Gaussian Random Variables

A real random variable X having the probability density function

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

is a normalized *Gaussian random variable* with zero mean and unit variance. Using simple mathematics, the characteristic function is

$$\psi_X(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2 + j\nu x} dx = e^{-\nu^2/2}$$

The probability function itself is an even function and the moments are

$$E\{X^n\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-x^2/2} dx = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (n-1) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

The cummulative distribution function is

$$\begin{aligned} F_X(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx \\ &= 1 - \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right). \end{aligned}$$

where

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt$$

is the complementary error function. *What about error function?*

Consider the new random variable

$$Y = \sigma X + \mu$$

it follows that

$$m_Y = \mu$$

$$\sigma_Y^2 = E[(y-\mu)^2] = \sigma^2$$

The probability density function of Y is

$$p_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$$

The corresponding characteristic function is

$$\psi_Y(\nu) = \exp\left[j\nu\mu - \frac{\nu^2\sigma^2}{2}\right]$$

Multivariate Gaussian Distribution

For N independent identical Gaussian random variables of

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_N \end{bmatrix}$$

with zero mean and unit variance, the joint multivariate Gaussian probability density function is

$$p_{\vec{X}}(x_1, x_2, \cdots, x_N) = \frac{1}{(2\pi)^{N/2}} \exp\left[-\frac{x_1^2 + x_2^2 + \cdots + x_N^2}{2}\right] = \frac{1}{(2\pi)^{N/2}} \exp\left[-\frac{\vec{x}^T \cdot \vec{x}}{2}\right]$$

where $\vec{x} = (x_1, x_2, \cdots, x_N)^T$ and

$$\vec{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix}^T = (x_1, x_2, \cdots, x_N)$$

The characteristic function is

$$\psi_{\vec{X}}(j\vec{\nu}) = e^{-\frac{1}{2}|\vec{\nu}|^2}$$

where $\vec{\nu}^T = (\nu_1, \cdots, \nu_N)$.

If we have new random variables of $\vec{Y} = (Y_1, Y_2, \cdots, Y_N)^T$ given by the transfer of

$$\vec{Y} = \mathcal{A}\vec{X}$$

where $\mathcal{A}$ is an $N \times N$ matrix. It follows that

$$E[\vec{Y}] = 0$$

and the covariance matrix is

$$\mathcal{C}_{\vec{Y}} = \begin{bmatrix} E[Y_1^2] & E[Y_1Y_2] & \cdots & E[Y_1Y_N] \\ E[Y_2Y_1] & E[Y_2^2] & \cdots & E[Y_2Y_N] \\ \cdots & \cdots & \cdots & \cdots \\ E[Y_NY_1] & E[Y_NY_2] & \cdots & E[Y_N^2] \end{bmatrix} = \mathcal{A}\mathcal{A}^T$$

Note that

$$p_{\vec{X}}(\vec{x})d\vec{x} = p_{\vec{Y}}(\vec{y})d\vec{y}$$

It is obvious that

$$d\vec{y} = |\det[\mathcal{A}]|d\vec{x}$$

where $\det[\mathcal{A}]$ is the determinant of the matrix $\mathcal{A}$. Using the relationship $\vec{x} = \mathcal{A}^{-1}\vec{y}$, the probability density function of $\vec{Y}$ is

$$p_{\vec{Y}}(\vec{y}) = \frac{1}{(2\pi)^{N/2}|\det[\mathcal{A}]|} \exp\left[-\frac{1}{2}\vec{y}^T[\mathcal{A}^T\mathcal{A}]^{-1}\vec{y}\right]$$

This probability density function can be rewritten as

$$p_{\vec{Y}}(\vec{y}) = \frac{1}{(2\pi)^{N/2}\sqrt{\det[\mathcal{C}_{\vec{Y}}]}} \exp\left[-\frac{1}{2}\vec{y}^T[\mathcal{C}_{\vec{Y}}]^{-1}\vec{y}\right]$$

where $\det[\mathcal{C}_{\vec{Y}}] = \det[\mathcal{A}]\det[\mathcal{A}^T] = [\det[\mathcal{A}]]^2$.

