1.

(a)

$s_1(t) = -s_2(t)$ $\therefore$ 爲antipodal signal

$$\varepsilon_1 = \int_0^{T}\left[\sqrt{\frac{3E}{T}}\left(-\frac{2t}{T}+1\right)\right]^2 dt = E$$

$$\varepsilon_2 = \int_0^{T}\left[\sqrt{\frac{3E}{T}}\left(\frac{2t}{T}-1\right)\right]^2 dt = E$$

$S_1 = -\sqrt{E}$, $S_2 = \sqrt{E}$ if base function $= \sqrt{\frac{3}{T}}\left(\frac{2t}{T}-1\right)$ $0 \le t \le T$

$d_{12} = 2\sqrt{E}$

$$P_{error} = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

(b)

$\because \int_0^{T} s_1(t)s_2(t)dt = 0$ $\therefore$ 爲orthogonal signals

若base function爲 $f_1 = \sqrt{\frac{1}{T}}$ $0 \le t \le T$, $f_2 = \sqrt{\frac{3}{T}}\left(\frac{2t}{T}-1\right)$ $0 \le t \le T$

$S_1 = \left(\sqrt{E}, 0\right)$, $S_1 = \left(0, \sqrt{E}\right)$

$d_{12} = \sqrt{2E}$

$$P_{error} = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{E}{N_0}}\right)$$

2.

$$f_{X_1X_2}(x_1,x_2)=\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2\sigma^2(1-\rho^2)}\left(x_1^2+x_2^2-2\rho x_1x_2\right)\right]$$

$$\psi(j\nu)=E\left[e^{j\nu Y}\right]=E\left[e^{j\nu X_1X_2}\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j\nu x_1x_2}f_{X_1X_2}(x_1,x_2)dx_1dx_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left(j\nu x_1x_2\right)\exp\left[-\frac{\left(x_1-\rho x_2\right)^2}{2\sigma^2\left(1-\rho^2\right)}\right]\exp\left(-\frac{x_2^2}{2\sigma^2}\right)dx_1dx_2$$

又 $\psi_x'(j\nu)=\int_{-\infty}^{\infty}\exp(j\nu x)\left\{\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{\left(x-m_x\right)^2}{2\sigma^2}\right]\right\}dx=\exp\left(j\nu m_x-\frac{1}{2}\nu^2\sigma^2\right)$

$$\therefore\psi(j\nu)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x_2^2}{2\sigma^2}\right)\exp\left(j\nu x_2\rho x_2-\frac{1}{2}\nu^2x_2^2\sigma^2\left(1-\rho^2\right)\right)dx_2$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{1}{\sigma^2}+\nu^2\sigma^2\left(1-\rho^2\right)-2j\nu\rho\right)x_2^2\right]dx_2$$

$$=\frac{1}{\sqrt{1+\nu^2\sigma^4\left(1-\rho^2\right)-2j\nu\rho\sigma^2}}$$

3.

(a)

known signal $=s(t)\quad 0\le t\le T$

$r(t)=\alpha s(t)\quad 0\le t\le T$

self-generate estimate function $=s(t,\alpha_2)=\alpha_2 s(t)\quad 0\le t\le T$

likelihood function $=\Lambda(\alpha_2)=\exp\left\{-\frac{1}{N_0}\int_{T_0}\left[r(t)-s(t,\alpha_2)\right]^2dt\right\}$

$$=\exp\left\{-\frac{1}{N_0}\int_0^T\left[r(t)-\alpha_2 s(t)\right]^2dt\right\}$$

$$=\exp\left\{-\frac{1}{N_0}\int_0^T r^2(t)-2r(t)\alpha_2 s(t)+\alpha_2^2s^2(t)dt\right\}$$

$$\Rightarrow\Lambda_L(\alpha_2)=\ln\left[\Lambda(\alpha_2)\right]=-\frac{1}{N_0}\int_0^T r^2(t)-2r(t)\alpha_2 s(t)+\alpha_2^2s^2(t)dt$$

