CHAPTER 9 FREQUENCY RESPONSE

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Frequency response of amplifiers

- **Midband:**
  - The frequency range of interest for amplifiers
  - Large capacitors act as short circuits and small capacitors act as open circuits
  - Gain is constant and can be obtained by small-signal analysis

- **Low-frequency band:**
  - Large capacitors no longer act as short circuits (small capacitors still act as open circuits)
  - The gain roll-off is mainly due to coupling and by-pass capacitors
  - Gain drops at frequencies lower than $f_L$

- **High-frequency band:**
  - Small capacitors no longer act as open circuits (large capacitors still act as short circuits)
  - The gain roll-off is mainly due to parasitic capacitances of the MOSFETs and BJTs
  - Gain drops at frequencies higher than $f_H$
9.1 Low-Frequency Response of the CS and CE Amplifiers

The CS amplifier

- Small-signal analysis for transfer function:
  \[
  V_g = \frac{R_G}{R_G + \frac{1}{sC_{c1}} + R_{sig}} V_{sig}
  \]
  \[
  I_d = \frac{1}{g_m + \left( \frac{1}{sC_s} || R_S \right)} V_g
  \]
  \[
  V_o = -\frac{R_D}{R_D + \left( R_L + \frac{1}{sC_{c2}} \right)} R_L I_d
  \]

- The transfer function is:
  \[
  \frac{V_o}{V_{sig}}(s) = \frac{V_o I_d}{I_d V_g V_{sig}} = A_M \left( \frac{s}{s + \omega_{p1}} \right) \left( \frac{s + \omega_{p2}}{s + \omega_{p3}} \right)
  \]
  \[
  A_M = -\frac{R_G}{R_G + R_{sig}} g_m (R_D || R_L)
  \]

- The corner frequencies are:
  \[
  \omega_{p1} = \frac{1}{C_{c1}(R_G + R_{sig})}
  \]
  \[
  \omega_{p2} = \frac{g_m + 1/R_S}{C_s}
  \]
  \[
  \omega_{p3} = \frac{1}{C_{c2}(R_D + R_L)}
  \]
  \[
  \omega_z = \frac{1}{C_S R_S}
  \]
The complete frequency response (gain and phase) are determined by the transfer function analysis.

Determining the lower 3-dB frequency ($f_L$) by dominant pole:
- Coupling and by-pass capacitors result in a high-pass frequency response with 3 poles and 1 zero.
- If poles are sufficiently separated (dominant-pole case):
  - Bode plot can be used to evaluate the response for simplicity.
  - The lower 3-dB frequency is the highest-frequency pole.
  - $f_{P2}$ is typically the highest-frequency pole due to small resistance of $1/g_m$.

$$R_G + R_{sig} \gg R_D + R_L \gg \frac{1}{g_m}$$

$$f_{P1} \ll f_{P3} \ll f_{P2} \approx f_L$$
Determining the lower 3-dB frequency \( f_L \) by poles and zeros

- If poles are located closely, the lower 3-dB frequency (at a frequency higher than poles and zeros) can be evaluated by the transfer function

\[
\frac{V_o}{V_{sig}}(s) = A_M \left( \frac{s}{s + \omega_{p1}} \right) \left( \frac{s + \omega_z}{s + \omega_{p2}} \right) \left( \frac{s}{s + \omega_{p3}} \right)
\]

\[
\left| \frac{V_o}{V_{sig}}(j\omega) \right| = \left| A_M \right| \left| \frac{1}{1 + \omega_{p1}/j\omega} \right| \left| \frac{1 + \omega_z/j\omega}{1 + \omega_{p2}/j\omega} \right| \left| \frac{1}{1 + \omega_{p3}/j\omega} \right|
\]

\[
\left| A_M \right| \left| \frac{1}{1 + f_{p1}/j f_L} \right| \left| \frac{1 + f_z/j f_L}{1 + f_{p2}/j f_L} \right| \left| \frac{1}{1 + f_{p3}/j f_L} \right| = \left| A_M \right| \frac{1}{\sqrt{2}}
\]

\[
(1 + \frac{f_{p1}^2}{f_L^2})(1 + \frac{f_{p2}^2}{f_L^2})(1 + \frac{f_{p3}^2}{f_L^2}) = 2 \left( 1 + \frac{f_z^2}{f_L^2} \right) \rightarrow 1 + \frac{f_{p1}^2}{f_L^2} + \frac{f_{p2}^2}{f_L^2} + \frac{f_{p3}^2}{f_L^2} \approx 2 \left( 1 + \frac{f_z^2}{f_L^2} \right)
\]

\[
f_L \approx \sqrt{f_{p1}^2 + f_{p2}^2 + f_{p3}^2 - 2f_z^2}
\]

Determining the pole and zero frequencies by inspection (capacitors do not interact)

- A pole is determined by the time constant of the capacitor

\[
\tau_1 = \frac{1}{g_m} \left( C_{C1} \right) R_G + R_w\)
\]

\[
\tau_2 = \frac{1}{g_m} \left( C_{C2} \right) \frac{1}{R_L}
\]

\[
\tau_3 = \frac{1}{g_m} \left( C_{C3} \right) \frac{1}{R_S}
\]

\[
\omega_{p1} = \frac{1}{\tau_1} = \frac{1}{C_{C1}(R_G + R_{sig})}
\]

\[
\omega_{p2} = \frac{1}{\tau_2} = \frac{1}{g_m + 1/R_S} \frac{1}{C_S}
\]

\[
\omega_{p3} = \frac{1}{\tau_3} = \frac{1}{C_{C2}(R_D + R_L)}
\]
A transmission zero is the value of \( s \) at which the input does not reach the output.

