TEM Waves and Transmission Lines

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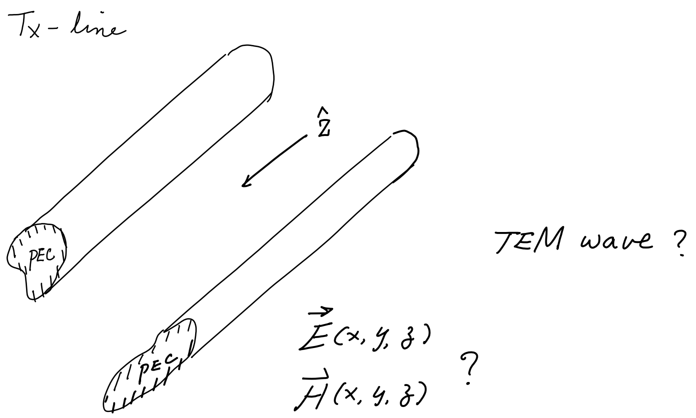
Textbook, 3.1, 3.6, 4.1, 4.7

Tx-Line & TEM wave



What will you learn?

- TEM 什麼時候存在，它向量E及H的一般解？
- 無損傳輸線的場論解？
- 如何用場論來說明無損傳輸線的各項物理量，含特徵阻抗、電能、磁能、功率、速度。
- 有限導電金屬時傳輸線的場論解。

TEM Waves and Transmission Lines

1.1 Plane TEM Waves

- Assumptions

  $e^{-j\omega t}$ dependence

  z axis: direction of wave propagation

  source-free: impressed sources, $\vec{J}_i = \rho_i = 0$

  Linear, isotropic, homogeneous region: (ε,μ)=scalar const.

$$\begin{cases} \vec{E} = \vec{E}_t + \hat{z}E_z = \\ \vec{H} = \vec{H}_t + \hat{z}H_z = \end{cases} \text{(transverse + longitudinal)} \begin{matrix}\text{components} \\ \text{parts}\end{matrix}$$

$\Rightarrow$ TEM waves: $E_z = H_z = 0,\ \vec{E} = \vec{E}_t,\ \vec{H} = \vec{H}_t$

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Maxwell Eqs. in TEM Waves

$$\begin{cases} \nabla \times \vec{E} = \nabla_t \times \vec{E}_t + \hat{z} \times \dfrac{\partial \vec{E}_t}{\partial z} = -j\omega\mu\vec{H}_t \\ \nabla \times \vec{H} = \nabla_t \times \vec{H}_t + \hat{z} \times \dfrac{\partial \vec{H}_t}{\partial z} = j\omega\varepsilon\vec{E}_t \\ \nabla \cdot \varepsilon\vec{E} = \varepsilon\nabla_t \cdot \vec{E}_t = 0 \\ \nabla \cdot \mu\vec{H} = \mu\nabla_t \cdot \vec{H}_t = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{z} \times \dfrac{\partial \vec{E}_t}{\partial z} = -j\omega\mu\vec{H}_t \\ \hat{z} \times \dfrac{\partial \vec{H}_t}{\partial z} = j\omega\varepsilon\vec{E}_t \end{cases}$$

$$\begin{cases} \nabla_t \times \vec{E}_t = 0 \\ \nabla_t \times \vec{H}_t = 0 \end{cases}$$

$$\begin{cases} \vec{E}_t(x,y,z) = g_1(z)\nabla_t\varphi_1(x,y) \\ \vec{H}_t(x,y,z) = g_2(z)\nabla_t\varphi_2(x,y) \end{cases}$$

$$\begin{cases} 0 = \nabla_t \cdot \vec{E}_t = g_1\nabla_t \cdot \nabla_t\varphi_1 \\ 0 = \nabla_t \cdot \vec{H}_t = g_2\nabla_t \cdot \nabla_t\varphi_2 \end{cases} \Rightarrow \boxed{\nabla_t^2\varphi_{1,2} = 0}$$

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TEM Wave Propagation

$$\begin{cases} \hat{z}\times\dfrac{\partial \vec{E}_t}{\partial z} = -j\omega\mu\vec{H}_t \\ \hat{z}\times\dfrac{\partial \vec{H}_t}{\partial z} = j\omega\varepsilon\vec{E}_t \end{cases} \Longrightarrow \hat{z}\times\left(\hat{z}\times\frac{\partial^2 \vec{E}_t}{\partial z^2}\right) = -j\omega\mu\left(\hat{z}\times\frac{\partial \vec{H}_t}{\partial z}\right) = -j\omega\mu\left(j\omega\varepsilon\vec{E}_t\right)$$

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\vec{E}_t = 0; \quad k^2 = \omega^2\mu\varepsilon$$

$$\underset{\vec{E}_t = g_1(z)\nabla_t\varphi_1(x,y)}{\Longrightarrow} \left(\frac{d^2}{dz^2}+k^2\right)g_1(z) = 0$$

$$\boxed{g_1(z) = A_{\pm}e^{\mp jkz}\ \left(\cdot e^{j\omega t}\right)}$$

wave propagating along $\pm z$ direction

$$g_2(z) = B_{\pm}e^{\mp jkz}$$

$$\begin{cases} k = \omega\sqrt{\mu\varepsilon} = \dfrac{\omega}{v_c} = \dfrac{2\pi f}{v_c} = \dfrac{2\pi}{\lambda} = \text{propagation constant} \\ v_c = \dfrac{1}{\sqrt{\mu\varepsilon}} = \text{light velocity in the medium } (\mu,\varepsilon) \\ \lambda = \dfrac{2\pi}{k} = \text{ wavelength in the media corresponding to } f \end{cases}$$

