Special Topics on Applied Mathematical Logic

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Lecture 08

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Outline

- Undecidability
  - Incompleteness
    - Gödel Numbering
    - Arithmetization of Metamathematics
    - Gödel's Argument

Incompleteness

Gödel's incompleteness theorems, proved by Kurt Gödel in 1931, are two theorems stating inherent limitations of formal systems for arithmetic. The theorems are of considerable importance to the philosophy of mathematics. They are widely regarded as showing that Hilbert's program to find a complete and consistent set of axioms for all of mathematics is impossible, thus giving a negative answer to Hilbert's second problem.
[See Wikipedia about "incompleteness theorems"]

Incompleteness Theorems

Theorem (Gödel's first incompleteness theorem)

Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory.

Theorem (Gödel's second incompleteness theorem)

For any formal effectively generated theory T including basic arithmetical truths and also certain truths about formal provability, T includes a statement of its own consistency if and only if T is inconsistent.

[Wiki about "incompleteness theorems"]

Incompleteness

To informally introduce incompleteness theorems, we follow *Gödel's Proof* by E. Nagel and J. Newman, 1958.

Gödel Numbering

It is possible to assign a unique number to each parameter, each logical symbol, each formula, and each proof.

E.g., in the language of number theory

symbols:

parameters (Gödel numbers)	logical symbols (Gödel numbers)
$\forall$ (0)	((1)
0 (2)	) (3)
S (4)	$\neg$ (5)
$<$ (6)	$\Rightarrow$ (7)
$+$ (8)	$=$ (9)
$\cdot$ (10)	v_1 (11)
E (12)	v_2 (13)
	$\vdots$

Gödel Numbering

E.g., (cont'd)

formulas:
Let φ be

$$\begin{array}{cccccccccc} \neg & \forall & v_1 & \neg & (& v_1 & = & S & v_2 &) \\ 5 & 0 & 11 & 5 & 1 & 11 & 9 & 4 & 13 & 3 \end{array}$$

The Gödel number of φ is
$2^5 \cdot 3^0 \cdot 5^{11} \cdot 7^5 \cdot 11^1 \cdot 13^{11} \cdot 17^9 \cdot 19^4 \cdot 23^{13} \cdot 29^3$

- Primes are used to record the order of a sequence

proofs:
A proof $\alpha_0 \Rightarrow \alpha_1 \Rightarrow \ldots \Rightarrow \alpha_n$ can be flatten into a big formula.
So its Gödel number is obtainable as well.

Gödel Numbering

Given a Gödel number, one can uniquely determine its represented parameter, logical symbol, formula, or proof.

Arithmetization of Metamathematics

- Since every expression in the language is associated with a Gödel number, a metamathematical statement about expressions and their relations to one another can be constructed as a statement about the corresponding Gödel numbers and their arithmetical relations to one another.

  E.g., the Gödel number of φ_1 is a factor of the Gödel number of φ_2 if and only if φ_1 is an initial part of φ_2 (for instance, $\varphi_1 = (v_1 = v_2)$ and $\varphi_2 = ((v_1 = v_2) \Rightarrow (v_1 = v_3)))$

Arithmetization of Metamathematics

- Let $\mathrm{Pv}(x, z)$ denote the metamathematical statement: "The sequence of formulas with Gödel number x is a proof of the formula with Gödel number z."
  - The statement can be represented by a formula in the language representing a purely arithmetical relation between x and z
  - $\mathrm{Pv}(x, z)$ is a relation on a pair of Gödel numbers
    - What do $\mathrm{Pv}(x, z)$ and $\neg\mathrm{Pv}(x, z)$ mean?

Arithmetization of Metamathematics

Consider the formula $\exists x(x = Sy)$, i.e., formally $\neg\forall v_1 \neg(v_1 = Sv_2)$. Let m be its Gödel number.

- What does $\exists x(x = Sm)$ mean?
- What is the Gödel number of the formula $\exists x(x = Sm)$?