- Coupling capacitor \( C_{C1} \): \( V_o = 0 \) as the capacitor is open (zero at \( s = 0 \))
- By-pass capacitor \( C_S \): \( V_o = 0 \) as \( Z_S \) is open (zero at \( s = -1/R_SC_S \))
- Coupling capacitor \( C_{C2} \): \( V_o = 0 \) as the capacitor is open (zero at \( s = 0 \))

Selecting values for coupling and by-pass capacitors:
- These capacitors are typically required for discrete amplifier designs.
- It is desirable to minimize the total capacitance for coupling and by-pass purposes.
- \( C_S \) is first determined to satisfy \( f_{P2} \approx f_L \).
- \( C_{C1} \) and \( C_{C2} \) are chosen such that poles \( f_{P1} \) and \( f_{P3} \) are 5 to 10 times lower than \( f_L \).
The method of short-circuit time constant

- Useful technique to estimate the lower 3-dB frequency for circuits where the capacitors interact
- Time-constant method is used to evaluate $f_L$ without deriving the transfer function
- The method is predicated on the assumption that one of the poles is dominant
- The estimation is usually very good even a dominant pole does not exist
  - Set independent source to zero ($V_{\text{sig}} = 0$)
  - Consider each capacitor one at a time by set all other capacitors ideal (as short circuit)
  - For each capacitor $C_i$, find the total equivalent resistance $R_i$ seen by $C_i$
  - Calculate the 3-dB frequency $f_L$ using:

$$f_L = \frac{1}{2\pi} \sum_{i=1}^{n} \frac{1}{C_i R_i}$$

- For the CS amplifier example:
  - $C_1 = C_{C1} \rightarrow R_1 = R_G + R_{\text{sig}}$
  - $C_2 = C_S \rightarrow R_2 = R_S || (1/g_m)$
  - $C_3 = C_{C2} \rightarrow R_3 = R_D + R_L$
  - $f_L = \frac{1}{2\pi} \sum_{i=1}^{n} \frac{1}{C_i R_i}$
  - $f_L = \frac{1}{2\pi} \left[ \frac{1}{C_{C1}(R_G + R_{\text{sig}})} + \frac{1}{C_S \left( R_S || \frac{1}{g_m} \right)} + \frac{1}{C_{C2}(R_D + R_L)} \right]$
The CE amplifier

- Small-signal equivalent circuit:

- The capacitors interact (3 capacitors not necessarily for 3 poles)
- Transfer function can be derived but is rather complex

- Method of short-circuit time constants to evaluate $f_L$:
  - Considering only $C_1 = C_{C1}$:
    - Set $V_{sig} = 0$ and treat $C_E$ and $C_{C2}$ as short
    - The equivalent resistance:
      $$R_1 = R_{C1} = R_{sig} + R_B || r_\pi$$
Considering only $C_2 = C_E$:
- Set $V_{\text{sig}} = 0$ and treat $C_{C1}$ and $C_{C2}$ as short
- The equivalent resistance:
  $$R_2 = R_{CE} = r_e + (R_{\text{sig}} || R_B)/(1+\beta)$$

Considering only $C_3 = C_{C2}$:
- Set $V_{\text{sig}} = 0$ and treat $C_{C1}$ and $C_E$ as short
- The equivalent resistance:
  $$R_3 = R_{C2} = R_C + R_L$$

Estimate the lower 3-dB frequency:
$$f_L \approx \left[1/\tau_1 + 1/\tau_2 + 1/\tau_3\right]/2\pi$$
$$= \left[1/C_{C1}R_{C1} + 1/C_E R_{CE} + 1/C_{C2}R_{C2}\right]/2\pi$$

Selecting values for the capacitors
- Typically required for discrete amplifier designs
- $C_E$ is first determined to satisfy needed $f_L$
  (choose $1/C_E R_{CE} = 80\%$ of $\omega_L$)
- $C_{C1}$ and $C_{C2}$ are chosen for smaller time constant
  ($1/C_{C1}R_{C2}$ and $1/C_{C2}R_{C2} \approx 10\%$ of $\omega_L$)
9.2 Internal Capacitive Effects and the High-Frequency Model

The MOSFET device

- There are basically two types of internal capacitance in the MOSFET
  - Gate capacitance effect: the gate electrode forms a parallel-plate capacitor with gate oxide
  - Junction capacitance effect: the source/body and drain/body are \( pn \)-junctions at reverse bias

- The gate capacitive effect
  - MOSFET in triode region:
    \[ C_{gs} = C_{gd} = \frac{1}{2} W L C_{ox} + C_{ov} \]
  - MOSFET in saturation region:
    \[ C_{gs} = \frac{2}{3} W L C_{ox} + C_{ov} \text{ and } C_{gd} = C_{ov} \]
  - MOSFET in cutoff region:
    \[ C_{gs} = C_{gd} = C_{ov} \text{ and } C_{gb} = W L C_{ox} \]
  - Overlap capacitance:
    \[ C_{ov} = W L_{ov} C_{ox} \]

- The junction capacitance
  - Includes components from the bottom and from side walls
  - The simplified expression are given by
    \[ C_{sb} = \frac{C_{sbo}}{\sqrt{1 + V_{SB}/V_0}} \text{ and } C_{db} = \frac{C_{dbo}}{\sqrt{1 + V_{DB}/V_0}} \]
The high-frequency MOSFET model
- Based on low-frequency small-signal models including $g_m$ and $r_o$
- All parasitic capacitances ($C_{gs}$, $C_{gd}$, $C_{sb}$ and $C_{db}$) for MOSFET in saturation are included

Simplified high-frequency MOSFET model
- Source and body terminals are shorted
- $C_{db}$ is also neglected to simplify the analysis
- $C_{gs}$ is typically the largest parasitic capacitance and plays the most important role in amplifier frequency response
The unity-gain frequency ($f_T$)
- The frequency at which the current gain of the MOSFET becomes unity
- Is typically used as an indicator to evaluate the high-frequency capability
- Smaller parasitic capacitances $C_{gs}$ and $C_{gd}$ are desirable for higher unity-gain frequency

\[ I_i = sC_{gs}V_{gs} + sC_{gd}V_{gs} \]
\[ sC_{gd}V_{gs} + I_o = g_mV_{gs} \]
\[ \rightarrow I_o = \frac{g_m - sC_{gd}}{s(C_{gs} + C_{gd})} \approx \frac{g_m}{s(C_{gs} + C_{gd})} \]
\[ \rightarrow f_T = \frac{1}{2\pi C_{gs} + C_{gd}} \]