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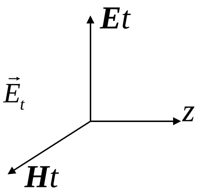
Relation between E and H

$$\begin{cases} \vec{E}_t(x,y,z) = A_{\pm}e^{\mp jkz}\nabla_t\varphi_1(x,y) \\ \vec{H}_t(x,y,z) = B_{\pm}e^{\mp jkz}\nabla_t\varphi_2(x,y) \end{cases}$$

$$\hat{z}\times\frac{\partial \vec{E}_t}{\partial z} = -j\omega\mu\vec{H}_t \Rightarrow \vec{H}_t = \frac{\mp jk}{-j\omega\mu}\hat{z}\times\vec{E}_t = \pm\frac{1}{\eta}\hat{z}\times\vec{E}_t$$

$$\vec{E}_t = \mp\eta\hat{z}\times\vec{H}_t; \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Intrinsic impedance



Summary

$$\begin{cases} \vec{E}_t(x,y,z) = A_{\pm}\nabla_t\varphi(x,y)e^{\mp jkz}; \quad \vec{H}_t = \pm\dfrac{1}{\eta}\hat{z}\times\vec{E}_t \\ \nabla_t^2\varphi(x,y) = 0; \quad \text{Laplace equation} \\ k = \omega\sqrt{\mu\varepsilon}; \quad \eta = \sqrt{\dfrac{\mu}{\varepsilon}} \end{cases}$$

➔Field in this form satisfy source-free Maxwell equation!

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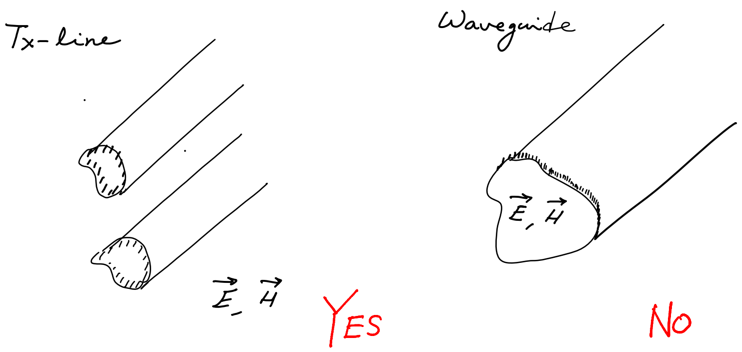
Question

- Can TEM wave solution be supported by the structures?



1.2 Transmission Lines

- Conventional transmission lines
  - Two-wire lines, multi-wire line, coaxial lines
- Ideal (lossless) line
  - Fundamental mode, $f_c = 0$
    - TEM mode, $E_z = H_z = 0$,
    - Circuit theory & Field theory
  - Higher-order mode, $f_c \neq 0$
    - TE modes, $E_z = 0$, $H_z \neq 0$
    - TM modes, $E_z \neq 0$, $H_z = 0$
    - Hybrid modes, $E_z \neq 0$, $H_z \neq 0$
    - Field theory

Transmission Lines -2

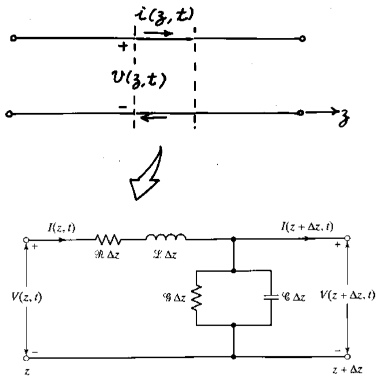
- Practical (lossy) line ~ ideal (lossless) line

  } Hybrid mode

  Quasi-TEM mode ⟵ TEM mode

- Transmission-line theory
  - Circuit theory
    - Given distributed line parameters *C, L, G, R*
    - *V(z), I(z),* Z_c, γ
  - *Field theory*
    - Given line conductor configuration
    - Line parameters *C, L, G, R*

1.3 Circuit Theory for Lossy Tx-Lines



$$\begin{cases} -\dfrac{\partial v}{\partial z} = R \cdot i + L \cdot \dfrac{\partial i}{\partial t} \\ -\dfrac{\partial i}{\partial z} = G \cdot v + C \cdot \dfrac{\partial v}{\partial t} \end{cases}$$

$$\Downarrow \quad v(z,t) = \mathcal{R}_e[V(z)e^{j\omega t}], \quad i(z,t) = \mathcal{R}_e[I(z)e^{j\omega t}]$$

$$\begin{cases} -\dfrac{dV}{dz} = (R + j\omega L) \cdot I \\ -\dfrac{dI}{dz} = (G + j\omega C) \cdot V \end{cases}$$

Solution of Tx-Line Equations

$$\frac{d^2V}{dz^2} = \gamma^2 V \quad ; \quad \gamma \equiv \sqrt{(R+j\omega L)\cdot(G+j\omega C)} ;$$

propagation constant.