The characteristic function is

$$\psi_{\vec{Y}}(j\vec{\nu}) = \exp\left(-\frac{1}{2}\vec{\nu}^T\mathcal{C}_{\vec{Y}}\vec{\nu}\right)$$

If the random variables are not zero mean and

$$E[\vec{Y}] = \vec{\mu}_{\vec{Y}},$$

the probability density function is

$$p_{\vec{Y}}(\vec{y}) = \frac{1}{(2\pi)^{N/2}\sqrt{|\det[\mathcal{C}_{\vec{Y}}]|}} \exp\left[-\frac{1}{2}(\vec{y}-\vec{\mu}_{\vec{Y}})^T[\mathcal{C}_{\vec{Y}}]^{-1}(\vec{y}-\vec{\mu}_{\vec{Y}})\right]$$

with characteristic function of

$$\psi_{\vec{Y}}(j\vec{\nu}) = \exp\left(j\vec{\nu}^T\vec{\mu}_{\vec{Y}} - \frac{1}{2}\vec{\nu}^T\mathcal{C}_{\vec{Y}}\vec{\nu}\right)$$

Bivariate Gaussian Distribution

For two zero-mean random variables with the following covariance matrix

$$\mathcal{C}_{\vec{Y}} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where the correlation coefficient of Y_1 and Y_2 is

$$\rho = \frac{E[Y_1Y_2]}{\sigma_1\sigma_2}$$

The determinant of $\mathcal{C}_{\vec{Y}}$ is

$$\det[\mathcal{C}_{\vec{Y}}] = \begin{vmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2(1-\rho^2)$$

and the inverse of $\mathcal{C}_{\vec{Y}}$ is

$$[\mathcal{C}_{\vec{Y}}]^{-1} = \frac{1}{\det[\mathcal{C}_{\vec{Y}}]}\begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} = \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

Therefore, the probability density is

$$p_{(Y_1,Y_2)}(y_1,y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y_1^2}{\sigma_1^2} - \frac{2\rho y_1y_2}{\sigma_1\sigma_2} + \frac{y_2^2}{\sigma_2^2}\right)\right]$$

If the random variables are not zero mean, the probability density function is

$$p_{(Y_1,Y_2)}(y_1,y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{(1-\rho^2)}\left(\frac{(y_1-\mu_1)^2}{2\sigma_1^2} - \frac{\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{2\sigma_2^2}\right)\right]$$

with $E[Y_1] = \mu_1$ and $E[Y_2] = \mu_2$.

Random variables derived from Gaussian random variables

χ^2 **distribution**: If X is a zero-mean Gaussian random variables with variance of σ^2, the distribution of $Y = X^2$ is

$$p_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/(2\sigma^2)}, \qquad y \geq 0$$

with characteristic function of

$$\psi(j\nu) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2} + j\nu x^2} dx = \frac{1}{(1 - 2j\nu\sigma^2)^{1/2}}.$$

When the random variable Y is

$$Y = \sum_{i=1}^{n} X_i^2$$

where X_i are statistically independent and identically distributed zero mean Gaussian random variables with variance of σ^2. The characteristic function of Y is

$$\psi_Y(j\nu) = \frac{1}{(1 - 2j\nu\sigma^2)^{n/2}}.$$

The probability density function is

$$p_Y(y) = \frac{1}{\sigma^n 2^{n/2} \Gamma(n/2)} y^{n/2-1} e^{-y/2\sigma^2}, \qquad y \geq 0$$

This is the chi-square, χ^2 (or Gamma, Γ) distribution with n degrees of freedom.

Noncentral χ^2 distribution: If $Y = X^2$ but X is with a mean of m_x, the characteristic function of Y is

$$\psi_Y(j\nu) = \frac{1}{(1 - 2j\nu\sigma^2)^{1/2}} \exp\left(\frac{j m_x^2 \nu}{1 - 2j\nu\sigma^2} \right)$$

For $Y = \sum_{i=1}^{n} X_i^2$ with X_i has a mean of m_i, the characteristic function of Y is

$$\psi_Y(j\nu) = \frac{1}{(1 - 2j\nu\sigma^2)^{n/2}} \exp\left(\frac{j\nu \sum_{i=1}^{n} m_i^2}{1 - 2j\nu\sigma^2} \right)$$

with characteristic function of

$$p_Y(y) = \frac{1}{2\sigma^2} \left(\frac{y}{s^2} \right) \exp\left(-\frac{s^2 + y}{2\sigma^2} \right) I_{n/2-1}\left(\sqrt{y} \frac{s}{\sigma^2} \right), \qquad y \geq 0$$

with $s^2 = \sum_{i=1}^{n} m_i^2$ as the noncentral χ^2 probability density with n degrees of freedom, where $I_\nu(\cdot)$ is the νth-order modified Bessel function of the first kind.