- The unity-gain frequency can also be expressed as

\[ f_T = \frac{1}{2\pi C_{gs} + C_{gd}} \approx \frac{1}{2\pi} \frac{\mu_n C_{ox} \left( \frac{W}{L} \right) V_{OV}}{2 \frac{2}{3} W/L C_{ox}} = \frac{3\mu_n V_{OV}}{4\pi L^2} \]

- The unity-gain frequency is strongly influenced by the channel length
- Minimizing the channel length effectively increases the unity-gain frequency of MOSFET
- Higher unity-gain frequency can be achieved for a given MOSFET by increasing the bias current ($I_D$) or the overdrive voltage ($V_{OV}$)
The BJT Device

- High-frequency small-signal model:
  - Includes all parasitic capacitances in the hybrid-\(\pi\) model or T-model
  - Parasitic capacitances:
    - The base-charging or diffusion capacitance (\(C_{de}\))
    - The base-emitter junction capacitance (\(C_{je}\))
    - The total base-emitter capacitance (\(C_\pi = C_{de} + C_{je}\))
    - The collector-base junction capacitance (\(C_\mu\))
  - \(C_\pi\) is typically in the range of a few pF to a few tens of pF
  - \(C_\mu\) is typically in the range of a fraction of a pF to a few pF
The cutoff (unity-gain) frequency:

- The frequency at which the current gain of the MOSFET becomes unity
- Is typically used as an indicator to evaluate the high-frequency capability

\[ V_π = \frac{I_b}{1/r_π + sC_π + sC_μ} \]

\[ g_mV_π = I_c + sC_μV_π \]

\[ |h_{fe}| = \frac{I_c}{I_b} = \frac{g_m - sC_μ}{1/r_π + s(C_π + C_μ)} \approx \frac{g_mr_π}{1 + s(C_π + C_μ)r_π} = \frac{β_0}{1 + s/ω_β} \]

\[ f_T = \frac{1}{2π} \frac{g_m}{C_π + C_μ} = \frac{1}{2π} \frac{I_c}{(C_π + C_μ)V_T} \]
9.3 High-Frequency Response of the CS and CE Amplifiers

The common-source amplifier

- The coupling and by-pass capacitors act as short circuits at high frequencies
- The MOSFET is represented by its high-frequency small-signal model

\[
\begin{align*}
V_{0\omega} & = V_{0\omega} + \frac{R_G}{R_D + R_G} + \frac{R_{0\omega}}{R_{0\omega}} \\
R_{0\omega} & = R_G || R_{0\omega} \\
V_{0\omega} & = V_{0\omega} R_G/(R_G + R_{0\omega})
\end{align*}
\]
Analysis of the high-frequency transfer function:

\[ \frac{V_{\text{sig}} - V_{gs}}{R_{\text{sig}}} = \frac{V_{gs}}{R_G} + sC_{gs}V_{gs} + sC_{gd}(V_{gs} - V_o) \]

\[ sC_{gd}(V_{gs} - V_o) = g_m V_{gs} + \frac{V_{gs}}{R'_L} \]

\[ \frac{V_o}{V_{\text{sig}}}(s) = -\left( \frac{g_m R'_L R_G}{R_G + R_{\text{sig}}} \right) \frac{1}{\left[ 1 + s\left[ (C_{gs} + C_{gd}(1 + g_m R'_L))R'_{\text{sig}} + (C_{gd} + C_L)R'_L \right] + s^2(C_{gs}C_{gd} + C_{gs}C_L + C_{gd}C_L)R'_{\text{sig}} R'_L \right]} \]

\[ \frac{V_o}{V_{\text{sig}}}(s) = -\left( \frac{g_m R'_L R_G}{R_G + R_{\text{sig}}} \right) \frac{1}{\left[ 1 + s\left[ (C_{gs} + C_{gd}(1 + g_m R'_L))R'_{\text{sig}} + (C_{gd} + C_L)R'_L \right] + s^2(C_{gs}C_{gd} + C_{gs}C_L + C_{gd}C_L)R'_{\text{sig}} R'_L \right]} \]

\[ A_M = \frac{1 - s/\omega_z}{(1 + s/\omega_{p1})(1 + s/\omega_{p2})} \]

■ The transfer function is a second-order one with two poles and one zero

■ The poles and zero result in gain roll-off at higher frequencies

■ The high-frequency response can be evaluated by

\[ \frac{V_o}{V_{\text{sig}}}(j\omega) = -\left( \frac{g_m R'_L R_G}{R_G + R_{\text{sig}}} \right) \left[ 1 - j\omega(C_{gd}/g_m) \right] \]

\[ \frac{1}{\left[ 1 - \omega^2(C_{gs}C_{gd} + C_{gs}C_L + C_{gd}C_L)R'_{\text{sig}} R'_L \right] + j\omega\left[ (C_{gs} + C_{gd}(1 + g_m R'_L))R'_{\text{sig}} + (C_{gd} + C_L)R'_L \right] \]

■ The bandwidth is typically defined by the 3-dB frequency \( f_{\text{H}} \)
Determining the upper 3-dB frequency ($f_H$):

- The 3-dB frequency is defined by the dominant pole (lowest-frequency) of the transfer function.
- If dominant pole does not exist, the 3-dB frequency can be evaluated by the poles and zeros of the high-frequency response:

$$A(s) = A_M F_H(s) = A_M \frac{(1 + s/\omega_{Z1})(1 + s/\omega_{Z2})}{(1 + s/\omega_{P1})(1 + s/\omega_{P2})}$$

$$|F_H(j\omega_H)|^2 = \frac{1}{2} \frac{(1 + \omega_H^2/\omega_{Z1}^2)(1 + \omega_H^2/\omega_{Z2}^2)}{(1 + \omega_H^2/\omega_{P1}^2)(1 + \omega_H^2/\omega_{P2}^2)} \approx \frac{1 + \omega_H^2}{1 + \omega_H^2}
\frac{\frac{1}{\omega_{P1}^2} + \frac{1}{\omega_{P2}^2}}{\frac{1}{\omega_{Z1}^2} + \frac{1}{\omega_{Z2}^2}}$$

$$f_H \approx \frac{1}{2\pi} \frac{1}{\sqrt{\left(\frac{1}{\omega_{P1}^2} + \frac{1}{\omega_{P2}^2}\right) - 2 \left(\frac{1}{\omega_{Z1}^2} + \frac{1}{\omega_{Z2}^2}\right)}}$$

$$f_H \approx \frac{\omega_{P1}}{2\pi} \quad \text{(for dominant-pole case)}$$
The common-emitter amplifier

