Special Topics on Applied Mathematical Logic

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Lecture 06

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Outline

- First-Order Logic
  - Soundness and Completeness
  - Compactness

Soundness and Completeness

- entailment $\models$ vs. deduction $\vdash$
- $\left\{ \begin{array}{ll} \text{Soundness} & \Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi \\ \text{Completeness} & \Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi \end{array} \right.$

Validity of Logical Axioms

Lemma
Every logical axiom is valid.

Proof.
Consider logical axioms that are not generalizations of other axioms. (Any generalization of a valid formula is valid.)

Ax1 If ϕ tautologically implies α, then ϕ logically implies α (check §2.4 Ex3)

Ax3 $\forall x(\alpha \Rightarrow \beta) \models \forall x\alpha \Rightarrow \forall x\beta$ (check §2.2 Ex3)

Ax4 $\alpha \models \forall x\alpha$ for x does not occur free in α (check §2.2 Ex4)

Ax5 $\models x = x$

Validity of Logical Axioms

Proof (cont'd).

Ax6 To show $\{x = y, \alpha\} \models \alpha'$, where α' is obtained from atomic formula α by replacing x at some places by y. For any $\mathfrak{A}$ and s such that $\models_{\mathfrak{A}} x = y[s]$, i.e., $s(x) = s(y)$, then we have $\overline{s}(t) = \overline{s}(t')$, where t is any term and t' is obtained from t by replacing x at some places by y.

If α is $t_1 = t_2$, then α' is $t_1' = t_2'$
$\models_{\mathfrak{A}} \alpha[s]$ iff $\overline{s}(t_1) = \overline{s}(t_2)$ iff $\overline{s}(t_1') = \overline{s}(t_2')$ iff $\models_{\mathfrak{A}} \alpha'[s]$

If α is $Pt_1 \cdots t_n$, then α' is $Pt_1' \cdots t_n'$
$\models_{\mathfrak{A}} \alpha[s]$ iff $\langle \overline{s}(t_1), \ldots, \overline{s}(t_n) \rangle \in P^{\mathfrak{A}}$ iff $\langle \overline{s}(t_1'), \ldots, \overline{s}(t_n') \rangle \in P^{\mathfrak{A}}$ iff $\models_{\mathfrak{A}} \alpha'[s]$

Validity of Logical Axioms

Proof (cont'd).

Ax2 ($\forall x \alpha \Rightarrow \alpha^x_t$ for t substitutable for free x in α)

Lemma (25B)

For any term u, let u^x_t be obtained from u by replacing variable x in u by term t. (Always substitutable!) *Then $\overline{s}(u^x_t) = \overline{s(x|\overline{s}(t))}(u)$.*

Proof.

By induction on term u. □

$$
\begin{tikzcd}
\text{terms} \arrow[rr, "\text{substitute } t \text{ for } x"] \arrow[dr, "\overline{s(x|\overline{s}(t))}"'] & & \text{terms} \arrow[dl, "\overline{s}"] \\
& |\mathfrak{A}| &
\end{tikzcd}
$$

Validity of Logical Axioms

Proof (cont'd).

Lemma (Substitution Lemma)

If the term t is substitutable for the variable x in the wff φ, then $\models_{\mathfrak{A}} \varphi^x_t[s]$ *iff* $\models_{\mathfrak{A}} \varphi[s(x|\overline{s}(t))]$

Proof of Substitution Lemma.

By induction on φ,

case 1 φ is atomic

E.g., $\varphi = Pu$

$\models_{\mathfrak{A}} Pu^x_t[s]$ iff $\overline{s}(u^x_t) \in P^{\mathfrak{A}}$ iff $\overline{s(x|\overline{s}(t))}(u) \in P^{\mathfrak{A}}$ iff $\models_{\mathfrak{A}} Pu[s(x|\overline{s}(t))]$

case 2 φ is $\neg\psi$ or $\psi \Rightarrow \theta$

The proof follows from inductive hypotheses (IH) for ψ and θ

case 3 φ is $\forall y \psi$, with x does not occur free in φ

$\varphi^x_t = \varphi$

Validity of Logical Axioms

Proof of Substitution Lemma (cont'd).

case 4 φ is $\forall y \psi$, with x occurs free in φ

For t substitutable for x in φ, (1) y must not occur in t and (2) t is substitutable for x in ψ

By (1), for every $d \in |\mathfrak{A}|$,

$$\overline{s}(t) = \overline{s(y|d)}(t) \qquad (*)$$

Since $x \neq y$, $\varphi^x_t = \forall y \psi^x_t$

$\models_{\mathfrak{A}} \varphi^x_t[s]$ iff for every d, $\models_{\mathfrak{A}} \psi^x_t[s(y|d)]$ by $(*)$ iff for every d, $\models_{\mathfrak{A}} \psi[s(y|d)(x|\overline{s}(t))]$ by IH iff $\models_{\mathfrak{A}} \varphi[s(x|\overline{s}(t))]$

By induction, Substitution Lemma holds for all φ.