$$\begin{cases} V(z) = V^+(z) + V^-(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ I(z) = I^+(z) + I^-(z) = \frac{1}{Z_0}\left(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right) ; \end{cases}$$

$$Z_0 \equiv \frac{V^+}{I^+} = \frac{V^-}{-I^-} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

characteristic impedance

Low-loss line $(R \ll \omega L \; ; \; G \ll \omega C)$

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\cdot\sqrt{1+\frac{R}{j\omega L}}\cdot\sqrt{1+\frac{G}{j\omega C}} \simeq j\omega\sqrt{LC} + \frac{1}{2}\left(R\cdot\sqrt{\frac{C}{L}} + G\cdot\sqrt{\frac{L}{C}}\right)$$

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{LC}} \quad ; \quad Z_0 = \sqrt{\frac{L}{C}}\cdot\sqrt{\frac{1+R/j\omega L}{1+G/j\omega C}} \simeq \sqrt{\frac{L}{C}}\left(1+\frac{1}{2j\omega}\left(\frac{R}{L}-\frac{G}{C}\right)\right).$$

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Summary

$$\begin{cases} \frac{\partial V}{\partial z} = -(R+j\omega L)I \\ \frac{\partial I}{\partial z} = -(G+j\omega C)V \end{cases} ;$$

R	series resistance	
L	inductance	= per unit length
C	capacitance	
G	shunt conductance	

Wave equations:

$$\left(\frac{\partial^2}{\partial z^2} - \gamma^2\right)\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

Relation between $V(z)$ and $I(z)$

$$\begin{cases} V_\pm(z) = V_\div e^{\mp\gamma z} = \pm Z_c I_\pm(z) \\ I_\pm(z) = I_\div e^{\mp\gamma z} = \pm Y_c V_\pm(z) \end{cases}; \quad Z_c = \frac{1}{Y_c} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

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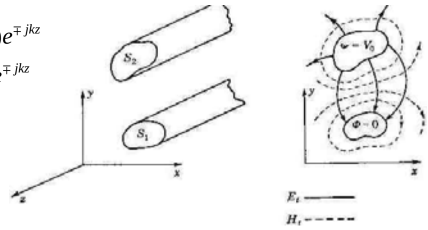
1.4 Field Theory for Lossless Tx_Lines

- Conductors S_1, S_2: PEC with uniform cross-section
  Medium: linear, isotropic, homogeneous, lossless
- Fundamental TEM mode

$$\begin{cases} \vec{E}_t(x,y,z) = -\nabla_t \varphi(x,y) e^{\mp jkz} = \vec{e}(x,y) e^{\mp jkz} \\ \vec{H}_t(x,y,z) = (\mp 1/\eta)\hat{z}\times\vec{E}_t = \pm\vec{h}(x,y)e^{\mp jkz} \end{cases}$$

$$\begin{cases} \vec{e}(x,y) = -\nabla_t \varphi(x,y) \\ \vec{h}(x,y) = \frac{1}{\eta}\hat{z}\times\vec{e}(x,y) \end{cases}; \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$



$$\because \nabla_t \times \vec{e}(x,y) = 0 \Rightarrow$$

$$-\int_{S_1}^{S_2} \vec{e}\cdot d\vec{\ell} = \int_{S_1}^{S_2} \nabla_t \varphi(x,y)\cdot d\vec{\ell} \underset{\text{indep. of path}}{=} \varphi\big|_{S_2} - \varphi\big|_{S_1} = V_0$$

Electric field wave $\qquad$ Unique voltage wave

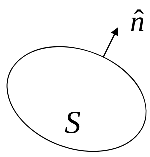
$$\vec{E}(x,y,z) = \vec{e}(x,y)e^{-jkz} \quad \Rightarrow \quad V(z) = V_0 e^{-jkz}$$

Field Theory for Lossless Tx-Lines -2

On PEC (S_1 & S_2) surfaces: $\hat{n}\times\vec{E}_t = \hat{n}\cdot\vec{H}_t = 0$

$\rho_s = \varepsilon\hat{n}\cdot\vec{E}_t =$ surface charge

$\vec{J}_s = \hat{n}\times\vec{H}_t =$ surface current



$$\left|\vec{J}_s\right| = \left|\hat{n}\times\vec{H}_t\right| \underset{\vec{H}_t = Y\hat{z}\times\vec{E}_t}{=} \frac{1}{\eta}\left|\hat{n}\cdot\vec{E}_t\right| = \frac{\rho_s}{\eta\varepsilon} \underset{\eta=\sqrt{\frac{\mu}{\varepsilon}}}{=} \frac{\rho_s}{\sqrt{\mu\varepsilon}} = v_c \rho_s$$

Total axial current I_0 on S_2 :

$$I_0 = \int_{S_2} \left|\vec{J}_s\right| d\ell = v_c \int_{S_2} \rho_s d\ell = v_c Q$$

Q = total charge on S_2 per unit length

$$v_c = \frac{1}{\sqrt{\mu\varepsilon}} = \text{light velocity in medium } (\varepsilon, \mu)$$

Magnetic field wave $\qquad$ Unique current wave

$$\vec{H}(x,y,z) = \vec{h}(x,y)e^{-jkz} \quad \Rightarrow \quad I(z) = I_0 e^{-jkz}$$

