[A]Phase Distribution of Gaussian Random Variables

With only additive Gaussian noise, the baseband representation of the received signal is

$$s(t) = A + n(t), \tag{39}$$

where A is a real-number representing the transmitted signal and $n(t)$ is complex Gaussian noise with $n(t) = n_1(t) + jn_2(t)$. For a noise variance of $E\{n_1^2(t)\} = E\{n_2^2(t)\} = \sigma_n^2$, the p.d.f. of the received signal is

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_n^2} \exp\left[-\frac{(x_1 - A)^2 + x_2^2}{2\sigma_n^2}\right], \tag{40}$$

with characteristic function of

$$\Psi(\omega) = \exp\left(j\omega_1 A - \frac{1}{2}\sigma_n^2 |\omega|^2\right). \tag{41}$$

where $\omega = \omega_1 + j\omega_2$ as a complex number.

The received signal can convert to polar representation of $s(t) = r(t)e^{j\theta_n(t)}$ with $x_1 = r(t)\cos\theta_n(t)$ and $x_2 = r(t)\sin\theta_n(t)$, we obtain the distribution of $r(t)$ and $\theta_n(t)$ as

$$\begin{aligned} p_{R,\Theta_n}(r,\theta) &= \frac{r}{2\pi\sigma_n^2}\exp\left(-\frac{(r\cos\theta - A)^2 + r^2\sin^2\theta}{2\sigma_n^2}\right) \\ &= \frac{r}{2\pi\sigma_n^2}\exp\left(-\frac{r^2 + A^2 + 2r\cos\theta}{2\sigma_n^2}\right). \end{aligned} \tag{42}$$

The distribution of the received amplitude R is (Rice, 1944, 1948)

$$p_R(r) = \int_{-\pi}^{+\pi} p_{R,\Theta_n}(r,\theta)\mathrm{d}\theta = \frac{r}{\sigma_n^2}\exp\left(-\frac{r^2 + A^2}{2\sigma_n^2}\right) I_0\left(\frac{rA}{\sigma_n^2}\right) \tag{43}$$

as the Rice distribution, where $I_0(\cdot)$ is zero-order modified Bessel function of first kind given by $I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\mathrm{d}\theta$ (Gradshteyn and Ryzhik, 1980, §8.431).

The received intensity is $Y = R^2$ with a noncentral chi-square (χ^2) distribution of

$$p_Y(y) = \frac{1}{2\sigma_n^2}\exp\left(-\frac{y + A^2}{2\sigma_n^2}\right) I_0\left(\frac{\sqrt{y}A}{\sigma_n^2}\right). \tag{44}$$

For phase modulation, the phase component of the received electric field has a distribution of

$$\begin{aligned} p_{\Theta_n}(\theta) &= \int_0^\infty p_{R,\Theta_n}(r,\theta)\mathrm{d}r \\ &= \frac{1}{2\pi}e^{-\rho_s} + \sqrt{\frac{\rho_s}{4\pi}}\cos\theta e^{-\rho_s\sin^2\theta}\mathrm{erfc}\left(-\sqrt{\rho_s}\cos\theta\right). \end{aligned} \tag{45}$$

The distribution of Eq. (45) is a function of $\cos\theta$ and thus a even periodic function with a period of 2π. We can expand the p.d.f. of Eq. (45) as a Fourier series of

$$p_{\Theta_n}(\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} c_m \exp(-jm\theta). \tag{46}$$

It is difficult to find the coefficients of c_m directly from the p.d.f. of Eq. (45), we can calculate the coefficients according to

$$\begin{aligned} c_m &= \int_{-\pi}^{\pi} p_{\Theta_n}(\theta) e^{jm\theta} \mathrm{d}\theta \\ &= \int_0^{\infty} \int_{-\pi}^{\pi} p_{R,\Theta_n}(r,\theta) e^{jm\theta} \mathrm{d}\theta \mathrm{d}r, \quad m \geq 0, \end{aligned} \tag{47}$$

and $c_{-m} = c_m$. Using Gradshteyn and Ryzhik (1980, §8.431), we obtain

$$\int_{-\pi}^{\pi} p_{R,\Theta_n}(r,\theta) e^{jm\theta} \mathrm{d}\theta = \frac{r}{\sigma_n^2} \exp\left(-\frac{r^2 + A^2}{2\sigma_n^2}\right) I_m\left(\frac{rA}{\sigma_n^2}\right). \tag{48}$$

