Special Topics on Applied Mathematical Logic

Spring 2012

Lecture 07

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April 23, 2012

Outline

First-Order Logic
- Models of Theories
- Decision Problems and Finite Structures
- Size of Models
- Theories

Models of Theories

Models of theories in the light of soundness and completeness theorems

- The sentence $\forall v_1 \forall v_2 \exists v_3 (v_1 < v_2 \Rightarrow (v_3 \neq v_2 \wedge v_2 < v_3))$ has only infinite models (i.e., $|\mathfrak{A}|$ infinite)
- The sentence $\forall v_1 \forall v_2 v_1 = v_2$ has only finite models (singleton $|\mathfrak{A}|$)

If a sentence σ has only infinite models, then $\neg\sigma$ is *finitely valid*, i.e., true in every finite structure. ($\because$ either $\models_{\mathfrak{A}} \sigma$ or $\models_{\mathfrak{A}} \neg\sigma$ for any sentence σ)

Models of Theories

Theorem (26A)

If a set Σ of sentences has arbitrarily large finite models, then it has an infinite model.

Proof.

Let λ_k be

$$\exists v_1 \cdots \exists v_k (v_1 \neq v_2 \wedge \cdots \wedge v_1 \neq v_k \wedge v_2 \neq v_3 \wedge \cdots \wedge v_2 \neq v_k \wedge \cdots \wedge v_{k-1} \neq v_k)$$

for $k \geq 2$. Then any finite subset of $\Sigma \cup \{\lambda_2, \lambda_3, \ldots\}$ has a model. By compactness, the entire set has an infinite model. $\square$

Decision Problems and Finite Structures

Definition

For a structure $\mathfrak{A}$, the **theory of** $\mathfrak{A}$, written $\mathrm{Th}\mathfrak{A}$, is the set of all sentences true in $\mathfrak{A}$.

We study if $\mathrm{Th}\mathfrak{A}$ is decidable for any finite structure, and if the set of sentences having finite models is decidable.

Decision Problems and Finite Structures

Observations:

1. Any finite structure $\mathfrak{A}$ is isomorphic to a structure with universe $\{1, 2, \ldots, n\}$ for n being the size of $\mathfrak{A}$
2. A finite structure for a *finite language* (with finitely many parameters) can be specified by a finite string of symbols
3. Given a finite structure for a finite language, a wff φ, and an assignment s, we can effectively decide if $\models_{\mathfrak{A}} \varphi[s]$. Restricting ourselves to sentences, we can effectively decide if $\mathfrak{A}$ is a model of σ.

Theorem (26C)

For a finite structure $\mathfrak{A}$ in a finite language, $\mathrm{Th}\mathfrak{A}$ is decidable

($\because$ either $\models_{\mathfrak{A}} \sigma$ or $\models_{\mathfrak{A}} \neg\sigma$ for any sentence σ)

Decision Problems and Finite Structures

Observations (cont'd):

4. Given a sentence σ and a positive integer n, we can effectively decide if σ has an n-element model. That is, the relation

$$\{\langle \sigma, n \rangle \mid \sigma \text{ has a model of size } n\}$$

is decidable.

(Note that there are only finitely many structures to check. E.g., if the language has only parameters $\forall$ and a 2-place predicate symbol E, then there are 2^{n^2} different structures. By Observation 3, we can decide if σ has a model of size n.)

Decision Problems and Finite Structures

Observations (cont'd):

5. The set $\{n \mid \sigma \text{ has a model of size } n\}$ of any sentence σ is a decidable set of positive integers

Theorem (26D)

For a finite language, $\{\sigma \mid \sigma \text{ has a finite model}\}$ is effectively enumerable

Proof.

Given σ, first check if σ has a model of size one by Observation 4. If not, try size 2, and so on. $\square$

Decision Problems and Finite Structures

Corollary (26E)

For a finite language, let Φ be the set of sentences true in every finite structure. Then its complement $\overline{\Phi}$ is effectively enumerable.

Proof.

$\sigma \in \overline{\Phi}$ iff $(\neg\sigma)$ has a finite model. We can apply the semidecision procedure of the previous theorem to $(\neg\sigma)$. $\square$

Decision Problems and Finite Structures

Theorem (Trakhtenbrot, 1950)

The set of sentences

$$\Phi = \{\sigma \mid \sigma \textit{ is true in every finite structure}\}$$

(i.e., σ is valid for finite structures) is not decidable or effectively enumerable

- As a consequence of Trakhtenbrot's theorem, Enumerability Theorem for finite structures only does not hold
  - Recall Enumerability Theorem says: For a reasonable language, the set of valid wffs can be effectively enumerated.

Size of Models

In the proof of Completeness Theorem, if the language is countable, then $|\mathfrak{A}/E|$ is a countable set. Hence a consistent set of sentences in a countable language has a countable model.