Characteristic Impedance Z_c

Z_c = characteristic impedance of tx-line

$$\equiv \left|\frac{\text{voltage between conductors}}{\text{total current on conductor}}\right| \quad (\text{for } \pm z\text{-propagating wave})$$

$$= \frac{V_0}{I_0} = \frac{V_0}{v_c Q} = \frac{1}{v_c C} = \frac{\sqrt{\mu\varepsilon}}{C} = \frac{\varepsilon}{C}\eta;$$

where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ = TEM wave impedance

$C \equiv \frac{Q}{V_0}$ =static capacitacne between S_1 & S_2 per unit length

$$\sim \begin{cases} \vec{e}(x,y) = -\nabla_t \varphi(x,y) \\ \nabla_t^2 \varphi(x,y) = 0 \end{cases} ; \text{ static field}$$

$\therefore$ Static $C \Rightarrow$ Characteristic impedance Z_c

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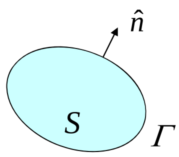
Lemma

$$\nabla_t \cdot \left(\phi \nabla_t \varphi\right) = \nabla_t \phi \cdot \nabla_t \varphi + \phi \nabla_t^2 \varphi$$



2nd divergence theorem

$$\iint_S \left(\nabla_t \phi \cdot \nabla_t \varphi + \phi \nabla_t^2 \varphi\right) dS = \iint_S \nabla_t \cdot \left(\phi \nabla_t \varphi\right) dS = \oint_\Gamma \phi \nabla_t \varphi \cdot \hat{n}\, d\ell$$

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Time-average Electric Energy W_e per unit length

$$W_e = \frac{\varepsilon}{4}\iint_{S_\infty} \vec{E}_t \cdot \vec{E}_t^* dS \int_0^1 dz$$

$$\underset{\vec{E}_t = -\nabla_t \varphi e^{-jkz}}{=} \frac{\varepsilon}{4}\iint_{S_\infty} \nabla_t\varphi \cdot \nabla_t\varphi dS$$

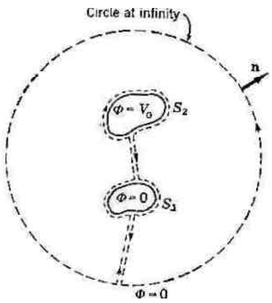
$$= \frac{\varepsilon}{4}\iint_{S_\infty} \left(\nabla_t \cdot (\varphi\nabla_t\varphi) - \varphi\nabla_t^2\varphi\right) dS \quad (\nabla_t^2\varphi \to 0)$$

$$= \frac{\varepsilon}{4}\left[\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_\infty}\right]\varphi\nabla_t\varphi \cdot \hat{n}' d\ell$$

$$\varphi|_{S_1} = 0 \qquad \varphi|_{S_2} = V_0 \qquad \varphi\nabla_t\varphi \to 1/r^2 \ \text{since } Q_1 = -Q_2$$

$$\hat{n}' \cdot \nabla_t\varphi = -\hat{n}\cdot\nabla_t\varphi = \hat{n}\cdot\vec{e} = \rho_s/\varepsilon$$

$$= \frac{\varepsilon}{4}\int_{\Gamma_2} V_0 \nabla_t\varphi \cdot \hat{n}' d\ell = \frac{\varepsilon}{4} V_0 \int_{\Gamma_2} \frac{\rho_s}{\varepsilon} d\ell = \frac{1}{4} V_0 Q \underset{Q = CV_0}{=} \frac{1}{4} C V_0^2$$



Time-average Magnetic Energy W_m per unit length

$$W_m = \frac{\mu}{4}\iint_{S_\infty} \vec{H}_t \cdot \vec{H}_t^* dS \int_0^1 dz \underset{\vec{H}_t = Y\hat{z}\times\vec{E}_t}{=} \frac{\mu}{4}\cdot\frac{\varepsilon}{\mu}\iint_{S_\infty} \vec{E}_t \cdot \vec{E}_t^* dS = W_e$$

Def.: Inductance L per unit length: $W_m = \frac{1}{4} L I_0^2$

then $L I_0^2 = 4W_m = 4W_e = CV_0^2 \qquad \therefore Z_c = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}}$

$$L = CZ_c^2 = C\left(\frac{\sqrt{\mu\varepsilon}}{C}\right)^2 = \frac{\mu\varepsilon}{C}$$

$$\therefore LC = \mu\varepsilon; \quad v_c = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{LC}}$$

Line configuration → static C → L

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Time-average Power Flow P along Tx-Line

$$P=\frac{1}{2}\mathrm{Re}\left[\iint_{S_\infty}\vec{E}_t\times\vec{H}_t^*\cdot\hat{z}\,dS\right]\underset{\vec{H}_t=Y\hat{z}\times\vec{E}_t}{=}\frac{1}{2\eta}\iint_{S_\infty}\vec{E}_t\cdot\vec{E}_t^*\,dS=\frac{2}{\varepsilon\eta}W_e$$

$$\underset{Z_c=\frac{\varepsilon}{C}Z}{=}\frac{2}{CZ_c}W_e\underset{W_e=\frac{1}{4}CV_0^2}{=}\frac{V_0^2}{2Z_c}\underset{V_0=Z_cI_0}{=}\frac{1}{2}V_0I_0=\frac{1}{2}\mathrm{Re}\left[V(z)I^*(z)\right]$$