Using Gradshteyn and Ryzhik (1980, §6.614, §9.220), the integration of Eq. (48) over amplitude r is equal to

$$\begin{aligned} c_m &= \frac{\rho_s^{\frac{m}{2}}}{m!} \Gamma\left(\frac{m}{2} + 1\right) {}_1F_1\left(\frac{m}{2}; m+1; -\rho_s\right) \\ &= \frac{\sqrt{\pi \rho_s}}{2} e^{-\rho_s/2} \left[I_{\frac{m-1}{2}}\left(\frac{\rho_s}{2}\right) + I_{\frac{m+1}{2}}\left(\frac{\rho_s}{2}\right) \right], \end{aligned} \tag{49}$$

where $\Gamma(\cdot)$ is the Gamma function, ${}_1F_1(a; b; \cdot)$ is the confluent hypergeometric function of the first kind , and $I_k(\cdot)$ is the k-th order modified Bessel function of the first kind. The p.d.f. of Eq. (45) is equal to

$$p_{\Theta_n}(\theta) = \frac{1}{2\pi} + \frac{e^{-\frac{\rho_s}{2}}}{2} \sqrt{\frac{\rho_s}{\pi}} \sum_{m=1}^{\infty} \left[I_{\frac{m-1}{2}}\left(\frac{\rho_s}{2}\right) + I_{\frac{m+1}{2}}\left(\frac{\rho_s}{2}\right) \right] \cos(m\theta). \tag{50}$$

The series representation of the phase distribution Eq. (50) can be found in Middleton (1960, §9.2-2) and Blachman (1981, 1988), Jain (1974), Jain and Blachman (1973), Prabhu (1969). The conversion from hypergeometric function to Bessel function is given by Blachman (1981, 1988), Jain (1974), Jain and Blachman (1973), Prabhu (1969).

The variance of the phase of amplifier noise is equal to

$$\begin{aligned} \sigma_{\Theta_n}^2 &= 2 \int_0^{\pi} \theta^2 p_{\Theta_n}(\theta) \mathrm{d}\theta \\ &= \frac{\pi^2}{3} + 2e^{-\frac{\rho_s}{2}} \sqrt{\pi \rho_s} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \left[I_{\frac{m-1}{2}}\left(\frac{\rho_s}{2}\right) + I_{\frac{m+1}{2}}\left(\frac{\rho_s}{2}\right) \right]. \end{aligned} \tag{51}$$

For high SNR, the p.d.f. of Eq. (45) can be approximated as

$$p_{\Theta_n}(\theta) \approx \sqrt{\frac{\rho_s}{\pi}} \cos\theta e^{-\rho_s \sin^2\theta} \tag{52}$$

with variance of

$$\sigma_{\Theta_n}^2 \approx \frac{1}{2\rho_s}. \tag{53}$$

Figure 10 shows the exact phase variance of Eq. (51) and the approximated phase variance of Eq. (53) as a function of SNR ρ_s. In high SNR of $\rho_s > 10$ dB, the exact [Eq. (51)] and approximated [Eq. (53)] phase variance are almost the same.

Figure 11 shows the p.d.f. of $p_{\Theta_n}(\theta)$ [either Eq. (45) or Eq. (50)] for a SNR of $\rho_s = 11, 18, 25$ (10.4 12.6 14.0 dB). The zero-mean Gaussian approximation with a variance of Eq. (53) is also plotted in Fig. 11