Size of Models

Theorem (Löwenheim-Skolem Theorem, 1915)

(a) *Let Γ be a satisfiable set of formulas in a countable language. Then Γ is satisfiable in some countable structure.*

(b) *Let Σ be a set of sentences in a countable language. If Σ has any model, then it has a countable model.*

Proof.

Γ must be consistent (by Soundness Theorem). Then Γ can be satisfied in a countable structure (by Completeness Theorem with the remark of the previous slide). $\square$

Size of Models

Theorem

For any structure $\mathfrak{A}$ for a countable language, there is a countable elementarily equivalent structure $\mathfrak{B}$

Proof.

If $\mathfrak{B}$ is a (countable) model of $\mathrm{Th}\mathfrak{A}$, then
$\models_{\mathfrak{A}} \sigma \;\Rightarrow\; \sigma \in \mathrm{Th}\mathfrak{A} \;\Rightarrow\; \models_{\mathfrak{B}} \sigma$ and
$\not\models_{\mathfrak{A}} \sigma \;\Rightarrow\; \models_{\mathfrak{A}} \neg\sigma \;\Rightarrow\; (\neg\sigma) \in \mathrm{Th}\mathfrak{A} \;\Rightarrow\; \models_{\mathfrak{B}} \neg\sigma \;\Rightarrow\; \not\models_{\mathfrak{B}} \sigma$.
Hence $\mathfrak{A} \equiv \mathfrak{B}$. $\square$

Size of Models

Theorem (Löwenheim-Skolem Theorem)

(a) *Let Γ be a satisfiable set of formulas in a language of cardinality λ. Then Γ is satisfiable in some structure of size no greater than λ.*

(b) *Let Σ be a set of sentences in a language of cardinality λ. If Σ has any model, then it has a model of cardinality no greater than λ.*

Size of Models

Let $\mathfrak{B}$ be a countable structure. Is there an uncountable $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{B}$?

Yes, if $\mathfrak{B}$ is infinite. No, otherwise.

Size of Models

Theorem (L-S-Tarski Theorem)

Let Γ be a satisfiable set of formulas in a language of cardinality λ, and assume Γ is satisfiable in some infinite structure. Then for every cardinal $\kappa \geq \lambda$, there is a structure of cardinality κ in which Γ is satisfiable.

Size of Models

Corollary (26F)

(a) *Let Σ be a set of sentences in a countable language. If Σ has some infinite model, then Σ has models of every infinite cardinality.*

(b) *Let $\mathfrak{A}$ be an infinite structure for a countable language. Then for any infinite cardinal λ, there is a structure $\mathfrak{B}$ of cardinality λ such that $\mathfrak{B} \equiv \mathfrak{A}$.*

Mod vs. Th

$$\begin{cases} \mathrm{Mod}\tau : & \text{the } \textit{class} \text{ of all models of sentence } \tau \\ \mathrm{Mod}\Sigma : & \text{the } \textit{class} \text{ of all models of all sentences in } \Sigma \end{cases}$$

$$\begin{cases} \mathrm{Th}\mathfrak{A} : & \text{the } \textit{set} \text{ of all sentences true in } \mathfrak{A} \\ \mathrm{Th}\mathcal{K} : & \text{the } \textit{set} \text{ of all sentences true in every member of } \mathcal{K}, \\ & \text{where } \mathcal{K} \text{ is a class of structures} \end{cases}$$

Theories

Definition

A **theory** is a set of *sentences* closed under logical implication

- For a theory T, if $T \models \sigma$, then $\sigma \in T$
- E.g.,
  the smallest theory: the set of valid sentences of the language
  the largest theory: the set of all the sentences of the language
  (the only unsatisfiable theory)
- "theory" vs. "theorem"

Theories

Definition

For a class $\mathcal{K}$ of structures for the language, the **theory of** $\mathcal{K}$ is
$\mathrm{Th}\mathcal{K} = \{\sigma \mid \sigma \text{ is true in every member of } \mathcal{K}\}$

Theorem (26G)

$\mathrm{Th}\mathcal{K}$ *is indeed a theory*

Proof.

Suppose σ is true in every model of $\mathrm{Th}\mathcal{K}$. Since any member of $\mathcal{K}$ is a model of $\mathrm{Th}\mathcal{K}$, σ is true in every member of $\mathcal{K}$. $\therefore \sigma \in \mathrm{Th}\mathcal{K}$
($\sigma \notin \mathrm{Th}\mathcal{K}$, then $\exists \mathfrak{A} \in \mathcal{K}, \not\models_{\mathfrak{A}} \sigma$, then $\mathrm{Th}\mathcal{K} \not\models \sigma$) $\square$

Theories

Corollary (26B)

The class of all finite structures (for a fixed language) is not EC_Δ; the class of all infinite structures is not EC (but EC_Δ).

- This corollary refers to Theorem 26A

Theories

$\mathrm{ThMod}\Sigma$ is the set of all sentences *true in all models of* Σ. That is, the set of all sentences *logically implied by* Σ.