- The coupling and by-pass capacitors act as short circuits at high frequencies
- The BJT is represented by its high-frequency small-signal model

\[ V_{o} = V_{o}(R_{\text{se}} || r_{g})/R_{\text{se}} || r_{g} + R_{o} \]

\[ R_{B} = R_{B1} || R_{B2} \]

\[ R_{L}' = r_{o} || R_{C} || R_{L} \]

\[ V'_{\text{sig}} = V_{\text{sig}}(R_{g} || r_{\pi})/(R_{B} || r_{\pi} + R_{\text{sig}}) \]
Analysis of the high-frequency transfer function:

\[
\frac{V'_{\text{sig}} - V_n}{R'_{\text{sig}}} = sC_n V_n + sC_\mu (V_n - V_o)
\]

\[sC_\mu (V_n - V_o) = g_m V_n + \frac{V_o}{R'_L}\]

\[\frac{V_o}{V'_{\text{sig}}} (s) = -\frac{g_m R'_L [1 - s(C_\mu / g_m)]}{1 + s \{[C_\pi + C_\mu (1 + g_m R'_L)] R'_{\text{sig}} + C_\mu R'_L\} + s^2 C_\pi C_\mu R'_{\text{sig}} R'_L}\]

\[\frac{V_o}{V_{\text{sig}}} (s) = -\frac{g_m R'_L (R_B | r_\pi)}{(R_B | r_\pi) + R_{\text{sig}}} \left[ 1 - s (C_\mu / g_m) \right] \left[ C_\pi + C_\mu (1 + g_m R'_L) \right] R'_{\text{sig}} + C_\mu R'_L + \frac{1 - s/\omega}{(1 + s/\omega_{p1})(1 + s/\omega_{p2})}\]

- The analysis is very similar to common-source amplifier
- The transfer function is a second-order one with two poles and one zero
- Poles and zero result in gain roll-off at higher frequencies
- High-frequency response can be evaluated by

\[
\frac{V_o}{V_{\text{sig}}} (j\omega) = -\frac{g_m R'_L (R_B | r_\pi)}{(1 - \omega^2 C_\pi C_\mu R'_{\text{sig}} R'_L) + j\omega \{[C_\pi + C_\mu (1 + g_m R'_L)] R'_{\text{sig}} + C_\mu R'_L\}}\]

- The bandwidth is typically defined by the 3-dB frequency \(f_{11}\)
Miller’s Theorem:

- A technique to replace the bridging capacitance
- The bridging impedance $Z$ can be equivalently divided into shunt $Z_1$ at node 1 and shunt $Z_2$ at node 2
- The equivalent input and output impedances are:

$$I = \frac{V_1 - V_2}{Z} = \frac{V_1 - KV_1}{Z} = \frac{V_1}{Z/(1 - K)} = \frac{V_1}{Z_1}$$

$$I = \frac{V_1 - V_2}{Z} = \frac{V_2/K - V_2}{Z} = \frac{-V_2}{Z/(1 - K)} = \frac{-V_2}{Z_2}$$

$\rightarrow Z_1 = Z/(1 - K)$ and $Z_2 = Z/(1 - \frac{1}{K})$

Simplified analysis of CS amplifier by Miller Theorem:

- Assume the gain from $G$ to $D$ is nearly constant for frequencies close to the midband
- Bridging capacitance $C_{gd}$ is equivalently divided into $C_1 = (1-K)C_{gd}$ and $C_2 = (1-K^{-1})C_{gd}$
Transfer function of the CS amplifier by using Miller's theorem:

\[
\frac{V_o}{V_{sig}}(s) = -\frac{g_m R'_L}{[1 + sR'_L (C_{gs} + C_1)][1 + sR'_L (C_L + C_2)]}
\]

\[
\frac{V_o}{V_{sig}}(s) = -\left(\frac{g_m R'_L R_G}{R_G + R_{sig}}\right) \frac{1}{\left[1 + sR'_L (C_{gs} + C_{gd} (1 + g_m R'_L))\right]\left[1 + R'_L (C_L + C_{gd} (1 + 1/g_m R'_L))\right]}
\]

The transfer function has two poles:

\[
\omega_{p1} = \frac{1}{R'_{sig} [C_{gs} + C_{gd} (1 + g_m R'_L)]}
\]

\[
\omega_{p2} = \frac{1}{R'_L [C_L + C_{gd} \left(1 + \frac{1}{g_m R'_L}\right)]} \approx \frac{1}{R'_L (C_L + C_{gd})}
\]

The assumption that gain from G to D is constant no longer valid as frequency increases

The midband gain remains the same

The transfer function can not be used to predict high-frequency response very accurately

It can be used to simplify the analysis for the 3-dB frequency estimation if a dominant pole exists

The equivalent response of the CS amplifier is typically approximated by a STC circuit:

\[
A(s) \approx -\left(\frac{g_m R'_L R_G}{R_G + R_{sig}}\right) \frac{1}{\left[1 + sR'_L (C_{gs} + C_{gd} (1 + g_m R'_L))\right]} = \frac{A_M}{1 + s/\omega_H}
\]
Simplified analysis of CE amplifier by Miller Theorem:

- Assume the gain from $B$ to $C$ is nearly constant for frequencies close to the midband
- Bridging capacitance $C_\mu$ is equivalently divided into $C_1 = C_{eq}$ and $C_2$ (negligible)

