Probability System

- A sample space S
- A class $\mathcal{E}$ of events that are subsets of S
- A probability measure $P(\ )$ for each event A in $\mathcal{E}$, which has the following properties
  - $P(S) = 1$
  - $0 \cdot P(A) \cdot 1$
  - If $A \setminus B = \emptyset$, $P(A + B) = P(A) + P(B)$

- Example
  - $S = \{1, 2, 3, 4, 5, 6\}$
  - $\mathcal{E} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, 2, 3\}, \ldots S\}$
  - $P(A) = |A|/6$
  - $A = \{1, 2\}$, $B = \{3, 4\}$
  - $P(A+B) = 2/3 = P(A) + P(B)$

- Joint events and joint probabilities

  Two experiments

  A_i, $i = 1, 2, \ldots, n$

  B_i, $i = 1, 2, \ldots, m$

  With properties

  $0 \leq P(A_i, B_j) \leq 1$

$$\sum_{j=1}^{m} P(A_i, B_j) = P(A_i)$$

$$\sum_{i=1}^{n} P(A_i, B_j) = P(B_j)$$

- Conditional probabilities

  One experiment, two events

  $$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

- Statistical independence

  $$P(A|B) = P(A)$$

  $$P(A,B) = P(A)P(B)$$

- Examples

  $A = \{1\}$, $B = \{1, 3, 5\}$

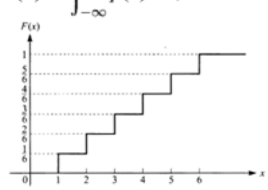
  $P(A|B) = 1/3$.

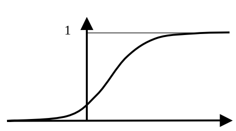
Probability Distribution
One-Dimensional Random Variable

- Cumulative distribution function (CDF)

  $$F(x) = P(X \leqslant x), \qquad -\infty < x < \infty$$

  $$F(x) = \int_{-\infty}^{x} p(u)\, du, \qquad -\infty < x < \infty$$



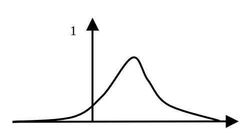


- Probability density function (PDF)

  $$p(x) = \frac{dF(x)}{dx}, \qquad -\infty < x < \infty$$

- Examples

  $$p(x) = \sum_{i=1}^{n} P(X = x_i)\delta(x - x_i)$$



Joint probability distributions

- Cumulative distribution function (CDF)

$$F(x_1, x_2) = P(X_1 \leqslant x_1, X_2 \leqslant x_2)$$
$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} p(u_1, u_2)\ du_1\ du_2$$

- *N*-dimensional CDF

$$F(x_1, x_2, \ldots, x_n) = P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n)$$
$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} p(u_1, u_2, \ldots, u_n)\, du_1\, du_2 \cdots du_n$$

- Independence

$$F(x_1, x_2, \ldots, x_n) = F(x_1)F(x_2) \cdots F(x_n)$$
$$p(x_1, x_2, \ldots, x_n) = p(x_1)p(x_2) \cdots p(x_n)$$

- Probability density function (PDF)

$$p(x_1, x_2) = \frac{\partial^2}{\partial x_1\ \partial x_2} F(x_1, x_2)$$

- Marginal PDF

$$\int_{-\infty}^{\infty} p(x_1, x_2)\ dx_1 = p(x_2)$$
$$\int_{-\infty}^{\infty} p(x_1, x_2)\ dx_2 = p(x_1)$$

- Conditional PDF

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$
$$p(x_1, x_2) = p(x_1|x_2)p(x_2)$$
$$= p(x_2|x_1)p(x_1)$$

Functions of Random Variables

- **One Dimension**

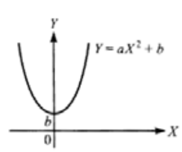
$$Y = g(X)$$
$$p_Y(y) = \sum_{i=1}^{n} \frac{p_X(x_i)}{|g'(x_i)|}$$

$y = g(x_i)$, $i = 1, \ldots,$ n

- **Example**

$$Y = aX^2 + b, \qquad a > 0$$

$$p_Y(y) = \frac{p_X[\sqrt{(y-b)/a}]}{2a\sqrt{[(y-b)/a]}} + \frac{p_X[-\sqrt{(y-b)/a}]}{2a\sqrt{[(y-b)/a]}}$$



