# Power Spectra of Digital Signals

# Linearly Modulated Signals

- The signal

$$v(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$$

- The autocorrelation function

$$\begin{aligned}\phi_{vv}(t+\tau; t) &= \tfrac{1}{2}E[v^*(t)v(t+\tau)] \\ &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[I_n^* I_m] g^*(t-nT)g(t+\tau-mT) \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \phi_{ii}(m-n) g^*(t-nT)g(t+\tau-mT) \\ &= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} g^*(t-nT)g(t+\tau-nT-mT)\end{aligned}$$

$$\phi_{ii}(m) = \tfrac{1}{2}E[I_n^* I_{n+m}]$$

- Cyclostationary

$$\phi_{vv}(t+T+\tau; t+T) = \phi_{vv}(t+\tau; t)$$

- Average autocorrelation function

$$\bar{\phi}_{vv}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \phi_{vv}(t+\tau; t)\, dt$$
$$= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} g^*(t-nT)g(t+\tau-nT-mT)\, dt$$
$$= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2-nT}^{T/2-nT} g^*(t)g(t+\tau-mT)\, dt$$
$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} \phi_{ii}(m)\phi_{gg}(\tau-mT)$$

$$\phi_{gg}(\tau) = \int_{-\infty}^{\infty} g^*(t)g(t+\tau)\, dt$$

- Power Spectrum

$$\Phi_{vv}(f) = \frac{1}{T}|G(f)|^2\Phi_{ii}(f)$$

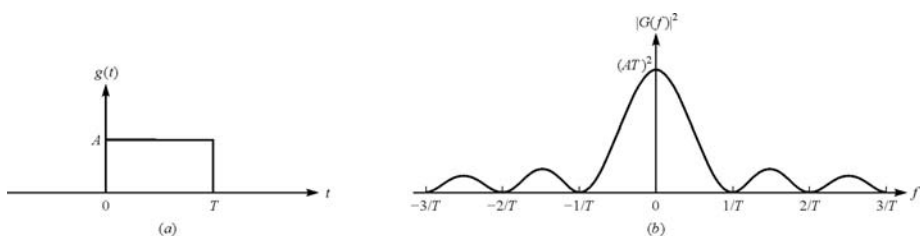
$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m)e^{-j2\pi fmT}$$

- Example

$$\phi_{ii}(m) = \begin{cases} \sigma_i^2 + \mu_i^2 & (m=0) \\ \mu_i^2 & (m \neq 0) \end{cases}$$

$$\Phi_{vv}(f) = \frac{\sigma_i^2}{T}|G(f)|^2 + \frac{\mu_i^2}{T^2}\sum_{m=-\infty}^{\infty}\left|G\left(\frac{m}{T}\right)\right|^2\delta\left(f-\frac{m}{T}\right)$$

## Rectangular Pulse




$$G(f) = AT\frac{\sin \pi fT}{\pi fT}e^{-j\pi fT}$$

$$\Phi_{vv}(f) = \sigma_i^2 A^2 T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 + A^2\mu_i^2\delta(f)$$

# Correlated Pulses
# Duobinary Signal

$$I_n = b_n + b_{n-1}$$

$$\phi_{ii}(m) = E(I_n I_{n+m}) = \begin{cases} 2 & (m=0) \\ 1 & (m=\pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\Phi_{ii}(f) = 2(1+\cos 2\pi f T) = 4\cos^2 \pi f T$$

$$\Phi_{vv}(f) = \frac{4}{T}|G(f)|^2 \cos^2 \pi f T$$

# CPFSK & CPM

- Outline the method
- CPM Signal

$$s(t;\mathbf{I}) = A\cos[2\pi f_c t + \phi(t;\mathbf{I})] \qquad \phi(t;\mathbf{I}) = 2\pi \sum_{k=-\infty}^{n} I_k h_k q(t-kT)$$

- Low-pass equivalent

$$v(t) = e^{j\phi(t;\mathbf{I})}$$

- Autocorrelation function

$$\phi_{vv}(t+\tau;t) = \tfrac{1}{2}E\left[\exp\left(j2\pi h \sum_{k=-\infty}^{\infty} I_k[q(t+\tau-kT) - q(t-kT)]\right)\right]$$

$$= \tfrac{1}{2}E\left(\prod_{k=-\infty}^{\infty} \exp\{j2\pi h I_k[q(t+\tau-kT) - q(t-kT)]\}\right)$$

$$= \tfrac{1}{2}\prod_{k=-\infty}^{\infty}\left(\sum_{\substack{n=-(M-1)\\ n\ \text{odd}}}^{M-1} P_n \exp\{j2\pi h n[q(t+\tau-kT) - q(t-kT)]\}\right)$$

$$P_n = P(I_k = n), \qquad n = \pm 1, \pm 3, \ldots, \pm(M-1)$$

$$= \frac{1}{2} \prod_{k=-\infty}^{\infty} \left( \sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} P_n \exp\{j2\pi hn[q(t+\tau-kT) - q(t-kT)]\} \right)$$