Q.E.D. (Substitution Lemma)

Validity of Logical Axioms

Proof (cont'd).
Back to Ax2 ($\forall x \varphi \Rightarrow \varphi^x_t$):
Assume $\mathfrak{A}$ satisfies $\forall x \varphi$ with s. To show $\models_{\mathfrak{A}} \varphi^x_t[s]$:
Since for any $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \varphi[s(x|d)]$, letting $d = \bar{s}(t)$ yields $\models_{\mathfrak{A}} \varphi[s(x|\bar{s}(t))]$.
By Substitution Lemma, $\models_{\mathfrak{A}} \varphi^x_t[s]$.
That is, Ax2 is valid.
Consequently, from the above we know every logical axiom is valid.

Q.E.D.

Soundness

Theorem (Soundness Theorem)
If $\Gamma \vdash \varphi$, *then* $\Gamma \models \varphi$

Proof.
Show by induction.

case 1 φ is a logical axiom, i.e., $\varphi \in \Lambda$
By the previous lemma, $\models \varphi$ and thus $\Gamma \models \varphi$

case 2 $\varphi \in \Gamma$
$\Gamma \models \varphi$

case 3 $\dfrac{\psi, \quad \psi \Rightarrow \varphi}{\varphi}$
By IH, $\Gamma \models \psi$ and $\Gamma \models \psi \Rightarrow \varphi$
It follows that $\Gamma \models \varphi$

$\square$

Soundness

Corollary (25C)

If $\vdash (\varphi \Leftrightarrow \psi)$, *then* φ *and* ψ *are logically equivalent, i.e.,* $\varphi \models\dashv \psi$

Proof.

$\vdash \varphi \Rightarrow \psi$ implies $\varphi \vdash \psi$ implies $\varphi \models \psi$
$\vdash \psi \Rightarrow \varphi$ implies $\psi \vdash \varphi$ implies $\psi \models \varphi$ $\square$

Soundness

Corollary (25D)

If φ' *is an alphabetic variant of* φ, *then* φ *and* φ' *are logically equivalent*

Soundness

Corollary (25E)

If Γ is satisfiable, then Γ is consistent

Proof.

Γ inconsistent

$$\Rightarrow \begin{cases} \Gamma \vdash \varphi \\ \Gamma \vdash \neg\varphi \end{cases}$$

$$\Rightarrow \begin{cases} \Gamma \models \varphi \\ \Gamma \models \neg\varphi \end{cases}$$

$\Rightarrow \Gamma$ unsatisfiable $\square$

(This corollary is equivalent to Soundness Theorem.)

Recall Γ is *consistent* iff there is no formula φ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ (syntactical)

Define Γ to be **satisfiable** iff there is some $\mathfrak{A}$ and s such that $\mathfrak{A}$ satisfies every member of Γ with s (semantical)

Completeness

Theorem (Completeness Theorem; Gödel, 1930)

(a) *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$*

(b) *If Γ is consistent, then Γ is satisfiable*

(a) and (b) are equivalent

$\because (a) \Leftrightarrow$ If $\Gamma \nvdash \varphi$, then $\Gamma \not\models \varphi$

$\Leftrightarrow$ If $\Gamma; \neg\varphi$ is consistent, then $\Gamma; \neg\varphi$ is satisfiable

$\Leftrightarrow (b)$

Completeness Theorem

Proof Outline (for (b)).