- ***N*-Dimension**

$$Y_i = g_i(X_1, X_2, \dots, X_n), \qquad i = 1, 2, \dots, n$$
$$X_i = g_i^{-1}(Y_1, Y_2, \dots, Y_n), \qquad i = 1, 2, \dots, n$$

$$p_Y(y_1, y_2, \dots, y_n) = p_X(x_1 = g_1^{-1}, x_2 = g_2^{-1}, \dots, x_n = g_n^{-1})|J|$$

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}}{\partial y_1} & \frac{\partial g_2^{-1}}{\partial y_2} & \cdots & \frac{\partial g_n^{-1}}{\partial y_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_1^{-1}}{\partial y_n} & \frac{\partial g_2^{-1}}{\partial y_n} & \cdots & \frac{\partial g_1^{-1}}{\partial y_n} \end{vmatrix}$$

- **Examples**

$$Y_i = \sum_{j=1}^{n} a_{ij} X_j, \qquad i = 1, 2, \dots, n$$
$$\mathbf{Y} = \mathbf{AX}$$
$$X_i = \sum_{j=1}^{n} b_{ij} Y_j, \qquad i = 1, 2, \dots, n$$

$$p_Y(y_1, y_2, \dots, y_n) = p_X\left(x_1 = \sum_{i=1}^{n} b_{1j}, y_j, x_2 = \sum_{i=1}^{n} b_{2j} y_j, \dots, x_n = \sum_{i=1}^{n} b_{nj} y_j\right) \frac{1}{|\det \mathbf{A}|}$$

Average/expectation

- Mean

$$E(X) \equiv m_x = \int_{-\infty}^{\infty} x p(x)\, dx$$

- *n*th moment

$$E(X^n) = \int_{-\infty}^{\infty} x^n p(x)\, dx$$

- *n*th central moment

$$E[(X - m_x)^n] = \int_{-\infty}^{\infty} (x - m_x)^n p(x)\, dx$$

- Variance

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2 p(x)\, dx$$
$$\sigma_x^2 = E(X^2) - [E(X)]^2 = E(X^2) - m_x^2$$

- Covariance matrix

$$\mu_{ij} \equiv E[(X_i - m_i)(X_j - m_j)] = E(X_i X_j) - m_i m_j$$

- Example

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
$$E[x^n] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-x^2/2} dx = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (n-1) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$Y = \sigma X + \mu$$
$$p_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$$

$$E[Y] = \mu$$
$$\mathrm{var}[Y] = E[(y-\mu)^2] = \sigma^2$$

Characteristic Functions

- Fourier transform of PDF

$$E(e^{jvX}) \equiv \psi(jv) = \int_{-\infty}^{\infty} e^{jvx} p(x)\, dx$$

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(jv) e^{-jvx} dv$$

- Moment vs. characteristic functions

$$E(X) = m_x = -j \left. \frac{d\psi(jv)}{dv} \right|_{v=0}$$

$$E(X^n) = (-j)^n \left. \frac{d^n \psi(jv)}{dv^n} \right|_{v=0}$$

- Sum of random variables

$$Y = \sum_{i=1}^{n} X_i$$

$$\begin{aligned} \psi_Y(jv) &= E(e^{jvY}) \\ &= E\left[\exp\left(jv \sum_{i=1}^{n} X_i\right)\right] \\ &= E\left[\prod_{i=1}^{n} (e^{jvX_i})\right] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} e^{jvx_i}\right) p(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n \end{aligned}$$

- Independence

$$p(x_1, x_2, \ldots, x_n) = p(x_1) p(x_2) \cdots p(x_n)$$

$$\psi_Y(jv) = \prod_{i=1}^{n} \psi_{X_i}(jv)$$

- Identical independent distributed

$$\psi_Y(jv) = [\psi_X(jv)]^n$$

Characteristic Functions

- N-Dimensional

$$\begin{aligned}\psi(jv_1, jv_2, \ldots, jv_n) &= E\left[\exp\left(j\sum_{i=1}^{n} v_i X_i\right)\right] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(j\sum_{i=1}^{n} v_i x_i\right) p(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n\end{aligned}$$

- Example

$$\psi(jv_1, jv_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(v_1 x_1 + v_2 x_2)} p(x_1, x_2) dx_1\, dx_2$$

$$E(X_1 X_2) = -\left.\frac{\partial^2 \psi(jv_1, jv_2)}{\partial v_1 \partial v_2}\right|_{v_1 = v_2 = 0}$$

Distribution Derived from Gaussian Distribution
X is Gaussian distributed

- Chi-square, χ^2 (Gamma, Γ) distribution n degrees of freedom

- Noncentral χ^2 distribution with n degrees of freedom

- Rayleigh Distribution

- Rice Distribution

- Nakagami *m*-distribution