- When we take the integration of

$$\bar{\phi}_{vv}(\tau) = \frac{1}{T} \int_0^T \phi_{vv}(t+\tau; t)\, dt$$

$t$ is between 0 and $T$ and only $1 - L < k < 1+\tau/T$ is meaningful for positive $\tau$.

- For $\tau > LT$, $\tau = \xi + mT$, $q(t-kT) = 0$ for $k > 1$ or $k - m > 1 - L$. $q(t+\tau-kT) = 1/2$ for $1 - L < k < 0$.

$$\bar{\phi}_{vv}(\xi + mT) = [\psi(jh)]^{m-L} \lambda(\xi), \qquad m \geqslant L, \qquad 0 \leqslant \xi < T$$

$$\lambda(\xi) = \frac{1}{2T} \int_0^T \prod_{k=1-L}^{0} \left( \sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} P_n \exp\{j2\pi hn[\tfrac{1}{2} - q(t-kT)]\} \right)$$

$$\times \prod_{k=1-L}^{1} \left( \sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} P_n \exp[j2\pi hnq(t+\xi-kT)] \right) dt, \qquad m \geqslant L$$

$$\psi(jh) = E(e^{j\pi h I_n}) = \sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} P_n e^{j\pi hn}$$

$$\Phi_{vv}(f) = 2\,\mathrm{Re}\left[ \int_0^{LT} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau + \frac{1}{1 - \psi(jh) e^{-j2\pi fT}} \int_{LT}^{(L+1)T} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau \right]$$

CPFSK:

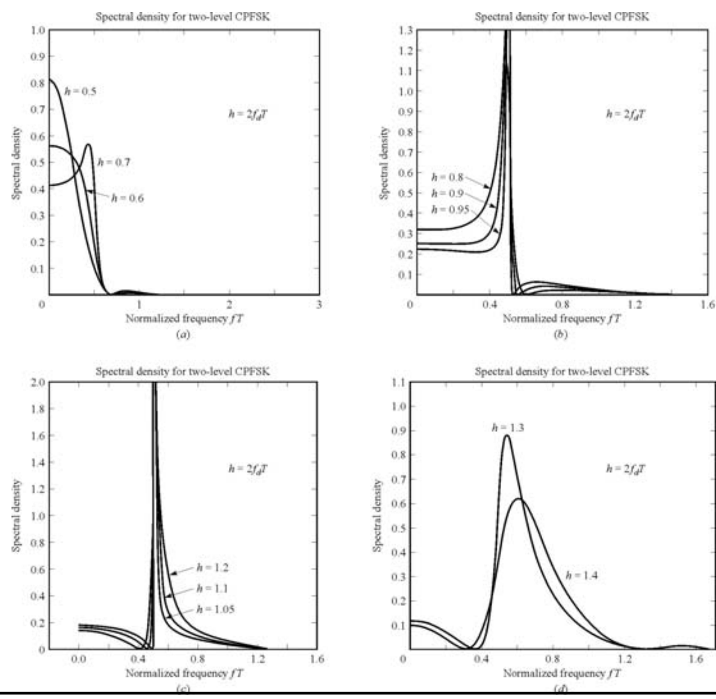
$$\Phi_{vv}(f) = T \left[ \frac{1}{M} \sum_{n=1}^{M} A_n^2(f) + \frac{2}{M^2} \sum_{n=1}^{M} \sum_{m=1}^{M} B_{nm}(f) A_n(f) A_m(f) \right]$$

$$A_n(f) = \frac{\sin \pi[fT - \frac{1}{2}(2n-1-M)h]}{\pi[fT - \frac{1}{2}(2n-1-M)h]}$$