Rayleigh Distribution: The distribution of $R = \sqrt{X_1^2 + X_2^2}$ for zero-mean Gaussian random varilables of X_1 and X_2 is

$$p_R(r) = \frac{r}{\sigma^2} e^{-r/2\sigma^2}, \qquad r \geq 0$$

with cummulative distribution function of

$$F_R(r) = 1 - e^{-r/2\sigma^2}, \qquad r \geq 0$$

Rice Distribution: The distribution of $R = \sqrt{X_1^2 + X_2^2}$ for Gaussian random varilables of X_1 and X_2 having means of m_1, m_2, respectively, is

$$p_R(r) = \frac{r}{\sigma^2} e^{-(r^2+s^2)/2\sigma^2} I_0\left(\frac{rs}{\sigma^2}\right), \qquad r \geq 0$$

Lognormal Distribution:
The distribution of $R = e^X$ for a Gaussian random variable X with a mean of m and variance of σ^2 is

$$p_R(r) = \frac{1}{\sqrt{2\pi}\sigma r} e^{-(\ln r - m)^2/2\sigma^2}, \qquad r \geq 0$$

Nakagami m-distribution: The probability density of

$$p_R(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m r^{2m-1} e^{-mr^2/\Omega}$$

with $\Omega = E\{R^2\}$ and

$$m = \frac{\Omega^2}{E\{(R^2 - \Omega)^2\}}$$

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Stochastic Processes

Stochastic process is random phenomena that occur as function of time. At any time instant, the value of a stochastic process is a random variable indexed by the paramter t and denoted as a process of $X(t)$. In general, the parameter t is continuous, X can be continuous or quantized. In any given measurement, the random process $X(t)$ gives a sample function of the stochatic function. The set of all possible sample function is equivalent to the stochastic process. The random variables $X_{t_i} = X(t_i)$, $i = 1, 2, \ldots, n$, obtained at time instants of $t_1 > t_2 > \cdots > t_n$ should have a joint PDF of $p_X(x_{t_1}, x_{t_2}, \ldots, x_{t_n})$. If the set of random variables of $X_{t_i+\tau}, i = 1, 2, \ldots, n$, has the same statistically properties as that of $X_{t_i} = X(t_i)$, $i = 1, 2, \ldots, n$, the random process of $X(t)$ is called stationary in strict sense.

For stationary random process, we get

$$p_X(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = p_X(x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau}).$$

Average and Autocorrelation Function: The mean and variance of a stochastic process are

$$m_{X(t_i)} = E\{X(t_i)\} = \int_{-\infty}^{+\infty} x_{t_i} p_X(x_{t_i}) dx_{t_i}$$

and

$$\sigma^2_{X(t_i)} = \int_{-\infty}^{+\infty} (x_{t_i} - m_{X(t_i)})^2 p_X(x_{t_i}) dx_{t_i}$$

For stationary stochastic process, the mean and variance are both independent of time instant and can be denotes as m_X and σ_X^2.

The autocorrelation function of a stochastic process is

$$\phi_X(t_1, t_2) = E\{X_{t_1} X_{t_2}\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_{t_1} x_{t_2} p_X(x_{t_1}, x_{t_2}) dx_{t_1} dx_{t_2}$$

For stationary stochastic process, we get

$$\phi_X(\tau) = \phi_X(t+\tau, t) = \phi_X(t, t-\tau) = \phi_X(\tau, 0)$$

If the stochastic process has the property of $\phi_X(t_1, t_2) = \phi_X(t_1 - t_2)$, it is called a wise-sense stationary stochastic process.

For stationary stochastic process, the autocovariance function is $\mu_X(\tau) = \phi_X(\tau) - m_X^2$.