The midband gain is defined as

$$A_M = \frac{g_mR'_L(R_B || r_\pi)}{R_{sig} + R_B || r_\pi}$$

- $C_{eq}$ is considered Miller capacitance multiplying $C_\mu$ by a factor of the midband gain
- $C_{in}$ is effectively the largest capacitance in the CE amplifier due to Miller Effect
- It is not necessary to derive the poles and zeros as the high-frequency response deviates
- The 3-dB frequency can be estimated by the dominant pole as

$$f_H = \frac{1}{2\pi R'_{sig}} \left[ \frac{1}{C_\mu + C_{eq}(1 + g_mR'_L)} \right]$$
9.4 Useful Tools for the Analysis of the High-Frequency Response of Amplifiers

Open-circuit time constant method to evaluate amplifier bandwidth

- General transfer function of the amplifier
  \[
  A(s) = A_M \frac{1 + \frac{s}{\omega_{z1}}}{1 + \frac{s}{\omega_{p1}}} \frac{1 + \frac{s}{\omega_{z2}}}{1 + \frac{s}{\omega_{p2}}} \cdots \frac{1 + \frac{s}{\omega_{zm}}}{1 + \frac{s}{\omega_{pn}}} = A_M \frac{1 + a_1 s + a_2 s^2 + \cdots + a_m s^m}{1 + a_1 s + a_2 s^2 + \cdots + a_n s^n}
  \]

- Open-circuit time constant:
  - Remove all independent sources
  - Consider the capacitors one at a time and the others are considered ideal (open-circuit)
  - Find the equivalent shunt resistance for each capacitor
  - The coefficient can be obtained as \( b_1 = C_1 R_1 + C_2 R_2 + \cdots + C_n R_n \)

- Dominant pole approximation:
  \[
  b_1 = \frac{1}{\omega_{p1}} + \frac{1}{\omega_{p2}} + \cdots + \frac{1}{\omega_{pn}}
  \]
  \[
  \omega_H \approx \omega_{p1} \approx \frac{1}{b_1} = \frac{1}{C_1 R_1 + C_2 R_2 + \cdots + C_n R_n}
  \]

- For an amplifier with \( 2^{nd} \) order transfer function and a dominant pole:
  \[
  A(s) = A_M \frac{1 + \frac{s}{\omega_{z1}}}{1 + \frac{s}{\omega_{p1}}} \frac{1 + \frac{s}{\omega_{z2}}}{1 + \frac{s}{\omega_{p2}}} = A_M \frac{1 + a_1 s + a_2 s^2}{1 + a_1 s + a_2 s^2}
  \]
  \[
  b_1 = \frac{1}{\omega_{p1}} + \frac{1}{\omega_{p2}} \approx \frac{1}{\omega_{p1}} \rightarrow \omega_{p1} \approx \frac{1}{b_1} \quad \text{and} \quad b_2 = \frac{1}{\omega_{p1} \omega_{p2}} \approx \frac{b_1}{\omega_{p1}} \rightarrow \omega_{p2} \approx \frac{b_1}{b_2}
  \]
Analysis of CS amplifier by using open-circuit time constants

Open-circuit time constants:

\[ R_{gs} = R_{s1}^{'} \]

\[ R_{gd} = R_{s1}^{'} (1 + g_m R_L^{''} + R_L^{''}) \]

\[ R_{cl} = R_L^{''} \]

\[ b_1 = C_{gs} R_{gs} + C_{gd} R_{gd} + C_L R_{cl} = C_{gs} R_{s1}^{'} + C_{gd} (g_m R_L^{''} R_{s1}^{'} + R_L^{''} + R_{s1}^{''}) + C_L R_L^{''} \]

\[ \omega_H \approx \frac{1}{b_1} = \frac{1}{C_{gs} R_{s1}^{'} + C_{gd} (g_m R_L^{''} R_{s1}^{'} + R_L^{''} + R_{s1}^{''}) + C_L R_L^{''}} \]

NTUEE Electronics – L.H. Lu
Analysis by using Miller’s Theorem and open-circuit time constants

- Bridging capacitance $C_{rd}$ is replaced by $C_1$ and $C_2$

\[
K \equiv \frac{V_o}{V_{gs}} \approx -g_m R'_L
\]

\[
C_1 = C_{gd}(1 - K) = C_{gd}(1 + g_m R'_L)
\]

\[
C_2 = C_{gd}(1 - K^{-1}) = C_{gd}(1 + 1/g_m R'_L) \approx C_{gd}
\]

- Transfer function and $\omega_H$:

\[
\frac{V_o}{V_{sig}} = \left(-g_m R'_L \frac{R_G}{R_G + R_{sig}}\right) \frac{1}{(1 + s/\omega_H)(1 + s/\omega_p)}
\]

\[
\omega_p = \frac{1}{C_{in} R'_{sig}} = \frac{1}{[C_{gs} + C_{gd}(1 + g_m R'_L)] R'_{sig}} \approx \omega_H
\]

\[
\omega_H = \frac{1}{b_1} = \frac{1}{[C_{gs} + C_{gd}(1 + g_m R'_L)] R'_{sig} + (C_L + C_{gd}) R'_L}
\]

- Open-circuit time constant analysis based on Miller’s equivalent circuit:

\[
R_{C_{in}} = R'_{sig}
\]

\[
R'_{C_L} = R'_L
\]

\[
b_1 = C_{in} R_{C_{in}} + C'_L R'_L = [C_{gs} + C_{gd}(1 + g_m R'_L)] R'_{sig} + (C_L + C_{gd}) R'_L
\]

\[
\omega_H \approx \frac{1}{b_1} \frac{1}{[C_{gs} + C_{gd}(1 + g_m R'_L)] R'_{sig} + (C_L + C_{gd}) R'_L}
\]
Analysis of CE amplifier by using open-circuit time constants

Open-circuit time constants:

\[ b_1 = C_\pi R_{\pi} + C_\mu R_\mu + C_L R_{CL} = C_\pi R'_{sig} + C_\mu \left( g_m R'_L R'_{sig} + R'_L + R'_{sig} \right) + C_L R'_L \]

\[ \omega_H \approx \frac{1}{b_1} = \frac{1}{C_\pi R'_{sig} + C_\mu \left( g_m R'_L R'_{sig} + R'_L + R'_{sig} \right) + C_L R'_L} \]
9.5 A Closer Look at the High-Frequency Response