Maximum likelihood $\Rightarrow\frac{d\Lambda_L(\alpha_2)}{d\alpha_2}=0$

$$\Rightarrow-\frac{1}{N_0}\left[0-2\int_0^T r(t)s(t)dt+2\int_0^T\alpha_2 s^2(t)dt\right]$$

$$\Rightarrow\alpha_2=\frac{\int_0^T r(t)s(t)dt}{\int_0^T s^2(t)dt}$$

(b)

$$r(t)=\alpha s(t)+n(t)$$

$$\Rightarrow \alpha_2=\frac{\int_0^T\left[\alpha s(t)+n(t)\right]s(t)dt}{\int_0^T s^2(t)dt}$$

$$E(\alpha_2)=E\left[\frac{\int_0^T\left[\alpha s(t)+n(t)\right]s(t)dt}{\int_0^T s^2(t)dt}\right]$$

$$=E\left[\frac{\int_0^T \alpha s^2(t)dt}{\int_0^T s^2(t)dt}\right]+E\left[\frac{\int_0^T n(t)s(t)dt}{\int_0^T s^2(t)dt}\right]$$

$$=\alpha+\frac{\int_0^T E\left[n(t)\right]s(t)dt}{\int_0^T s^2(t)dt}=\alpha$$

$$Var(\alpha_2)=E\left[(\alpha_2-\alpha)^2\right]$$

$$=E\left\{\frac{\left[\int_0^T \alpha s^2(t)dt+\int_0^T n(t)s(t)dt\right]^2}{\left[\int_0^T s^2(t)dt\right]^2}-2\alpha\frac{\int_0^T\left[\alpha s(t)+n(t)\right]s(t)dt}{\int_0^T s^2(t)dt}+\alpha^2\right\}$$

$$=E\left\{\alpha^2+\frac{2\alpha\int_0^T s^2(t)dt\int_0^T n(t)s(t)dt}{\left[\int_0^T s^2(t)dt\right]^2}+\frac{\left[\int_0^T n(t)s(t)dt\right]^2}{\left[\int_0^T s^2(t)dt\right]^2}-2\alpha^2-\frac{2\alpha\int_0^T n(t)s(t)dt}{\int_0^T s^2(t)dt}+\alpha^2\right\}$$

$$=E\left\{\frac{\left[\int_0^T n(t)s(t)dt\right]^2}{\left[\int_0^T s^2(t)dt\right]^2}\right\}$$

$$=\frac{1}{\left[\int_0^T s^2(t)dt\right]^2}\int_0^T\int_0^T E\left[n(t)n(\tau)\right]s(t)s(\tau)dtd\tau$$

$$=\frac{1}{\left[\int_0^T s^2(t)dt\right]^2}\int_0^T\int_0^T \frac{N_0}{2}\delta(t-\tau)s(t)s(\tau)dtd\tau$$

$$=\frac{N_0}{2\left[\int_0^T s^2(t)dt\right]^2}\int_0^T s^2(t)dt=\frac{N_0}{2\int_0^T s^2(t)dt}$$

4.

(a)

$$s(t)=\begin{cases}A & 0\le t\le T\\ 0 & \text{others}\end{cases}$$

$$\Rightarrow \text{match filter } h(t)=s(T-t)=\begin{cases}A & 0\le t\le T\\ 0 & \text{others}\end{cases}$$

$$SNR_o=\frac{\left[\int_0^T s(t)h(T-t)dt\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt}=\frac{\left[\int_0^T s^2(t)dt\right]^2}{\frac{N_0}{2}\int_0^T s^2(t)dt}$$

$$=\frac{2\int_0^T s^2(t)dt}{N_0}=\frac{2A^2T}{N_0}$$

(b)

$$h(t)=\begin{cases}e^{-\alpha t} & 0\le t\le T\\ 0 & \text{others}\end{cases}$$

$$SNR_o=\frac{\left[\int_0^T s(t)h(T-t)dt\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt}=\frac{\left[\int_0^T Ae^{-\alpha(T-t)}dt\right]^2}{\frac{N_0}{2}\int_0^T e^{-2\alpha(T-t)}dt}$$

$$=\frac{A^2\left(\int_0^T e^{\alpha t}dt\right)^2}{\frac{N_0}{2}\int_0^T e^{2\alpha t}dt}=\frac{4A^2}{\alpha N_0}\frac{e^{\alpha T}-1}{e^{\alpha T}+1}$$