Energy velocity v_e

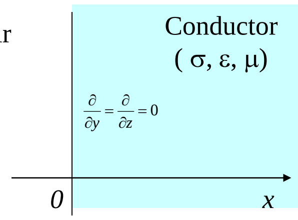
$$P=\frac{2}{\varepsilon\eta}W_e=\frac{1}{\varepsilon\eta}\left(W_e+W_m\right)\underset{\eta=\sqrt{\frac{\mu}{\varepsilon}}}{=}\frac{1}{\sqrt{\mu\varepsilon}}\left(W_e+W_m\right)$$

$$\therefore\ v_e\equiv\frac{P}{\left(W_e+W_m\right)}=\frac{1}{\sqrt{\mu\varepsilon}}=v_c$$

1.5 Power Loss in Planar Conductor

- Planar conductor

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial z}=0$$



Maxwell equations:

$$\begin{cases}\nabla\times\vec{E}=-j\omega\mu\vec{H}\\ \nabla\times\vec{H}=\vec{J}_i+\sigma\vec{E}+j\omega\varepsilon\vec{E}\underset{\text{good conductor}}{\cong}\sigma\vec{E}\end{cases}$$

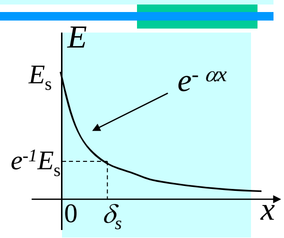
$$\nabla\times\nabla\times\vec{E}=-j\omega\mu\nabla\times\vec{H}=-j\omega\mu\sigma\vec{E};$$

$$\nabla\times\nabla\times\vec{E}=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}$$

Wave eq.:

$$\left(\nabla^2-j\omega\mu\sigma\right)\vec{E}=\left(\frac{d^2}{dx^2}-\gamma^2\right)\vec{E}=\vec{0};\quad \gamma^2=j\omega\mu\sigma$$

Plane Wave Field



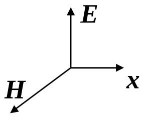
Electric filed $\vec{E}$:

$$\vec{E}(x) = \vec{E}_s e^{-\gamma x} = E_s e^{-\alpha x} e^{-j\beta x}$$

Propagation constant γ :

$$\gamma = \sqrt{j\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = \alpha + j\beta$$

Skin depth δ_s : $e^{-\alpha x}\Big|_{x=\delta_s} = e^{-1} \Rightarrow \delta_s \equiv \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$



Magnetic filed $\vec{H}$:

$$\vec{H}(x) = \frac{1}{-j\omega\mu}\nabla\times\vec{E} \underset{\vec{E}=\vec{E}_s e^{-\gamma x}}{=} \frac{-\gamma\hat{x}}{-j\omega\mu}\times E_s e^{-\alpha x}e^{-j\beta x} = \frac{1}{Z}\hat{x}\times\vec{E}$$

Wave impedance $Z \equiv \frac{|\mathrm{E}|}{|\mathrm{H}|} = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$

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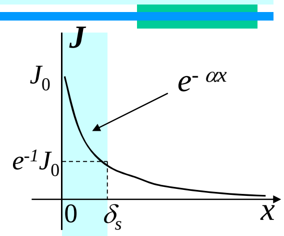
Surface Impedance



Current density $\vec{J}$ in conductor :

$$\vec{J}(x) = \sigma\vec{E} = \sigma\vec{E}_s e^{-\gamma x}$$

Surface current = current per unit width :

$$\vec{J}_s = \int_0^\infty \vec{J}dx\int_0^1 dy = \frac{\sigma}{\gamma}\vec{E}_s = \frac{1}{Z_s}\vec{E}_s$$

Surface impedance of the conductor :

$$Z_s = R_s + jX_s \equiv \frac{|\vec{E}_s|}{|\vec{J}_s|} = \frac{\gamma}{\sigma} \underset{\gamma=\sqrt{j\omega\mu\sigma}}{=} \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = Z$$

Surface resistance & reactance: $R_s = X_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \frac{1}{\sigma\delta_s}$ (Ω)

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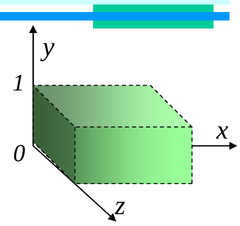
Power Loss P_{LC} in Conductor per unit area

$$P_{LC} = \frac{1}{2}\mathrm{Re}\left[\iint \vec{E}_s \times \vec{H}_s^* \cdot \hat{x}dydz\right]$$

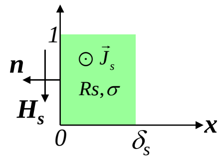
$$\underset{\vec{H}=\frac{1}{Z_s}\hat{x}\times\vec{E}}{=} \frac{1}{2}\cdot\frac{\mathrm{Re}[Z_s]}{|Z_s|^2}\left|\vec{E}_s\right|^2 \int_0^1 dy \int_0^1 dz = \frac{1}{2}R_s\left|\vec{J}_s\right|^2$$

surface resistance $R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$

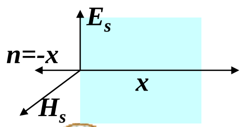


Surface current $\vec{J}_s$:

$$\vec{J}_s = \frac{1}{Z_s}\vec{E}_s = \frac{1}{Z_s}\left[Z(-\hat{x})\times\vec{H}_s\right] \underset{\hat{n}=-\hat{x}}{=} \hat{n}\times\vec{H}_s$$



