Non-Coherent Detection

Binary Signal with Random Phase

- Two signals

$$s_m(t) = \mathrm{Re}[s_{lm}(t)e^{j2\pi f_c t}], \qquad m = 1, 2, \qquad 0 \leqslant t \leqslant T$$

- Received signal

$$r(t) = \mathrm{Re}\{[s_{lm}(t)e^{j\phi} + z(t)]e^{j2\pi f_c t}\}$$

with low-pass representation

$$r_l(t) = s_{lm}(t)e^{j\phi} + z(t), \qquad 0 \leqslant t \leqslant T$$

- Output of matched filter

$$r_m = r_{mc} + jr_{ms}, \qquad m = 1, 2$$

- Examples

$$r_1 = 2\mathcal{E}\cos\phi + n_{1c} + j(2\mathcal{E}\sin\phi + n_{1s})$$
$$r_2 = 2\mathcal{E}|\rho|\cos(\phi - \alpha_0) + n_{2c} + j[2\mathcal{E}|\rho|\sin(\phi - \alpha_0) + n_{2s}]$$

$$\rho = \frac{1}{2\mathcal{E}}\int_0^T s_{l1}^*(t)s_{l2}(t)\,dt$$

Likelihood Ratio

- *A posteriori* probability

$$P(\mathbf{s}_m|\mathbf{r}) = \frac{p(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)}{p(\mathbf{r})}, \qquad m = 1, 2$$

- MAP decision rule

$$P(\mathbf{s}_1|\mathbf{r}) \underset{s_2}{\overset{s_1}{\gtrless}} P(\mathbf{s}_2|\mathbf{r})$$

or

$$\frac{p(\mathbf{r}|\mathbf{s}_1)}{p(\mathbf{r}|\mathbf{s}_2)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)}$$

- Likelihood ratio

$$\Lambda(\mathbf{r}) = \frac{p(\mathbf{r}|\mathbf{s}_1)}{p(\mathbf{r}|\mathbf{s}_2)}$$

PDF's

- p.d.f. for $\rho = 0$

$$\begin{aligned} r_1 &= r_{1c} + jr_{1s} \\ &= 2\mathcal{E}\cos\phi + n_{1c} + j(2\mathcal{E}\sin\phi + n_{1s}) \\ r_2 &= r_{2c} + jr_{2s} \\ &= n_{2c} + jn_{2s} \end{aligned}$$

$$p(r_{1c}, r_{1s}|\mathbf{s}_1, \phi) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(r_{1c} - 2\mathcal{E}\cos\phi)^2 + (r_{1s} - 2\mathcal{E}\sin\phi)^2}{2\sigma^2}\right]$$

$$p(r_{2c}, r_{2s}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_{2c}^2 + r_{2s}^2}{2\sigma^2}\right)$$

$$p(\mathbf{r}|\mathbf{s}_1, \phi) = p(r_{1c}, r_{1s}|\mathbf{s}_1, \phi)\, p(r_{2c}, r_{2s})$$

$$p(\mathbf{r}|\mathbf{s}_1) = \int_0^{2\pi} p(\mathbf{r}|\mathbf{s}_1, \phi)p(\phi)\, d\phi = p(r_{2c}, r_{2s}) \int_0^{2\pi} p(r_{1c}, r_{1s}|\mathbf{s}_1, \phi)p(\phi)\, d\phi$$

PDF's (cont)

$$\frac{1}{2\pi}\int_0^{2\pi} p(r_{1c}, r_{1s}|\mathbf{s}_1, \phi)\, d\phi$$

$$= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r_{1c}^2 + r_{1c}^2 + 4\mathcal{E}^2}{2\sigma^2}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\mathcal{E}(r_{1c}\cos\phi + r_{1s}\sin\phi}{\sigma^2}\right]\mathrm{d}\phi$$