Cross-correlation function: For two stochastic processes of $X(t)$ and $Y(t)$, the cross-correlation function is

$$\phi_{XY}(t_1, t_2) = E\{X_{t_1} Y_{t_2}\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_{t_1} y_{t_2} p_{XY}(x_{t_1}, y_{t_2}) dx_{t_1} dy_{t_2}$$

The cross-covariance function is $\mu_{XY}(t_1, t_2) = \phi_{XY}(t_1, t_2) - m_{X(t_1)} m_{X(t_2)}$. For stationary stochastic processes, the cross-correlation function is $\phi_{XY}(t_1, t_2) = \phi_{XY}(t_1 - t_2)$.

If the stochastic processes of $X(t)$ and $Y(t)$ are uncorrelated, we get

$$\mu_{XY}(t_1, t_2) = 0 \qquad \text{and} \qquad \phi_{XY}(t_1, t_2) = m_{X(t_1)} m_{X(t_2)}$$

Complex-valued stochastic process: A complex-valued stochastic process $Z(t)$ is defined as $Z(t) = X(t) + jY(t)$ where $X(t)$ and $Y(t)$ are stochastic processes. The autocorrelation function of $Z(t)$ is defined as

$$\begin{aligned}
\phi_Z(t_1, t_2) &= \frac{1}{2} E\{Z(t_1) Z^*(t_2)\} \\
&= \frac{1}{2} E\{(X(t_1) + jY(t_1))(X(t_2) - jY(t_2))\} \\
&= \frac{1}{2}\left[\phi_X(t_1, t_2) + \phi_Y(t_1, t_2) + j(\phi_{YX}(t_1, t_2) - \phi_{XY}(t_1, t_2))\right].
\end{aligned}$$

Power Spectral Density

For wide-sense stationary stochastic process, the distribution of power with frequency is givn by

$$\Phi_X(f) = \int_{-\infty}^{+\infty} \phi_X(\tau) e^{-j2\pi f\tau} d\tau$$

with inverse Fourier transform relationship as

$$\phi_X(\tau) = \int_{-\infty}^{+\infty} \Phi_X(f) e^{j2\pi f\tau} df$$

We can get

$$\int_{-\infty}^{+\infty} \Phi_X(f) df = \phi_X(0) = E\{X_t^2\} \geq 0$$

Because $\Phi(f)$ is the distribution of power as a function of frequency, it is called the power density spectrum of the stochastic process.

Wiener-Khintchine Theorem:

$$\Phi_X(f) = \lim_{T\to\infty} \frac{1}{T} E\left\{\left|\int_{-T/2}^{T/2} X(t) e^{-j2\pi f t} dt\right|^2\right\} = \int_{-\infty}^{+\infty} \phi_X(\tau) e^{-j2\pi f\tau} d\tau$$

We can also define the cross-power density spectrum as

$$\Phi_{XY}(f) = \int_{-\infty}^{+\infty} \phi_{XY}(\tau) e^{-j2\pi f\tau} d\tau$$

While $\Phi_X(f)$ is real number, $\Phi_{XY}(f)$ may be a complex number with the following relationship

$$\Phi_{XY}^*(f) = \Phi_{YX}(f) = \Phi_{XY}(-f)$$

Linear Systems: When a stochastic process is passed through a linear time-invariant system with impulse response of $h(t)$, the output is

$$Y(t) = \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau$$

If $X(t)$ is stationary, $Y(t)$ is also stationary with a mean value of $Y(t)$ of

$$m_Y = E\{Y(t)\} = \int_{-\infty}^{+\infty} h(\tau) E\{X(t-\tau)\} d\tau = m_X \int_{-\infty}^{+\infty} h(\tau) d\tau = m_X H(0)$$

Without going into detial, the power density spectrum of the output is

$$\Phi_Y(f) = \Phi_X(f) |H(f)|^2$$

and

$$\Phi_{YX}(f) = \Phi_X(f) H(f)$$

Discrete Stochastic Process: An ensemble of sample sequence $\{x(n)\}$ can be described as a discrete-time stochastic process $X(n)$ with autocorrelation sequence of

$$\phi_X(n,k) = \frac{1}{2} E\{X_n X_k^*\} = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_n x_k^* p_X(x_n, x_k) dx_n dx_k$$

For stationary discrete-time stochastic process, $\phi_X(n,k) = \phi_X(n-k)$ and the power density spectrum is

$$\Phi_X(f) = \sum_{n=-\infty}^{+\infty} \phi_X(n) e^{-j2\pi fn},$$

If the discrete pass a linear filter of $h(n)$ with frequency response of

$$H(f) = \sum_{n=-\infty}^{+\infty} h(n) e^{-j2\pi fn},$$

its output has a power density spectrum of

$$\Phi_Y(f) = \Phi_X(f) |H(f)|^2$$

Cyclostationary Processes: For digital signals, the corresponding stochastic process is

$$X(t) = \sum_{n=-\infty}^{+\infty} a_n g(t - nT)$$

where $\{a_n\}$ is a discrete-time sequence of complex random variables and the pulse of $g(t)$ is deterministic.