Analysis by transfer function

- Transfer function of the amplifier:
  \[
  \frac{V_o}{V_{sig}}(s) = -\frac{g_m R'_L [1 - s(C_{gd}/g_m)]}{1 + s \left\{ \left[ C_{gs} + C_{gd} (1 + g_m R'_L) \right] R'_{sig} + (C_{gd} + C_L) R'_L \right\} + s^2 \left( C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L \right) R'_{sig} R'_L}
  \]

- Poles and zero for dominant-pole approximation:
  \[
  A_M = -\left( \frac{g_m R'_L R_G}{R_G + R_{si}} \right)
  \]
  \[
  \omega_z = \frac{g_m}{C_{gd}}
  \]
  \[
  \frac{1}{b_1} \approx \frac{1}{\omega_p 1} = \frac{1}{\left[ C_{gs} + C_{gd} (1 + g_m R'_L) \right] R'_{sig} + (C_{gd} + C_L) R'_L}
  \]
  \[
  \frac{b_1}{b_2} \approx \frac{1}{\omega_p 2} = \frac{1}{C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L} \left( C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L \right) R'_{sig} R'_L
  \]

- The 3-dB frequency is given by \( \omega_1 \approx \omega_p \) (is typically limited by \( R_{sig} \) and Miller effect)
CS amplifier with a small source resistance

- For the case where source resistance is zero
- The transfer function of the amplifier:
  \[ \frac{V_o}{V_{sig}}(s) = -(g_mR'_L) \frac{1 - s(C_{gd}/g_m)}{1 + s(C_{gd} + C_L)R'_L} \]

  - Midband gain:
    \[ A_M = -g_mR'_L \]
  - The transfer function has one pole and one zero
    \[ \omega_{p1} = \frac{1}{(C_{gd} + C_L)R'_L} \]
    \[ \omega_z = \frac{g_m}{C_{gd}} \]

  - Midband gain and zero are virtually unchanged
  - The lowest-frequency pole no longer exists

- Gain-bandwidth product:
  - Gain rolls off beyond \( f_{11} \) (−20 dB/decade)
  - The gain becomes 0 dB at \( f_t \):
    \[ f_t = |A_M|f_{11} = \frac{g_m}{2\pi(C_{gd} + C_L)} \]
9.6 High-Frequency Response of the CG and Cascode Amplifiers

High-frequency response of the CG amplifier

- High-frequency circuit model for CS amplifier

- The frequency response by neglecting $r_o$:

$$\frac{V_o}{V_{sig}}(s) = \left( \frac{g_m R_L}{1 + g_m R_{sig}} \right) \frac{1}{(1 + s/\omega_{p1})(1 + s/\omega_{p2})}$$

$$\omega_{p1} = \frac{1}{C_{gs} \left( R_{sig} \parallel \frac{1}{g_m} \right)}$$

$$\omega_{p2} = \frac{1}{\left( C_{gd} + C_L \right) R_L}$$

- The upper 3-dB frequency is typically defined by $\omega_H \cong \omega_{p2} << \omega_{p1}$
- The poles of CG amplifier is usually much higher than the dominant pole of the CS amplifier

$\omega_{p1}$ $\cong \omega_{p2} \ll \omega_{p1}$

$\omega_{p1}$ $\cong \omega_{p2} \ll \omega_{p1}$
Time constant method to evaluate the 3-dB frequency of CG amplifier by neglecting $r_o$:

\[ b_1 = \tau_{gs} + \tau_{gd} = C_{gs}\left( R_{sig}||\frac{1}{g_m}\right) + (C_{gd} + C_L)R_L \]

\[ \omega_H \approx \frac{1}{b_1} = \frac{1}{C_{gs}\left( R_{sig}||\frac{1}{g_m}\right) + (C_{gd} + C_L)R_L} \]

The frequency response as $r_o$ included ($r_o$ has to be taken into account for IC amplifiers)
- The transfer function derivation is rather complex if $r_o$ is not negligible
- Time constant method can be used to evaluate the 3-dB frequency

\[ \tau_{gs} = C_{gs}R_{gs} = C_{gs}\left( R_{sig}||\frac{r_o + R_L}{1 + g_m r_o}\right) \]

\[ \tau_{gd} = (C_{gd} + C_L)R_{gd} = (C_{gd} + C_L)[R_L||\left(g_m r_o R_{sig} + r_o + R_{sig}\right)] \]
High-frequency response of the cascode amplifier

- Open-circuit time constant method:
  - Capacitance $C_{gs1}$ sees a resistance $R_{sig}$
  - Capacitance $C_{gd1}$ sees a resistance $R_{gd1}$
    \[ R_{gd1} = R_{sig}(1 + g_{m1}R_{d1}) + R_{d1} \]
  - Capacitance $(C_{db1} + C_{gs2})$ sees a resistance $R_{d1}$
    \[ R_{d1} = r_{o1}||R_{in2} = r_{o1}|| \left( \frac{r_{o2} + R_L}{1 + g_{m2}r_{o2}} \right) \]
  - Capacitance $(C_{gd2} + C_L)$ sees a resistance $R_o||R_L$
    \[ R_o||R_L = (g_{m2}r_{o2} + r_{o2} + r_{o1})||R_L \]
  - Effective time constant
    \[ b_1 = C_{gs1}R_{gs1} + C_{gd1}R_{gd1} + (C_{db1} + C_{gs2})R_{d1} + (C_{gd2} + C_L)(R_o||R_L) \]

- The upper 3-dB frequency:
  \[ \omega_H \approx \frac{1}{b_1} = \frac{1}{C_{gs1}R_{gs1} + C_{gd1}R_{gd1} + (C_{db1} + C_{gs2})R_{d1} + (C_{gd2} + C_L)(R_o||R_L)} \]
The effective time constant can be expressed as:

\[ \tau_H = R_{sig}[C_{gs1} + C_{gd1}(1 + g_{m1}R_{d1})] + R_{d1}(C_{gd1} + C_{db1} + C_{gs2}) + (R_o||R_L)(C_{gd2} + C_L) \]