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\mathcal{E}(r_{1c}\cos\phi + r_{1s}\sin\phi)}{\sigma^2}\right]d\phi = I_0\left(\frac{2\mathcal{E}\sqrt{r_{1c}^2 + r_{1s}^2}}{\sigma^2}\right)$$

$$p(r_{1c}, r_{1s}|\mathbf{s}_1) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r_{1c}^2 + r_{1c}^2 + 4\mathcal{E}^2}{2\sigma^2}\right) I_0\left(\frac{2\mathcal{E}\sqrt{r_{1c}^2 + r_{1s}^2}}{\sigma^2}\right)$$

$$p(r_{2c}, r_{2s}|\mathbf{s}_2) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r_{2c}^2 + r_{2s}^2 + 4\mathcal{E}^2}{2\sigma^2}\right) I_0\left(\frac{2\mathcal{E}\sqrt{r_{2c}^2 + r_{2s}^2}}{\sigma^2}\right)$$

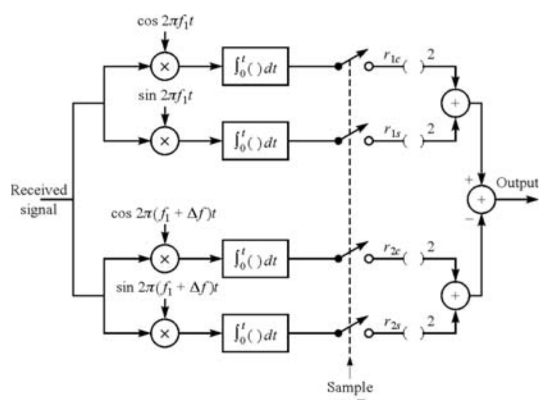
Binary Envelope Detector

- Likelihood Ratio

$$\Lambda(\mathbf{r}) = \frac{I_0(2\mathcal{E}\sqrt{r_{1c}^2 + r_{1s}^2}/\sigma^2)}{I_0(2\mathcal{E}\sqrt{r_{2c}^2 + r_{2s}^2}/\sigma^2)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)}$$

- Equal probable envelope detector

$$\sqrt{r_{1c}^2 + r_{1s}^2} \underset{s_2}{\overset{s_1}{\gtrless}} \sqrt{r_{2c}^2 + r_{2s}^2}$$



M-ary Orthogonal Signals

- M signals

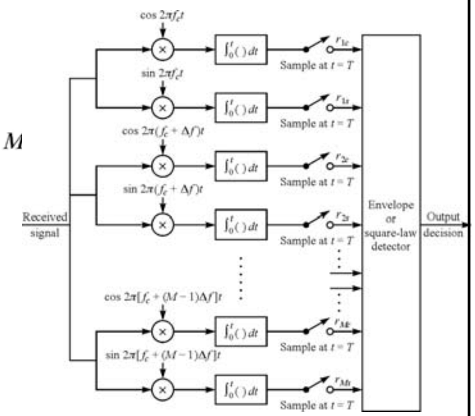
$$s_m(t) = \mathrm{Re}[s_{lm}(t)e^{j2\pi f_c t}]$$

$$m = 1, 2, \ldots, M, \qquad 0 \leqslant t \leqslant T$$

$$r_m = r_{mc} + jr_{ms} = \int_0^T r_l(t)s_{lm}^*(t)\,dt$$

- Selected the largest among

$$|r_m| = \sqrt{r_{mc}^2 + r_{ms}^2}, \qquad m = 1, 2, \ldots, M$$



Error probability

- M envelopes

$$|r_m| = \sqrt{r_{mc}^2 + r_{ms}^2} \qquad m = 1, 2, \ldots, M$$

$$r_{1c} = \sqrt{\mathcal{E}_s}\cos\phi_1 + n_{1c} \qquad r_{mc} = n_{mc}, \qquad r_{ms} = n_{ms}$$

$$r_{1s} = \sqrt{\mathcal{E}_s}\sin\phi_1 + n_{1s} \qquad m = 2, 3, \ldots, M$$

- PDF as

$$p_{\mathbf{r}_1}(r_{1c}, r_{1s}) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r_{1c}^2 + r_{1s}^2 + \mathcal{E}_s}{2\sigma^2}\right) I_0\left(\frac{\sqrt{\mathcal{E}_s(r_{1c}^2 + r_{1s}^2)}}{\sigma^2}\right)$$