Steps 1-3 Extend Γ to Δ such that

- (i) $\Gamma \subseteq \Delta$
- (ii) Δ is consistent and maximal in the sense that for any formula, either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$
- (iii) For any formula φ and variable x, there is a constant c such that $(\neg\forall x\varphi \Rightarrow \neg\varphi^x_c) \in \Delta$

Step 4 From a structure $\mathfrak{A}$ where members of Γ not containing $=$ can be satisfied. In particular, $|\mathfrak{A}|$ is the set of terms and $\langle t_1, \ldots, t_n\rangle \in P^{\mathfrak{A}}$ iff $Pt_1 \ldots t_n \in \Delta$

Steps 5,6 Modify $\mathfrak{A}$ to work for formulas containing $=$

Completeness Theorem

Proof.

Step 1 Expand the language by adding a countably infinite set of new constant symbols. Then Γ remains a consistent set of wffs in the new language.

Step 2 For every wff φ in the new language and every variable x, we add the wff $(\neg\forall x\varphi \Rightarrow \neg\varphi^x_c)$ to Γ, for c to be some new constant symbol. (So c provides a counterexample to φ if any.) This can be done such that Γ together with the set Θ of all the added wffs is still consistent.

Step 3 Extend $\Gamma \cup \Theta$ to a consistent set maximal in the sense that for any wff φ either $\varphi \in \Delta$ or $(\neg\varphi) \in \Delta$. Note that Δ is deductively closed. That is, $\Delta \vdash \varphi \Rightarrow \Delta \nvdash \neg\varphi \Rightarrow (\neg\varphi) \notin \Delta \Rightarrow \varphi \in \Delta$.

Completeness Theorem

Proof (cont'd).

Step 4 From Δ, we construct a structure $\mathfrak{A}$ for the new language, but with $=$ replaced by a new 2-place predicate symbol E. $\mathfrak{A}$ is such that

(a) $|\mathfrak{A}|$ = the set of all terms of the new language
(b) $\langle u, t \rangle \in E^{\mathfrak{A}}$ iff wff $(u = t) \in \Delta$
(c) $\langle t_1, \ldots, t_n \rangle \in P^{\mathfrak{A}}$ iff $Pt_1 \ldots t_n \in \Delta$
(d) $f^{\mathfrak{A}}(t_1, \ldots, t_n) = ft_1 \ldots t_n$ and $c^{\mathfrak{A}} = c$

Besides define $s : V \to |\mathfrak{A}|$ be the identity function, i.e., $s(x) = x$ on V. Then for any term t, $\bar{s}(t) = t$. For any wff φ, let φ^* be obtained from φ by replacing $=$ by E. Then $\models_{\mathfrak{A}} \varphi^*[s]$ iff $\varphi \in \Delta$. (Prove by induction on the # of places at which connective/quantifier symbols appear.)

Completeness Theorem

Proof (cont'd).

To see that $\mathfrak{A}$ cannot be used directly in the language, consider Γ containing a sentence $(c_1 = c_2)$, for c_1 and c_2 are distinct constant symbols. We have $\langle c_1^{\mathfrak{A}}, c_2^{\mathfrak{A}} \rangle \in E^{\mathfrak{A}}$ but $c_1^{\mathfrak{A}} \neq c_2^{\mathfrak{A}}$. It does not hold that $\models_{\mathfrak{A}} (c_1 = c_2)[s]$ iff $(c_1 = c_2) \in \Delta$. Rather we need a new structure $\mathfrak{B}$ such that $c_1^{\mathfrak{B}} = c_2^{\mathfrak{B}}$.

Step 5 We obtain $\mathfrak{B}$ as the quotient structure $\mathfrak{A}/E$ of $\mathfrak{A}$ modulo $E^{\mathfrak{A}}$. Note that $E^{\mathfrak{A}}$ is an equivalence relation on $|\mathfrak{A}|$ that forms a *congruence relation* for $\mathfrak{A}$:

(i) $E^{\mathfrak{A}}$ is an equivalence relation on $|\mathfrak{A}|$
(ii) $P^{\mathfrak{A}}$ is compatible with $E^{\mathfrak{A}}$: $\langle t_1, \ldots, t_n \rangle \in P^{\mathfrak{A}}$ and $\langle t_i, t_i' \rangle \in E^{\mathfrak{A}}$ for $1 \le i \le n$ implies $\langle t_1', \ldots, t_n' \rangle \in P^{\mathfrak{A}}$
(iii) $f^{\mathfrak{A}}$ is compatible with $E^{\mathfrak{A}}$: $\langle t_i, t_i' \rangle \in E^{\mathfrak{A}}$ for $1 \le i \le n$ implies $\langle f^{\mathfrak{A}}(t_1, \ldots, t_n), f^{\mathfrak{A}}(t_1', \ldots, t_n') \rangle \in E^{\mathfrak{A}}$

Completeness Theorem

Proof (cont'd).