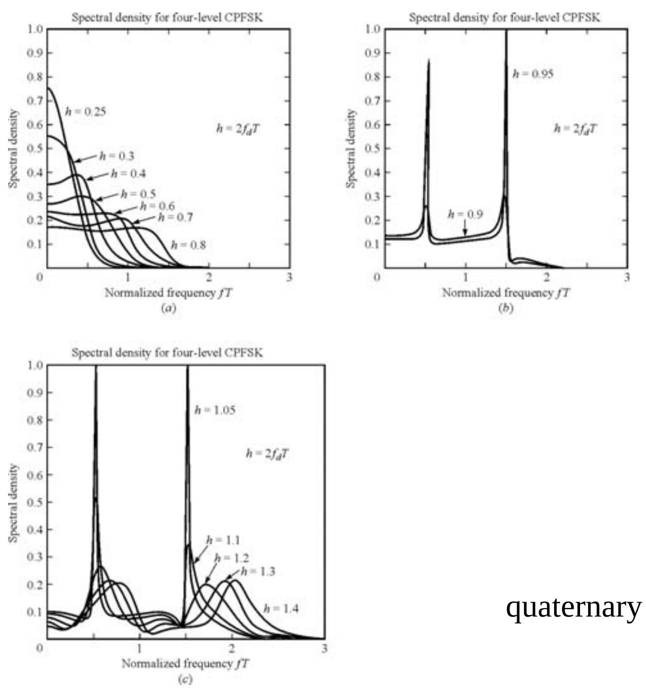
$$B_{nm}(f) = \frac{\cos(2\pi fT - \alpha_{nm}) - \psi \cos \alpha_{nm}}{1 + \psi^2 - 2\psi \cos 2\pi fT}$$

$$\alpha_{nm} = \pi h(m + n - 1 - M)$$

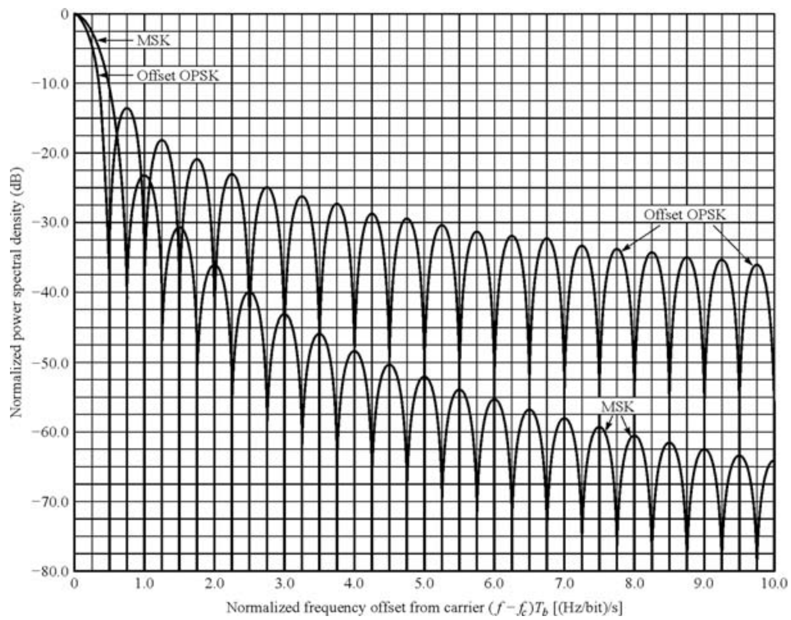
$$\psi \equiv \psi(jh) = \frac{\sin M\pi h}{M \sin \pi h}$$



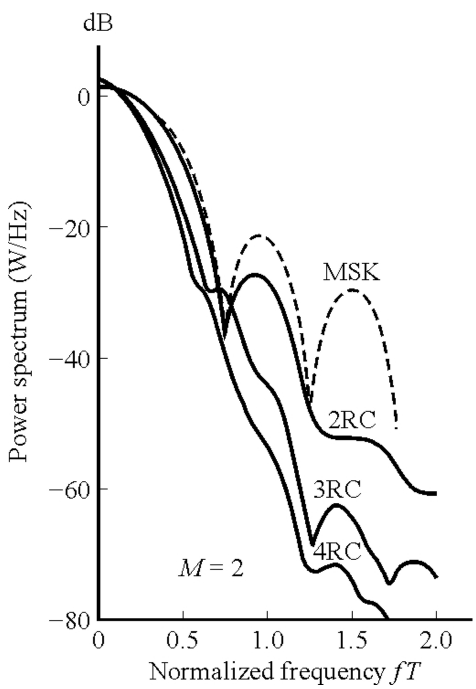

Binary CPFSK




quaternary CPFSK




MSK and offset QPSK




binary CPM with $h$ = ½ and different pulse shapes