- The 1\textsuperscript{st} term arises at input node; the 2\textsuperscript{nd} term at the middle node; the 3\textsuperscript{rd} term at output node

- In the case of a large \( R_{sig} \):
  - The first term dominates if the Miller multiplier is large (typically with large \( R_{di} \) and \( R_L \))
  - A small \( R_L \) (to the order of \( r_o \)) is needed for extended bandwidth
  - The midband gain drops as the value of \( R_L \) decreases
  - A trade-off exists between gain and bandwidth

- In the case of a small \( R_{sig} \):
  - The 1\textsuperscript{st} term becomes negligible
  - A large \( R_L \) (to the order of \( A_o r_o \)) can be used to boost the amplifier gain
  - The 3\textsuperscript{rd} term dominates

- In the case of zero \( R_{sig} \):
  - \( f_H \approx \frac{1}{2\pi(R_o||R_L)(C_{gd2} + C_L)} \)
  - Choose \( R_o||R_L \) larger than \( R'_L \), which is defined in CS amplifier, by a factor of \( A_o \)
  - The \( f_H \) of the cascade will be lower than that of the CS amplifier by the same factor \( A_o \)
  - The unity-gain frequency is unchanged at

\[ f_L \approx \frac{g_m}{2\pi(C_{gd2} + C_L)} \]
## Summary of CS and cascode amplifiers

<table>
<thead>
<tr>
<th></th>
<th>Common Source</th>
<th>Cascode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circuit</td>
<td><img src="image1" alt="Common Source Circuit" /></td>
<td><img src="image2" alt="Cascode Circuit" /></td>
</tr>
<tr>
<td>DC Gain</td>
<td>(-g_mR'_L)</td>
<td>(-A_0g_mR'_L)</td>
</tr>
<tr>
<td>(f_{3dB})</td>
<td>(\frac{1}{2\pi(C_L + C_{so})R'_L})</td>
<td>(\frac{1}{2\pi(C_L + C_{so})A_0R'_L})</td>
</tr>
<tr>
<td>(f_t)</td>
<td>(\frac{g_m}{2\pi(C_L + C_{so})})</td>
<td>(\frac{g_m}{2\pi(C_L + C_{so})})</td>
</tr>
</tbody>
</table>

Gain (dB) vs Frequency (log scale)

\[ A_0g_mR'_L \quad A_0 \]

\[ f_{3dB_{cascode}} \quad f_{3dB_{CS}} \quad f_t \]
High-frequency response of the cascode amplifier

- Open-circuit time constant method:
  - Capacitance $C_{\pi 1}$ sees a resistance $R_{\pi 1}$
    \[ R_{\pi 1} = R_{\text{sig}} \| r_{\pi 1} \]
  - Capacitance $C_{\mu 1}$ sees a resistance $R_{\mu 1}$
    \[ R_{\mu 1} = (R_{\text{sig}} \| r_{\pi 1})(1 + g_{m1}R_{c1}) + R_{c1} \]
  - Capacitance $(C_{c1} \| C_{\pi 2})$ sees a resistance $R_{c1}$
    \[ R_{c1} = r_{o1} \| R_{in2} = r_{o1} \| \left[ r_{e2} \left( \frac{r_{o2} + R_L}{r_{o2} + R_L/(\beta_2 + 1)} \right) \right] \]
  - Capacitance $(C_{\mu 2} \| C_L)$ sees a resistance $R_o \| R_L$
    \[ R_o \| R_L \approx (\beta_2 r_{o2}) \| R_L \]
  - Effective time constant
    \[ b_1 = C_{\pi 1} R_{\pi 1} + C_{\mu 1} R_{\mu 1} + (C_{c1} + C_{\pi 2}) R_{c1} + (C_{\mu 2} + C_L)(R_o \| R_L) \]

- High-frequency response:
  \[ A_M = -\frac{r_\pi}{r_\pi + R_{\text{sig}}} g_m (\beta r_o \| R_L) \]
  \[ \omega_H \approx \frac{1}{b_1} = \frac{1}{C_{\pi 1} R_{\pi 1} + C_{\mu 1} R_{\mu 1} + (C_{c1} + C_{\pi 2}) R_{c1} + (C_{\mu 2} + C_L)(R_o \| R_L)} \]
9.7 High-Frequency Response of the Source and Emitter Followers

The source follower:

- High-frequency transfer function:

\[
\frac{V_o}{V_{\text{sig}}}(s) = A_M \frac{1 + s/\omega_z}{1 + b_1s + b_2s^2}
\]

- \( A_M = \frac{R'_L / (R'_L + 1/g_m)}{1 + g_m R'_L} \approx 1 \)

- \( \omega_z = \frac{g_m}{C_{gs}} \)

- \( b_1 = \left( C_{gd} + \frac{C_{gs}}{1 + g_m R'_L} \right) R_{\text{sig}} + \left( \frac{C_{gs} + C_L}{1 + g_m R'_L} \right) R'_L \)

- \( b_2 = \left( C_{gs} C_L + C_{gd} C_L + C_{gs} C_{gd} \right) R_{\text{sig}} R'_L \)

\( R'_L = R_L || r_o || \frac{1}{g_{mn}} \)
Frequency response with dominant pole approximation:

\[ 1 + b_1 s + b_2 s^2 = (1 + s / \omega_1)(1 + s / \omega_2) \]

\[ \omega_p = \frac{1}{b_1} = \left[ \frac{C_{gs} + \frac{C_{gs} + C_L}{1 + g_m R'_L}}{R_{sig}} \right]^{-1} \]

\[ \omega_p = \frac{b_1}{b_2} \approx \frac{1}{b_2} = \left[ \frac{R_{sig}}{C_{gd} + \frac{C_{gs} + C_L}{1 + g_m R'_L}} + \left( \frac{C_{gs} + C_L}{1 + g_m R'_L} \right) \right] \]

Frequency response for non-dominant pole cases (real poles):

\[ f_H \approx \frac{1}{2\pi} \sqrt{\frac{1}{\frac{\omega_p^2}{\omega_1^2} + \frac{1}{\omega_p^2}} - 2 \left( \frac{1}{\omega_p^2} \right)} \]