$$p_{\mathbf{r}_m}(r_{mc}, r_{ms}) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r_{mc}^2 + r_{ms}^2}{2\sigma^2}\right), \qquad m = 2, 3, \ldots, M$$

$$p(R_1, \Theta_1) = \frac{R_1}{2\pi}\exp\left[-\frac{1}{2}\left(R_1^2 + 2\frac{\mathcal{E}_s}{N_0}\right)\right] I_0\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}R_1\right) \qquad R_m = \frac{\sqrt{r_{mc}^2 + r_{ms}^2}}{\sigma}$$

$$p(R_m, \Theta_m) = \frac{R_m}{2\pi}\exp(-\tfrac{1}{2}R_m^2), \qquad m = 2, 3, \ldots, M \qquad \Theta_m = \tan^{-1}\frac{r_{ms}}{r_{mc}}$$

Error probability (cont.)

- Correct prob.

$$P_c = P(R_2 < R_1, R_3 < R_1, \ldots, R_M < R_1)$$
$$= \int_0^\infty P(R_2 < R_1, R_3 < R_1, \ldots, R_M < R_1 | R_1 = x) p_{R_1}(x)\, dx$$
$$= \int_0^\infty [P(R_2 < R_1 | R_1 = x)]^{M-1} p_{R_1}(x)\, dx$$

- Because

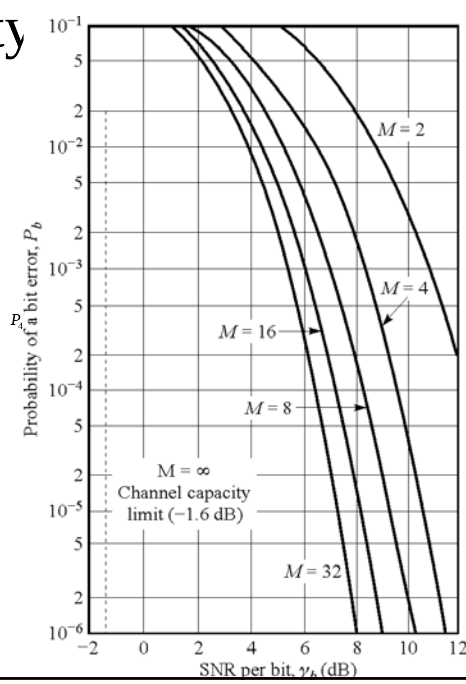
$$P(R_2 < R_1 | R_1 = x) = \int_0^x p_{R_2}(r_2)\, dr_2 = 1 - e^{-x^2/2}$$
$$(1 - e^{-x^2/2})^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-nx^2/2}$$

we get
$$P_c = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \frac{1}{n+1} \exp\left[\frac{-n\mathcal{E}_s}{(n+1)N_0}\right]$$
$$P_M = 1 - P_c = \sum_{n=1}^{M-1} (-1)^{n+1} \binom{M-1}{n} \frac{1}{n+1} \exp\left[-\frac{nk\mathcal{E}_b}{(n+1)N_0}\right]$$

Error probability of M-ary FSK

- $M = 2$

$$P_2 = \tfrac{1}{2} e^{-\mathcal{E}_b/2N_0}$$



DPSK vs. FSK

- FSK

$$P_b = P(R_2 > R_1) = P(R_2^2 > R_1^2) = P(R_2^2 - R_1^2 > 0)$$

$$P_2 = \tfrac{1}{2}e^{-\mathcal{E}_b/2N_0}$$

- DPSK (pg. 274)