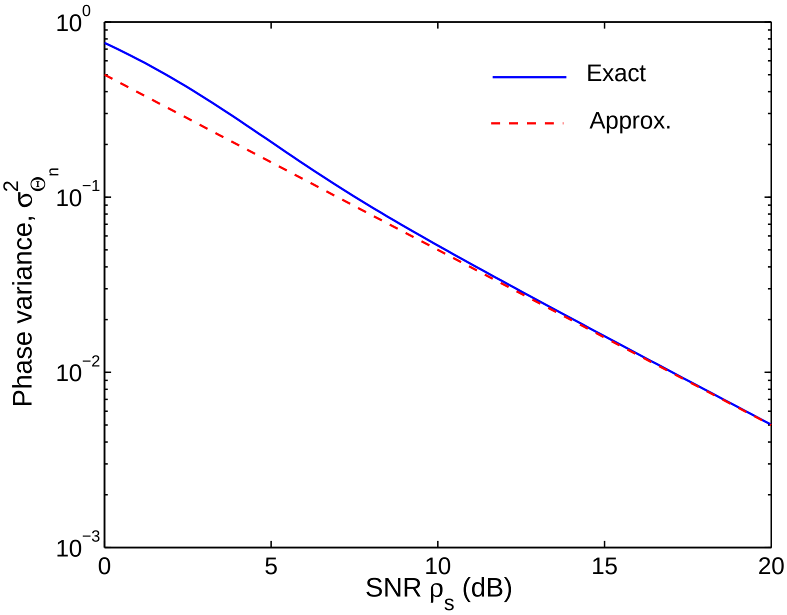

Figure 10: The phase variance of $\sigma^2_{\Theta_n}$ as a function of SNR ρ_s.

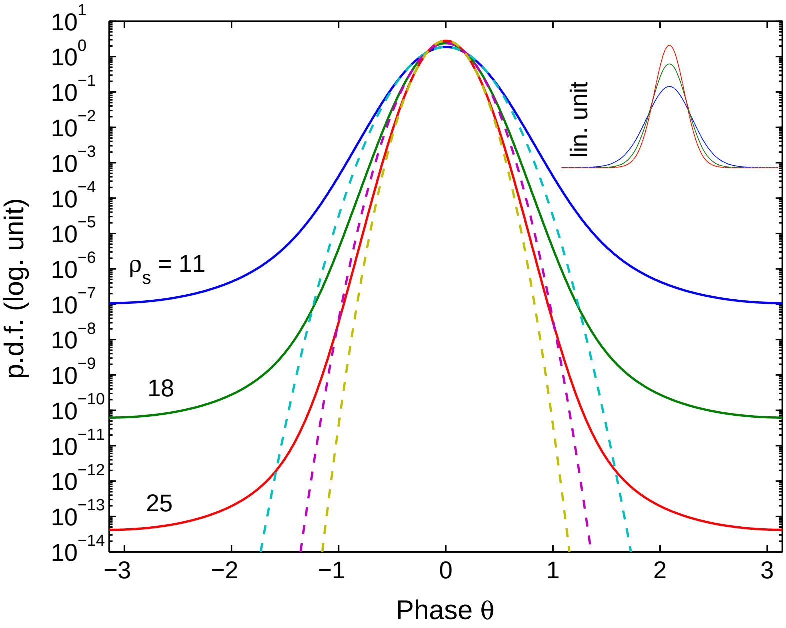

Figure 11: The p.d.f. of $p_{\Theta_n}(\theta)$ as comapred with the Gaussian approximation.

for comparison. Figure 11 plots in logarithmic scale to show the difference in the tail between the exact p.d.f. and Gaussian approximation. An inset plots the p.d.f. in linear scale. In the linear scale inset, the exact and approximated p.d.f. overlaps with each other and has not observable difference. While the phase distribution of $p_{\Theta_n}(\theta)$ [either Eq. (45) or Eq. (50)] is the same as Gaussian distribution in linear scale, Gaussian distribution cannot be used if a tail probability is interested. For optical communications interest in an error probability of 10^{-9} or lower, the tail probability is essential to evaluate the error probability.