Arithmetization of Metamathematics

- Let "$sub(m, 13, m)$" mean the "The Gödel number of the formula obtained from the formula with Gödel number m, by substituting for the variable with the Gödel number 13, i.e., v_2, the numeral for m."
  - Numerals are number names, e.g., number three is named by (Arabic) numeral "3"
  - What does "$sub(v_2, 13, v_2)$" mean?
    - The Gödel number of the formula obtained from the formula with Gödel number v_2, by substituting for the variable with Gödel number 13 the numeral for v_2

Gödel's Argument

(i) Consider $\forall v_1 \neg \mathrm{Pv}(v_1, sub(v_2, 13, v_2))$

- The formula is in the language of number theory and has a Gödel number that can be calculated, say n
- The corresponding metamathematical statement is "The formula $sub(v_2, 13, v_2)$ is not provable"

Let G denote $\forall v_1 \neg \mathrm{Pv}(v_1, sub(n, 13, n))$

- What is the Gödel number of G?
  - It equals $sub(n, 13, n)$
  - Since G is not provable, this formula can be constructed as asserting of itself that it is not provable

Gödel's Argument

(ii) We can show that

- If G were provable, then $\neg G$ would also be provable
- If $\neg G$ were provable, then G would also be provable

$\Longrightarrow$ G is provable iff $\neg G$ is provable
$\Longrightarrow$ The axioms (of number theory) are not consistent
$\Longrightarrow$ If the axioms are consistent, then G is *undecidable*, that is, neither G nor $\neg G$ can be deduced from the axioms

Gödel's Argument

(iii) Although G is undeciable if the axioms are consistent, it can be shown by *metamathematical reasoning* that G is true.
That is, G formulas a complex but definite numerical property that necessarily holds of all integers.
This is because

1. For consistent axioms, G is not provable
2. The statement "G is not provable" is represented within arithmetic by the very formula mentioned in the statement
3. Metamathematical statements have been mapped onto the arithmetical formula in such a way that true metamathematical statements correspond to true arithmetical formulas

Gödel's Argument

(iv) Definition

The axioms of a deductive system are *complete* if and only if every true statement that can be expressed in the system is formally deducible from the axioms

- Since G is a true formula of arithmetic (number theory) not formally deducible within it, the axioms of arithmetic are incomplete
- In fact, the incompleteness is essential. That is, even if G were added as a further axiom, the augmented set would still not suffice to yield formally all arithmetical truths. It shows a fundamental limitation in the power of the axiomatic method!

Gödel's Argument

(v) To show:
The conditional metamathematical statement "If arithmetic is consistent (denoted as C), it is incomplete (denoted as I)" taken as a whole is represented by a provable formula within formalized arithmetic

C: $\exists v_2 \forall v_1 \neg \mathrm{Pv}(v_1, v_2)$

- There is at least one formula of arithmetic that is not provable, i.e., arithmetic is consistent ($\because$ if inconsistent, everything can be proved.)

I: G

- There is a true arithmetical statement that is not formally provable in arithmetic

We will not show but it can be proved that the formula $C \Rightarrow I$ is provable. By modus ponens, C is not provable.

Gödel's Argument

- Note that if the metamathematical statement of C could be established by any argument that can be mapped onto a sequence of formulas which constitutes a proof in the language of number theory, the formula C would itself be provable. But this is impossible if arithmetic is consistent.
- So we conclude that if arithmetic is consistent, its consistency cannot be established by any metamathematical reasoning that can be represented within the formalism of arithmetic!

Gödel's Argument

Remark:
The above conclusion does not exclude a metamathematical proof of the consistency of arithmetic. What is excluded is a proof of consistency that can be "mirrored" by the formal deductions of arithmetic. In fact, metamathematical proofs of the consistency of arithmetic have been shown by Gentzen in 1936. The class of inference rules needs to be enlarged (not finitistic) if the consistency of arithmetic is to be established.