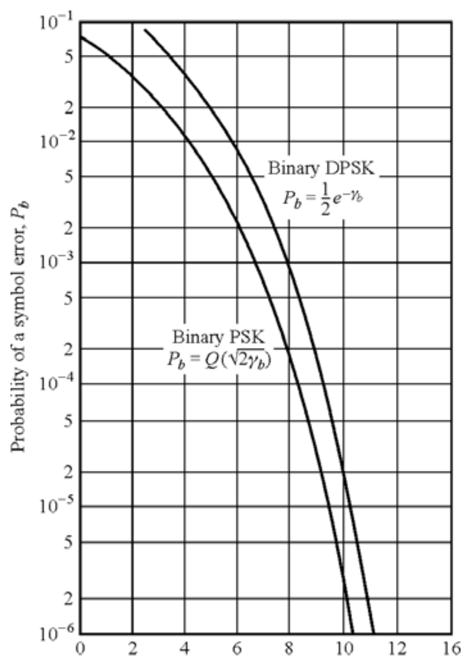
$$P_b = P(\mathrm{Re}(r_k r_{k-1}^*) < 0)$$

$$\mathrm{Re}(r_k r_{k-1}^*) = \tfrac{1}{2}(r_k r_{k-1}^* + r_k^* r_{k-1})$$

$$= \frac{1}{2}\left\{\left|\frac{r_k + r_{k-1}}{2}\right|^2 - \left|\frac{r_k - r_{k-1}}{2}\right|^2\right\}$$

$$P_b = \tfrac{1}{2}e^{-E/N_0}$$

DPSK vs. PSK



Correlated binary signals

- PDF's

$$p(R_m) = \begin{cases} \dfrac{R_m}{2\mathcal{E}_s N_0} \exp\left(-\dfrac{R_m^2 + \beta_m^2}{4\mathcal{E}N_0}\right) I_0\left(\dfrac{\beta_m R_m}{2\mathcal{E}N_0}\right) & (R_m > 0) \\ 0 & (R_m < 0) \end{cases}$$

$$m = 1, 2. \quad \beta_1 = 2\mathcal{E} \text{ and } \beta_2 = 2\mathcal{E}|\rho|.$$

$$P_b = P(R_2 > R_1) = \int_0^\infty \int_{x_1}^\infty p(x_1, x_2)\, dx_1\, dx_2$$

$$P_b = Q_1(a, b) - \tfrac{1}{2} e^{-(a^2+b^2)/2} I_0(ab)$$

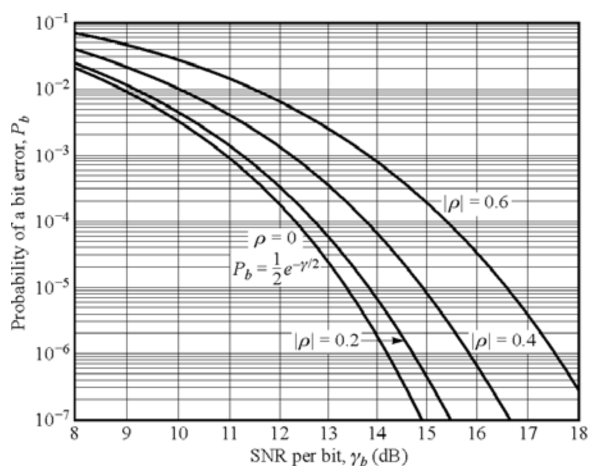
$$a = \sqrt{\frac{\mathcal{E}_b}{2N_0}\left(1 - \sqrt{1 - |\rho|^2}\right)}$$

$$b = \sqrt{\frac{\mathcal{E}_b}{2N_0}\left(1 + \sqrt{1 - |\rho|^2}\right)}$$

$$Q_1(a,b) = \int_b^\infty x e^{-(x^2+a^2)/2} I_0(ax)\, dx$$

$$= e^{-(a^2+b^2)/2} \sum_{k=0}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab), \qquad b > a > 0$$

Correlated binary signals



DQPSK

$$\rho = \tfrac{1}{2}\left(1 + e^{j\pi/2}\right) = \tfrac{1}{2}\left(1 + j\right)$$

$$|\rho| = \tfrac{1}{\sqrt{2}}$$

$$a = \sqrt{2\gamma_b(1 - \sqrt{\tfrac{1}{2}})}$$

$$b = \sqrt{2\gamma_b(1 + \sqrt{\tfrac{1}{2}})}$$