Frequency response for complex pole cases (Q > 0.5):

\[ 1 + b_1 s + b_2 s^2 = 1 + \frac{1}{Q \omega_0} + \frac{s^2}{\omega_0^2} \]

\[ \omega_0 = \frac{1}{\sqrt{b_2}} = \frac{g_m R'_L + 1}{\sqrt{R_{sig} R'_L (C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L)}} \]

\[ Q = \frac{\sqrt{b_2}}{b_1} = \frac{\sqrt{g_m R'_L + 1}}{\sqrt{R_{sig} R'_L (C_{gs} C_{gd} + C_{gs} C_L + C_{gd} C_L)}} - \frac{\sqrt{C_{gs} C_{gd} (g_m R'_L + 1) R_{sig} + (C_{gs} + C_L) R'_L}}{\sqrt{C_{gs} + C_{gd} (g_m R'_L + 1) R_{sig} + (C_{gs} + C_L) R'_L}} \]
The emitter follower:

![Emitter Follower Diagram]

- **High-frequency transfer function:**

\[
\frac{V_o}{V_{sig}}(s) = A_M \frac{1 + s/\omega_z}{1 + b_1 s + b_2 s^2}
\]

\[
A_M = \frac{R'_L}{R'_L + r_e + R'_{sig}/(\beta + 1)} \approx 1
\]

\[
\omega_Z = \frac{1}{C_\pi r_e}
\]

\[
b_1 = \frac{[C_\pi + C_\mu (1 + R'_L/r_e)] R'_{sig} + [C_\pi + C_L (1 + R'_{sig}/r_\pi)] R'_L}{1 + R'_L/r_e + R'_{sig}/r_\pi}
\]

\[
b_2 = \frac{(C_\pi C_L + C_\mu C_L + C_\pi C_\mu) R'_L R'_{sig}}{1 + R'_L/r_e + R'_{sig}/r_\pi}
\]
9.8 High-Frequency Response of Differential Amplifiers

Resistively Loaded MOS Differential Amplifier:

- **Differential-mode operation:**
  - Use differential half-circuit for analysis
  - Identical to the case for common-source amplifier
  - Can be approximated by a dominant-pole system

- **Common-mode operation:**
  - Use common-mode half-circuit for analysis
  - $C_{SS}$ is usually significant and cannot be neglected
  - $C_{SS}$ results in a zero at lower frequency
  - Other capacitance $\rightarrow$ high-frequency poles and zeros

\[
A_{cm} = -\frac{R_D \Delta R_D}{2R_{SS} R_D} \\
A_{cm}(s) = -\frac{R_D \Delta R_D}{2Z_{SS} R_D} = -\frac{R_D \Delta R_D}{2} \left( \frac{1}{R_{SS}} + sC_{SS} \right) \\
= -\frac{R_D \Delta R_D}{2R_{SS} R_D} (1 + sC_{SS}R_{SS}) = A_{cm} \left( 1 + \frac{s}{\omega_z} \right) \\
\omega_z = \frac{1}{C_{SS}R_{SS}}
\]
Common-mode rejection ratio (CMRR):
- The frequency dependence of CMRR can be evaluated
- CMRR decreases at higher frequencies due to the pole of $A_d$ and the zero of $A_{cm}$
Active-Loaded MOS Differential Amplifier:

- Differential-mode operation:
  - Equivalent capacitances:
    \[ C_m = C_{gd1} + C_{db1} + C_{gd3} + C_{gs3} + C_{gs4} \]
    \[ C_L = C_{gd2} + C_{db2} + C_{gd4} + C_{db4} + C_x \]
  - Transconductance \((G_m)\):
    \[ V_{g3} = -\frac{g_m V_{id}}{g_m + sC_m} \]
    \[ I_{d4} = -g_{m4} V_{g3} = \frac{g_m g_{m1} V_{id}}{g_m + sC_m} = \frac{g_m V_{id}}{1 + sC_m / g_m} \]
    \[ I_o = I_{d4} + I_{d2} = \frac{g_m V_{id}}{1 + sC_m / g_m} + g_m V_{id} / 2 \]
    \[ G_m \equiv \frac{I_o}{V_{id}} = g_m \frac{1 + sC_m / (2g_m)}{1 + sC_m / g_m} \]
  - The pole and zero of \(G_m(s)\) are at very high frequencies
  - Minor pole and zero near unity-gain frequency
    \[ f_{p2} = \frac{g_m}{2\pi C_m} \approx \frac{f_t}{2} \]
    \[ f_z = \frac{2g_m}{2\pi C_m} \approx f_t \]
The transfer function of the amplifier:

\[ V_o = G_m V_{id} \frac{R_o}{1 + s C_L R_o} = G_m V_{id} \frac{r_{o2} || r_{o4}}{1 + s C_L (r_{o2} || r_{o4})} \]

\[ \frac{V_o}{V_{id}} = g_{m1} R_o \frac{1 + s C_m / (2 g_m)}{1 + s C_m / g_m} \frac{1}{1 + s C_m / g_m} = g_m (r_{o2} || r_{o4}) \frac{1 + s C_m / (2 g_m)}{1 + s C_m / g_m} \frac{1}{1 + s C_L (r_{o2} || r_{o4})} \]

- The dominant pole is typically

\[ f_{p1} = \frac{1}{2 \pi C_L R_o} = \frac{1}{2 \pi C_L (r_{o2} || r_{o4})} \]

Common-mode operation:
- By taking \( C_{SS} \) into account for the mid-band common-mode gain:

\[ A_{cm} \approx - \frac{1}{2 g_{m2} R_{ss}} \frac{R_{ss}}{1 + s C_{ss} R_{ss}} = - \frac{1 + s C_{ss} R_{ss}}{2 g_{m2} R_{ss}} \]

- Contributes to a zero at lower frequency

Common-mode rejection ratio (CMRR):
- CMRR decreases at higher frequencies due to the dominant pole of \( A_d(s) \) and the zero of \( A_{cm}(s) \)