Definition

The set of **consequences** of Σ, $\mathrm{Cn}\Sigma = \{\sigma \mid \Sigma \models \sigma\} = \mathrm{ThMod}\Sigma$

- Hence a set T of sentences is a theory iff $T = \mathrm{Cn}\, T$
- E.g., set theory is the set of consequences of the axioms for set theory

Theories

Definition

A theory is **complete** iff for every sentence σ, either $\sigma \in T$ or $(\neg\sigma) \in T$

- E.g., $\mathrm{Th}\mathfrak{A}$ is always a complete theory for any structure $\mathfrak{A}$ ($\because$ either $\models_{\mathfrak{A}} \sigma$ or $\models_{\mathfrak{A}} \neg\sigma$ $\therefore$ either $\sigma \in \mathrm{Th}\mathfrak{A}$ or $\neg\sigma \in \mathrm{Th}\mathfrak{A}$)
- $\mathrm{Th}\mathcal{K}$ is a complete theory iff any two members of $\mathcal{K}$ are elementarily equivalent. ($\mathfrak{A} \equiv \mathfrak{B}$ iff $\forall\sigma, \models_{\mathfrak{A}} \sigma \Leftrightarrow \models_{\mathfrak{B}} \sigma$)
- A theory T is complete iff any two models of T are elementarily equivalent

Theories

Definition

A theory T is **axiomatizable** iff there is a *decidable* (existing effective procedures deciding membership) set Σ of sentences such that $T = \mathrm{Cn}\Sigma$

Definition

A theory T is **finitely axiomatizable** iff $T = \mathrm{Cn}\Sigma$ for some finite set Σ of sentences

($\mathrm{Cn}\Sigma = \mathrm{Cn}\sigma$ with $\sigma = \bigwedge_{\sigma_i \in \Sigma} \sigma_i$ for Σ finite)

Theories

Theorem (26H)

If $\mathrm{Cn}\Sigma$ *is finitely axiomatizable, then there is a finite* $\Sigma_0 \subseteq \Sigma$ *such that* $\mathrm{Cn}\Sigma_0 = \mathrm{Cn}\Sigma$

Proof.

$\because$ $\mathrm{Cn}\Sigma$ is finitely axiomatizable. There exists some σ such that $\mathrm{Cn}\Sigma = \mathrm{Cn}\sigma$. Besides, $\Sigma \models \sigma$. By compactness, for some finite $\Sigma_0 \subseteq \Sigma$ we have $\Sigma_0 \models \sigma$. $\therefore$ $\mathrm{Cn}\sigma \subseteq \mathrm{Cn}\Sigma_0 \subseteq \mathrm{Cn}\Sigma$. Hence $\mathrm{Cn}\Sigma_0 = \mathrm{Cn}\Sigma$. □

From the definition of "finitely axiomatizable," it only says there is some Σ_0 with $\mathrm{Cn}\Sigma_0 = \mathrm{Cn}\Sigma$. However we don't know if $\Sigma_0 = \Sigma$.

Theories

Corollary (26I)

(a) *An axiomatizable theory (in a reasonable language) is effectively enumerable*

(b) *A complete axiomatizable theory (in a reasonable language) is decidable*

(Recall Corollary 25F and 25G)

$$
\begin{tikzcd}
\text{decidable} \arrow[d, Rightarrow] & \text{finitely axiomatizable} \arrow[d, Rightarrow] \\
\text{effectively enumerable} \arrow[r, Leftrightarrow] & \text{axiomatizable} \arrow[ul, "\text{if complete}"']
\end{tikzcd}
$$

Theories

- (§3.7) Set theory (if consistent) is not decidable and not complete
- (§3.5) Number theory is complete but not effectively enumerable and hence not axiomatizable

Theories

Definition

A theory T is κ-**categorical** for a cardinal κ iff all models of T having cardinality κ are isomorphic

(If T is a theory in a language of cardinality λ, then we must demand $\lambda \leq \kappa$)

- A theory T is $\aleph_0$-**categorical** iff all the infinite countable models of T are isomorphic

Theories

Theorem (Łoś-Vaught Test, 1954)

Let T be a theory in a countable language. Assume T has no finite models.

(a) *If T is $\aleph_0$-categorical, then T is complete*

(b) *If T is κ-categorical for some infinite cardinal κ, then T is complete*

Proof.

By LST Theorem, for any 2 infinite models $\mathfrak{A}$ and $\mathfrak{B}$, there exist structures $\mathfrak{A}' \equiv \mathfrak{A}$ and $\mathfrak{B}' \equiv \mathfrak{B}$ with cardinality κ. Since $\mathfrak{A}' \cong \mathfrak{B}'$, we have $\mathfrak{A} \equiv \mathfrak{A}' \cong \mathfrak{B}' \equiv \mathfrak{B}$. $\therefore \mathfrak{A} \equiv \mathfrak{B}$ □

(The converse is not true as there are complete theories not κ-categorical for any κ)