of* $\forall x \varphi$ *from* Γ *in which* c *does not occur.*

Deduction Strategy

Corollary (Rule EI)

Assume that constant symbol c *does not occur in* φ*,* ψ*, and* Γ*, and that* $\Gamma; \varphi_c^x \vdash \psi$*. Then* $\Gamma; \exists x \varphi \vdash \psi$ *and there is a deduction of* ψ *from* $\Gamma; \exists x \varphi$ *in which* c *does not occur.*