Let $[t]$ be the equivalence class of term t in $|\mathfrak{A}|$. We define $\mathfrak{A}/E$ as follows.

(1) $|\mathfrak{A}/E|$ is the set of all equivalence classes of members of $|\mathfrak{A}|$

(2) $\langle [t_1], \ldots, [t_n] \rangle \in P^{\mathfrak{A}/E}$ iff $\langle t_1, \ldots, t_n \rangle \in P^{\mathfrak{A}}$

(3) $f^{\mathfrak{A}/E}([t_1], \ldots, [t_n]) = [f^{\mathfrak{A}}(t_1, \ldots, t_n)]$, $c^{\mathfrak{A}/E} = [c^{\mathfrak{A}}]$

Let $h : |\mathfrak{A}| \to |\mathfrak{A}/E|$ such that $h(t) = [t]$. So h is a homomorphism of $\mathfrak{A}$ *onto* $\mathfrak{A}/E$.

Hence for any φ:

$$\begin{aligned} \varphi \in \Delta \quad &\Leftrightarrow \quad \models_{\mathfrak{A}} \varphi^*[s] \\ &\Leftrightarrow \quad \models_{\mathfrak{A}/E} \varphi^*[h \circ s] \text{ (by Homomorphism Theorem)} \\ &\Leftrightarrow \quad \models_{\mathfrak{A}/E} \varphi[h \circ s] \ (\langle [t], [t'] \rangle \in E^{\mathfrak{A}/E} \text{ iff } \langle t, t' \rangle \in E^{\mathfrak{A}} \text{ iff } [t] = [t'] \end{aligned}$$

That is, $\mathfrak{A}/E$ satisfies every member of Δ with $h \circ s$.

Completeness Theorem

Proof (cont'd).

Step 6 Restrict $\mathfrak{A}/E$ to the *original* language. $\mathfrak{A}/E$ satisfies every member of Γ with $h \circ s$.

Q.E.D.

Compactness

Theorem (Compactness Theorem)

(a) *If* $\Gamma \models \varphi$, *then for some finite* $\Gamma_0 \subseteq \Gamma$ *we have* $\Gamma_0 \models \varphi$

(b) *If every finite subset* Γ_0 *of* Γ *is satisfiable, then* Γ *is satisfiable*

In particular, a set Σ *of sentences has a model iff every finite subset has a model. (Similar to that in sentential logic.)*

Proof.

(a)

$$\begin{aligned} \Gamma \models \varphi &\Rightarrow \Gamma \vdash \varphi \\ &\Rightarrow \Gamma_0 \vdash \varphi \text{ (deduction is finite)} \\ &\Rightarrow \Gamma_0 \models \varphi \end{aligned}$$

Compactness Theorem

Proof (cont'd).

(b) If every finite subset Γ_0 of Γ is satisfiable, then by Soundness Theorem every Γ_0 is consistent. Since deduction is finite, Γ is consistent. By Completeness Theorem, Γ is satisfiable.

Q.E.D.

Note that

- (a) and (b) of Compactness Theorem are equivalent
- Compactness Theorem involves only semantical notions

Enumerability

Theorem (Enumerability Theorem)

For a reasonable language, the set of valid wffs can be effectively enumerated

A language is *reasonable* if its set of parameters can be effectively enumerated and

$$\{\langle P, n\rangle \mid \ P \text{ is an } n\text{-place predicate symbol}\} \text{ and}$$
$$\{\langle f, n\rangle \mid \ f \text{ is an } n\text{-place function symbol}\}$$

are decidable

Enumerability

Corollary (25F)

Let Γ be a decidable set of formulas in a reasonable language.

(a) *The set $\{\varphi \mid \Gamma \vdash \varphi\}$ of theorems of Γ is effectively enumerable*

(b) *The set $\{\varphi \mid \Gamma \models \varphi\}$ of formulas logically implied by Γ is effectively enumerable*

Enumerability

Corollary (25G)

Assume that Γ is a decidable set of formulas in a reasonable language, and for any sentence σ either $\Gamma \models \sigma$ or $\Gamma \models \neg\sigma$. Then the set of sentences implied by Γ is decidable.

(related to Corollary 26I)