Optimal value of α:

$$\Rightarrow \frac{dSNR_o(\alpha)}{d\alpha}=0 \qquad \Rightarrow \frac{d\left[\frac{e^{\alpha T}-1}{\alpha\left(e^{\alpha T}+1\right)}\right]}{d\alpha}=0$$

$$\Rightarrow \frac{Te^{\alpha T}\left[\alpha\left(e^{\alpha T}+1\right)\right]-\left(e^{\alpha T}-1\right)\left(e^{\alpha T}+1+\alpha Te^{\alpha T}\right)}{\left[\alpha\left(e^{\alpha T}+1\right)\right]^2}=0$$

$$\Rightarrow 2\alpha Te^{\alpha T}-e^{2\alpha T}+1=0$$

$$\Rightarrow \alpha=0$$

(c)

$$h(t)=e^{-\alpha t}\quad t\ge 0$$

$$SNR_o=\frac{\left[\int_0^T s(t)h(T-t)dt\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt}=\frac{\left[\int_0^T Ae^{-\alpha(T-t)}dt\right]^2}{\frac{N_0}{2}\int_{-\infty}^T e^{-2\alpha(T-t)}dt}$$

$$=\frac{4A^2}{\alpha N_0}\frac{e^{2\alpha T}-2e^{\alpha T}+1}{e^{2\alpha T}}$$

$\because h(t)=e^{-\alpha t} \quad t\geq 0\,,\ \alpha>0 \quad \Rightarrow e^{\alpha T}>1$

令 $x=e^{\alpha T}$ ，若 $SNR_o(c)\leq SNR_o(b)$

$\Leftrightarrow \dfrac{(x-1)^2}{x^2}\leq\dfrac{x-1}{x+1}$ 　又 $\because x>1$

$\Leftrightarrow \dfrac{x-1}{x^2}\leq\dfrac{1}{x+1}\Leftrightarrow x^2-1\leq x^2$ 恆成立, 得證!

5.

(a)

$$\rho=\frac{1}{\sqrt{E}\sqrt{E}}\int_0^T\left(\sqrt{\frac{2E}{T}}\right)^2\cos\left[2\pi\left(f_c+\frac{\Delta f}{2}\right)t\right]\cos\left[2\pi\left(f_c-\frac{\Delta f}{2}\right)t\right]dt$$

$$=\frac{1}{T}\int_0^T\left[\cos(4\pi f_c t)+\cos(2\pi\Delta f t)\right]dt$$

$$=\frac{\sin(2\pi\Delta fT)}{2\pi\Delta fT}$$

(b)

$\rho=0\Rightarrow \Delta fT=\dfrac{m}{2}$ 　　$m\in$ integer

minimum value of $\Delta f=\dfrac{1}{2T}$

(c)

To find the minimum vlaue of ρ

$$\Rightarrow\frac{\partial\rho}{\partial\Delta f}=0$$

$$\Rightarrow\frac{\cos(2\pi\Delta fT)2\pi T}{2\pi\Delta fT}-\frac{\sin(2\pi\Delta fT)2\pi T}{(2\pi\Delta fT)^2}=0$$

$$\Rightarrow 2\pi\Delta fT=\tan(2\pi\Delta fT)$$

$$\Rightarrow \Delta f=\frac{0.7151}{T}$$

$$\rho=-0.2172$$

(d)

when $\Delta f=\dfrac{0.7151}{T}$, $\rho=-0.2172\Rightarrow P_e=Q\left[\sqrt{\dfrac{2E(1-\rho)}{2N_0}}\right]=Q\left[\sqrt{\dfrac{1.2172E}{N_0}}\right]$

when $\Delta f=\dfrac{1}{2T}$, $\rho=0\Rightarrow P_e=Q\left[\sqrt{\dfrac{E}{N_0}}\right]$

$$\Rightarrow (P_e)_{(c)}<(P_e)_{(b)}$$