Based on the series expansion of Eq. (50), the error probability of binary PSK signal [see Eq. (3.76)] is also equal to

$$\begin{aligned} p_e &= 1 - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_{\Theta_n}(\theta) \mathrm{d}\theta \\ &= \frac{1}{2} - \frac{e^{-\frac{\rho_s}{2}}}{2} \sqrt{\frac{\rho_s}{\pi}} \sum_{m=1}^{\infty} \frac{2\sin(m\pi/2)}{m} \left[I_{\frac{m-1}{2}}\left(\frac{\rho_s}{2}\right) + I_{\frac{m+1}{2}}\left(\frac{\rho_s}{2}\right)\right] \\ &= \frac{1}{2} - \exp\left(-\frac{\rho_s}{2}\right) \sqrt{\frac{\rho_s}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left[I_k\left(\frac{\rho_s}{2}\right) + I_{k+1}\left(\frac{\rho_s}{2}\right)\right]. \end{aligned} \tag{54}$$

Because $\sin(m\pi/2) = 0$ for m as even number, the error probability of Eq. (54) is simplified.

From the factor of $(-1)^k$, the terms of Eq. (54) oscillate between positive and negative values. Although the summation of Eq. (54) converges, the calculation is numerically challenging for small error probability. Note that the multiplication factor of the summation is a small value in the order of $e^{-\frac{\rho_s}{2}}\sqrt{\rho_s}$, the summation has a value in the order of $e^{\frac{\rho_s}{2}}/\sqrt{\rho_s}$ for small error probability. For large SNR ρ_s, the summation has very large terms although the error probability is small. The error probability is the difference between 1/2 and a value a little bit smaller than 1/2. The series summation of Eq. (54) can be calculated to an error probability of 10^{-13} to 10^{-14} with an accuracy of three to four significant digits. Symbolic mathematical software can provide better accuracy by using variable precision arithmetic in the calculation of low error probability.

A DPSK signal is demodulated using the differential phase of

$$\Delta\Theta_n = \Theta_n(t) - \Theta_n(t-T), \tag{55}$$

where $\Theta_n(\cdot)$ is the phase of amplifier noise as a function of time and T is the symbol interval. The phases of $\Theta_n(t)$ and $\Theta_n(t-T)$ are two identical independently distributed random variables with p.d.f given by $p_{\Theta_n}(\theta)$ of either Eq. (45) and Eq. (50).

When two independently distributed random variables are summed (or subtracted) together, the characteristic function of the sum (or difference) is equal to the product of its individual characteristic functions. For the series expansion like Eq. (50), the sum has a Fourier coefficient that is the product of the corresponding Fourier coefficients. Based on the series expansion of Eq. (50), the differential phase of Eq. (55) has a p.d.f. of

$$p_{\Delta\Theta_n}(\theta) = \frac{1}{2\pi} + \frac{\rho_s e^{-\rho_s}}{4} \sum_{m=1}^{\infty} \left[I_{\frac{m-1}{2}}\left(\frac{\rho_s}{2}\right) + I_{\frac{m+1}{2}}\left(\frac{\rho_s}{2}\right)\right]^2 \cos(m\theta). \tag{56}$$

The error probability corresponding to Eq. (54) for DPSK signal [see Eq. (3.103) is

$$p_e = \frac{1}{2} - \frac{\rho_s e^{-\rho_s}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left[I_k\left(\frac{\rho_s}{2}\right) + I_{k+1}\left(\frac{\rho_s}{2}\right)\right]^2. \tag{57}$$

Comparing the error probability of Eq. (3.76) with Eq. (54) for PSK signal and Eq. (3.103) with Eq. (57) for DPSK signal, the series of Eq. (50) is not very useful for performance analysis of PSK signal. However, when the system has additive phase noise that is independent of the additive Gaussian noise, the series of Eq. (50) is very useful. The summed phase noise has Fourier coefficients that are the multiplication of the corresponding coefficients of each individual phase noise components. Using the series of Eq. (50), the error probability of PSK or DPSK signals was derived by Prabhu (1969) for multilevel signals, Jain (1974), Jain and Blachman (1973), Lindsey and Simon (1973), Prabhu (1976) with phase-locked loop noise, Blachman (1981) with phase error, Nicholson (1984) with laser phase noise, and Iwashita and Matsumoto (1987), Jacobsen and Garrett (1987) with phase error and laser phase noise.