The autocorrelation function of $X(t)$ is

$$\begin{aligned}\phi_X(t+\tau,t) &= \frac{1}{2}E\{X(t+\tau)X^*(t)\}\\ &= \frac{1}{2}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}E\{a_n a^*\}g^*(t-nT)g(t+\tau-mT)\\ &= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\phi_A(m-n)g^*(t-nT)g(t+\tau-mT)\end{aligned}$$

We get

$$\phi_X(t+\tau+T,t+T)=\phi_X(t+\tau,t)$$

as a periodic function of t with a period of T.

Because the autocorrelation function is a periodic function, we can defined the time-average autocorrelation function as

$$\bar{\phi}_X(\tau)=\frac{1}{T}\int_{-T/2}^{T/2}\phi_X(t+\tau,t)dt$$

The power density spectrum is

$$\Phi_X(f)=\int_{-\infty}^{+\infty}\bar{\phi}_X(\tau)e^{-j2\pi f\tau}d\tau.$$

Bandpass signal and bandpass stochastic process.

For a bandpass signal around the frequency of $\pm f_c$, it can always be represented by two low-pass (or baseband) signal $x(t)$ $y(t)$ as

$$\begin{aligned}s(t) &= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t\\ &= \Re\left\{[x(t)+jy(t)]e^{j2\pi f_c t}\right\}\\ &= \Re\left[s_l(t)e^{j2\pi f_c t}\right]\end{aligned}$$

where $s_l(t)=x(t)+jy(t)=e^{-j2\pi f_c t}[s(t)+j\hat{s}(t)]$ with $\hat{s}(t)$ as the Hilbert transform of $s(t)$.

In the polar representaton, $s_l(t)=a(t)e^{j\theta(t)}$, where

$$\begin{aligned}a(t) &= \sqrt{x^2(t)+y^2(t)}\\ \theta(t) &= \tan^{-1}\frac{y(t)}{x(t)}\end{aligned}$$

and

$$s(t)=a(t)\cos[2\pi f_c t+\theta(t)]$$

We can also find that

$$S(f) = \frac{1}{2}\left[S_l(f-f_c) + S_l^*(-f-f_c)\right]$$

Similarly, the impulse response of a bandpass system can also be represented as $h(t) = 2\Re\{h_l(t)e^{2\pi f_c t}\}$ such that the output signal is $R_l(f) = H_l(f)S_l(f)$.

A bandpass stochastic process can also represented as

$$\begin{aligned} N(t) &= A(t)\cos[2\pi f_c t + \Theta_n(t)] \\ &= X(t)\cos 2\pi f_c t - Y(t)\sin 2\pi f_c t \\ &= \Re\left[Z(t)e^{j2\pi f_c t}\right] \end{aligned}$$

The autocorrelation function of $N(t)$ is

$$\begin{aligned} E\{N(t)N(t+\tau)\} &= \frac{1}{2}[\phi_X(\tau)+\phi_Y(\tau)]\cos 2\pi f_c\tau + \frac{1}{2}[\phi_X(\tau)-\phi_Y(\tau)]\cos 2\pi f_c(2t+\tau) \\ &\quad -\frac{1}{2}[\phi_{YX}(\tau)-\phi_{XY}(\tau)]\sin 2\pi f_c\tau - \frac{1}{2}[\phi_{YX}(\tau)+\phi_{XY}(\tau)]\sin 2\pi f_c(2t+\tau) \end{aligned}$$

For stationary narrowband stochastic process,

$$\phi_X(\tau) = \phi_Y(\tau), \quad \text{and} \quad \phi_{YX}(\tau) = -\phi_{XY}(\tau)$$

and

$$\phi_N(\tau) = \phi_X(\tau)\cos 2\pi f_c\tau - \phi_{YX}(\tau)\sin 2\pi f_c\tau$$

The baseband representation of $Z(t) = X(t) + jY(t)$ autocorrelation function of

$$\phi_Z(\tau) = \phi_X(\tau) + j\phi_{YX}(\tau)$$

and

$$\phi_N(\tau) = \Re\left[\phi_Z(\tau)e^{j2\pi f_c\tau}\right]$$

Previously, we have $\phi_{YX}(\tau) = \phi_{XY}(-\tau)$, we get $\phi_{XY}(\tau) = -\phi_{XY}(-\tau)$.