$\vec{H}_s$ = magnetic field at conductor surface



P_{LC} per unit surface area: $P_{LC} = \frac{1}{2}R_s\left|\hat{n}\times\vec{H}_s\right|^2$

1.6 Field Theory for Lossy Tx-Lines

- two conductors with small loss

- Lossy surrounding medium (perfect conductors)

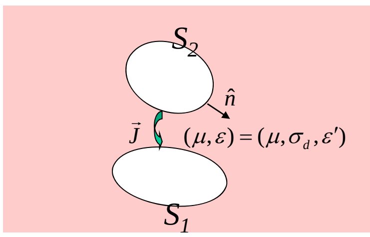
Current density $\vec{J}$ in lossy medium

$$\vec{J} = (\sigma_d + j\omega\varepsilon')\vec{E} \equiv j\omega\varepsilon\vec{E};$$

Effective permittivity of the medium

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon'\left(1 - j\frac{\sigma_d}{\omega\varepsilon'}\right) = \varepsilon'(1 - j\tan\delta_\ell)$$

Loss tangent: $\tan\delta_\ell = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma_d}{\omega\varepsilon'}$



PEC S_1 & S_2 → Fundamental mode = TEM mode

Total p.u.l. shunt current I_s from S_2 to S_1

$$I_s = \int_{S_2}\vec{J}\cdot\hat{n}d\ell\int_0^1 dz = (\sigma_d + j\omega\varepsilon')\int_{S_2}\vec{E}\cdot\hat{n}d\ell \underset{\varepsilon'\vec{E}\cdot\hat{n}=\rho_s}{=} \frac{(\sigma_d + j\omega\varepsilon')}{\varepsilon'}\int_{S_2}\rho_s d\ell$$

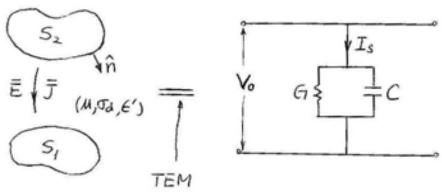
$$= \frac{(\sigma_d + j\omega\varepsilon')}{\varepsilon'}Q \underset{Q=CV_0}{=} \frac{(\sigma_d + j\omega\varepsilon')}{\varepsilon'}CV_0 = (G + j\omega C)V_0$$

Eq. Circuit and Line Parameters

Shunt p.u.l. conductance G:

$$G=\frac{\sigma_d}{\varepsilon'}C=\frac{\omega\varepsilon''}{\varepsilon'}C;\ \text{or}\ \frac{G}{\omega C}=\tan\delta_\ell$$

Line configuration $\rightarrow$ static $C \rightarrow G$



Propagation constant γ:

$$\gamma = j\omega\sqrt{\mu\varepsilon}=\sqrt{(j\omega)^2\mu\varepsilon'(1-j\tan\delta_\ell)}$$

$$\underset{\text{TEM: }\mu\varepsilon'=LC}{=}\sqrt{(j\omega)^2 LC\left(1-j\frac{G}{\omega C}\right)}=\underbrace{\sqrt{j\omega L(G+j\omega C)}}_{\text{Field theory}} \rightleftarrows \underbrace{\sqrt{(\cancel{R}+j\omega L)(G+j\omega C)}}_{\text{Circuit theory}}$$

($R \rightarrow 0$)

Characteristic impedance Z_c :

$$|\vec J_s|=|\hat n\times\vec H_t| \underset{TEM}{=} \sqrt{\frac{\varepsilon}{\mu}}|\vec E_t| \underset{PEC}{=}\sqrt{\frac{\varepsilon}{\mu}}|\hat n\cdot\vec E_t|=\sqrt{\frac{\varepsilon}{\mu}}\frac{\rho_s}{\varepsilon'}$$

$$I_0=\int_{S_2}|\vec J_s|\,d\ell=\sqrt{\frac{\varepsilon}{\mu}}\int_{S_2}\frac{\rho_s}{\varepsilon'}\,d\ell=\sqrt{\frac{\varepsilon}{\mu}}\frac{Q}{\varepsilon'}=V_0\frac{C}{\varepsilon'}\sqrt{\frac{\varepsilon}{\mu}}$$

$$\therefore Z_c=\frac{V_0}{I_0}=\frac{\varepsilon'}{C}\sqrt{\frac{\mu}{\varepsilon}}=\frac{\sqrt{\mu\varepsilon'}}{C\sqrt{1-j\tan\delta_\ell}}=\frac{\sqrt{LC}}{C\sqrt{1-jG/\omega C}}=\underbrace{\sqrt{\frac{j\omega L}{G+j\omega C}}}_{\text{Field theory}} \rightleftarrows \underbrace{\sqrt{\frac{\cancel{R}+j\omega L}{G+j\omega C}}}_{\text{Circuit theory}}$$

Tx-Lines with Non-perfect Conductors

- General case
  - Conductors: finite conductivity σ
  - Conductor loss → Poynting vector comp. into conductors → $E_z \neq 0$
  - $E_z \neq 0$ → $J_z >> |\boldsymbol{J}_t|$ → $H_z \sim 0$
  - Boundary value problem: difficult to solve
- Perturbation technique

Quasi-TEM approximation: $|\vec E_t| \gg E_z \doteq 0,\ |\vec H_t| \gg H_z \doteq 0$

Lossy-line fields → Ideal-line fields

Hybrid mode | Small losses | TEM mode

Perturbed Quasi-TEM field ← Unperturbed fields

Series Resistance R per unit length

Conductor power loss P_{LC} per unit length:

$$P_{LC} = \frac{1}{2} R_s \int_{S_1+S_2} \left|\vec{J}_s\right|^2 d\ell \quad \text{(planar approximation)}$$

$$R_s = \sqrt{\frac{\omega\mu'}{2\sigma}} = \frac{1}{\sigma\delta_s}; \quad \delta_s = \sqrt{\frac{2}{\omega\mu'\sigma}}$$

$$\vec{J}_s = \hat{n} \times \vec{H}_s; \quad \vec{H}_s \doteq \text{unperturbed magnetic field at conductor surface}$$

Series resistance R per unit length:

$$\frac{1}{2} I_0^2 R = \frac{1}{2} R \left(\int_{S_2} \left|\vec{J}_s\right| d\ell \right)^2 \equiv P_{LC} = \frac{1}{2} R_s \int_{S_1+S_2} \left|\vec{J}_s\right|^2 d\ell$$

$$R = \left(R_s \int_{S_1+S_2} \left|\vec{J}_s\right|^2 d\ell \right) \Big/ \left(\int_{S_2} \left|\vec{J}_s\right| d\ell \right)^2$$

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Attenuation Constant α

$$\vec{E}_t = \vec{E}_0 e^{-\gamma z} = \vec{E}_0 e^{-\alpha z} e^{-j\beta z}$$

Time-average power flow: $P = \frac{1}{2} \mathrm{Re}\left[\iint \vec{E}_t \times \vec{H}_t \cdot \hat{z} dS \right] = P_0 e^{-2\alpha z}$

Power loss P_L per unit length: P_L = rate of decrease of power $= -\frac{\Delta P}{\Delta z} = -\frac{dP}{dz} = 2\alpha P$

P_L = (conductor + dielectric) losses = $P_{LC} + P_{LD}$

$$= \frac{1}{2} R I_0^2 + \frac{1}{2} \sigma_d \iint_{S\infty} \vec{E}_t \cdot \vec{E}_t^* dS \underset{\sigma_d = \omega\varepsilon'' = G\varepsilon'/C}{=} \frac{1}{2} R I_0^2 + \frac{1}{2} G V_0^2$$

$$P_{LC} = \frac{1}{2} R I_0^2 = \frac{1}{2} R_s \int_{S_1+S_2} \left|\vec{J}_s\right|^2 d\ell \underset{?}{\equiv} \frac{1}{2} \omega\mu'' \iint_{S\infty} \vec{H}_t \cdot \vec{H}_t^* dS$$

Attenuation constant α:

$$\alpha = \frac{P_L}{2P} = \frac{\frac{1}{2} R I_0^2 + \frac{1}{2} G V_0^2}{V_0 I_0} = \frac{R}{2} \cdot \frac{I_0}{V_0} + \frac{G}{2} \cdot \frac{V_0}{I_0} = \frac{1}{2}\left(R Y_c + G Z_c \right)$$

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Electric Characteristics & Eq. Circuit

Propagation constant γ:

$$\gamma = \alpha + j\beta \underset{\text{Quasi-TEM}}{\doteq} jk = j\omega\sqrt{\mu\varepsilon} = j\omega\sqrt{\mu'\varepsilon'}\sqrt{\frac{(\mu' - j\mu'')(\varepsilon' - j\varepsilon'')}{\mu'\varepsilon'}}$$

$$\underset{\text{TEM: } \mu\varepsilon' = LC}{=} \underbrace{j\omega\sqrt{LC}\sqrt{\left(1 - j\frac{R}{\omega L}\right)\left(1 - j\frac{G}{\omega C}\right)}}_{\text{Field theory}} \rightleftarrows \underbrace{\sqrt{(R + j\omega L)(G + j\omega C)}}_{\text{Circuit theory}}$$

Characteristic impedance Z_c :

$$Z_c \doteq \frac{\varepsilon'}{C}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sqrt{\mu'\varepsilon'}}{C}\sqrt{\frac{\mu\varepsilon'}{\varepsilon\mu'}} = \underbrace{\frac{\sqrt{LC}}{C}\sqrt{\frac{1 - jR/\omega L}{1 - jG/\omega C}}}_{\text{Field theory}} \rightleftarrows \underbrace{\sqrt{\frac{R + j\omega L}{G + j\omega C}}}_{\text{Circuit theory}}$$

1.7 General Two-Conductor Tx-Lines

- Two parallel PEC A, B with uniform cross-sections
- Linear, isotropic, homogeneous medium
- C (capacitance), G (conductance), L (inductance) per unit length satisfy:

$$\frac{G}{C} = \frac{\sigma_d}{\varepsilon'}; \quad LC = \mu\varepsilon'$$

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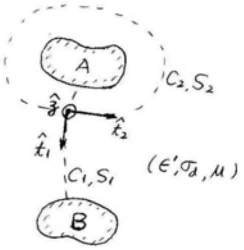
Relation between Parameters

$$C=\frac{Q}{V}=\frac{\varepsilon'\oint_{S_2}\vec{E}\cdot d\vec{S}}{\int_{C_1:A}^{B}\vec{E}\cdot d\vec{\ell}}=\frac{\varepsilon'\oint_{C_2}\vec{E}\cdot\hat{t}_1 d\ell\int_0^1 dz}{\int_{C_1}\vec{E}\cdot\hat{t}_1 d\vec{\ell}}$$