Bandpass White Noise: Bandpass white noise has a lowpass power density spectrum of

$$\Phi_Z(f) = \begin{cases} N_0 & |f| \le \frac{B}{2} \\ 0 & |f| > \frac{B}{2} \end{cases}$$

with

$$\phi_Z(\tau) = N_0\frac{\sin(\pi B\tau)}{\pi\tau}$$

and

$$\phi_Z(\tau) = \phi_X(\tau) = \phi_Y(\tau)$$

Gaussian Random Process

A random process is said to be a *Gaussian random process* if, for every finite set of time instant t_n, the random variables $\vec{X} = (X(t_1), X(t_2), \cdots, X(t_n), \cdots, X(t_N))^T$ have a joint Gaussian probability density function.

The probability density function of $\vec{X} = (X(t_1), X(t_2), \cdots, X(t_n), \cdots, X(t_N))^T$ is given by

$$p_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{N/2}\sqrt{|\det[\mathcal{C}_{\vec{X}}]|}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu}_{\vec{X}})^T[\mathcal{C}_{\vec{X}}]^{-1}(\vec{x}-\vec{\mu}_{\vec{X}})\right]$$

where

$$\vec{\mu}_{\vec{X}} = \begin{bmatrix} E[X(t_1)] \\ E[X(t_2)] \\ \cdots \\ E[X(t_n)] \end{bmatrix} = \begin{bmatrix} \mu_{X(t_1)} \\ \mu_{X(t_2)} \\ \cdots \\ \mu_{X(t_n)} \end{bmatrix}$$

and

$$\begin{aligned} \mathcal{C}_{\vec{X}} &= \begin{bmatrix} E[X(t_1)^2] & E[X(t_1)X(t_2)] & \cdots & E[X(t_1)X(t_N)] \\ E[X(t_2)X(t_1)] & E[X(t_2)^2] & \cdots & E[X(t_2)X(t_N)] \\ \cdots & \cdots & \cdots & \cdots \\ E[X(t_N)X(t_1)] & E[X(t_N)X(t_2)] & \cdots & E[X(t_N)^2] \end{bmatrix} - \vec{\mu}_{\vec{X}}\vec{\mu}_{\vec{X}}^T \\ &= \begin{bmatrix} R_X(t_1,t_1) - \mu^2_{X(t_1)} & R_X(t_1,t_2) - \mu_{X(t_1)}\mu_{X(t_2)} & \cdots & R_X(t_1,t_N) - \mu_{X(t_1)}\mu_{X(t_N)} \\ R_X(t_2,t_1) - \mu_{X(t_2)}\mu_{X(t_1)} & R_X(t_2,t_2) - \mu^2_{X(t_2)} & \cdots & R_X(t_2,t_N) - \mu_{X(t_2)}\mu_{X(t_N)} \\ \cdots & \cdots & \cdots & \cdots \\ R_X(t_N,t_1) - \mu_{X(t_N)}\mu_{X(t_1)} & R_X(t_N,t_2) - \mu_{X(t_N)}\mu_{X(t_2)} & \cdots & R_X(t_N,t_N) - \mu^2_{X(t_N)} \end{bmatrix} \end{aligned}$$

Although the covariance matrix looks complicated, it is a standard correlation relationship in a random process, i.e., $C_X(t_i,t_j) = R_X(t_i,t_j) - \mu_{X(t_i)}\mu_{X(t_j)}$.

Properties of a Gaussian Process

1. *If a Gaussian process $X(t)$ is applied to a linear time-invariant system, then the output random process is also Gaussian process.*
2. *A Gaussian process is uniquely specified by its means and autocovariance functions.*
3. *If a Gaussian process is wide-sense stationary, then the process is also stationary in the strict sense.*
4. *Uncorrelated random samples are also statistically independent.*