$$G=\frac{I_c}{V}=\frac{\sigma_d\oint_{S_2}\vec{E}\cdot d\vec{S}}{\int_{C_1:A}^{B}\vec{E}\cdot d\vec{\ell}}=\frac{\sigma_d\oint_{C_2}\vec{E}\cdot\hat{t}_1 d\ell\int_0^1 dz}{\int_{C_1}\vec{E}\cdot\hat{t}_1 d\vec{\ell}};$$

$$\therefore\frac{G}{C}=\frac{\sigma_d}{\varepsilon'}$$



$$L=\frac{\Psi}{I}=\frac{\mu\int_{S_1}\vec{H}\cdot d\vec{S}}{\oint_{C_2}\vec{H}\cdot d\vec{\ell}}=\frac{\mu\int_{C_1:A}^{B}\vec{H}\cdot\hat{t}_2 d\ell\int_0^1 dz}{\oint_{C_2}\vec{H}\cdot d\vec{\ell}}\underset{\vec{H}=Y\hat{z}\times\vec{E}}{=}\frac{\mu\int_{C_1:A}^{B}\vec{E}\cdot\hat{t}_1 d\vec{\ell}}{\oint_{C_2}\vec{E}\cdot\hat{t}_1 d\ell};$$

$$\therefore LC=\mu\varepsilon$$

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Mechanism in Conductor

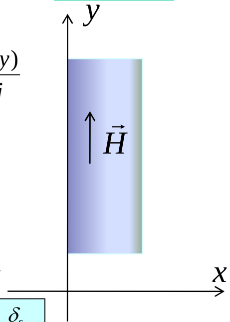
Near conductor boundary ($x\geq 0$)

$$\vec{H}\approx\hat{y}H_0(y)e^{-(1+j)x/\delta_s}$$

$$\vec{J}\approx\hat{z}J_0(y)e^{-(1+j)x/\delta_s}\Rightarrow\quad J_s(y)=\int_0^\infty J_0(y)e^{-(1+j)x/\delta_s}dx=\frac{\delta_s J_0(y)}{1+j}$$

$$\vec{E}=\vec{J}/\sigma\qquad\text{as }\sigma\to\infty,\ J_s(y)=-H_0(y)$$

$$\therefore J_0(y)=-\frac{1+j}{\delta_s}H_0(y)$$



Stored magnetic energy inside conductor

$$\tfrac{1}{4}L_{\text{int}}|I|^2=W_m=\tfrac{\mu}{4}\int\int_0^\infty\vec{H}\cdot\vec{H}^*dxdy=\tfrac{\mu}{4}\int|H_0(y)|^2dy\cdot\tfrac{\delta_s}{2}$$

$$=\boxed{W_m\text{ stored in }0\leq x\leq\tfrac{\delta_s}{2}\text{ when PEC is placed at }x=\tfrac{\delta_s}{2}}$$

Dissipated power inside conductor

$$\tfrac{1}{2}R_{ac}|I|^2=\tfrac{1}{2\sigma}\int\int_0^\infty\vec{J}\cdot\vec{J}^*dxdy=\tfrac{1}{2\sigma}\int|J_0(y)|^2dy\cdot\tfrac{\delta_s}{2}$$

$$=\tfrac{1}{2\sigma}\cdot\tfrac{2}{\delta_s^2}\cdot\int|H_0(y)|^2dy\cdot\tfrac{\delta_s}{2}=\tfrac{\omega\mu}{2}\int|H_0(y)|^2dy\cdot\tfrac{\delta_s}{2}=\tfrac{\omega}{2}L_{\text{int}}|I|^2\Rightarrow\boxed{R_{ac}\approx\omega L_{\text{int}}}$$

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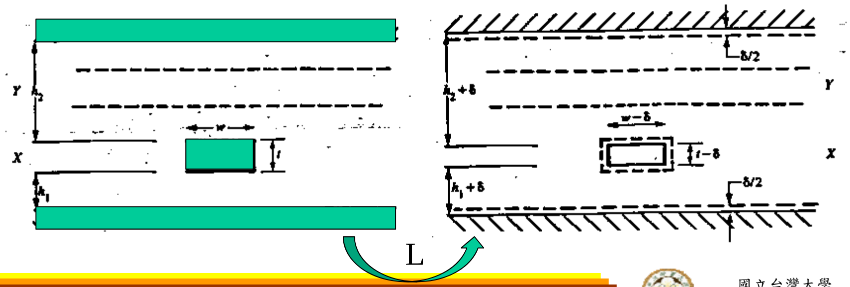
Incremental Inductance Rule (Wheeler, 1942)

- At high frequencies, say, δ < 0.1t, current crowds to conductor surface within a skin depth of δ.

$$R = R_s(f) = \omega \frac{\delta}{2} \cdot \frac{\partial L}{\partial n}$$

$$L = L_{ext} + L_{\mathrm{int}}(f) = L_{ext} + \frac{\delta}{2} \cdot \frac{\partial L}{\partial n}$$



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Related Further Reading

- R. B. Wu and J. C. Yang, "Boundary integral equation formulation of skin effect problems in multiple transmission lines," *IEEE T-MAG*, vol. 25, pp. 3013-3015, July 1989
- K. M. Coperich, A. E. Ruehli, and A. Cangellaris, “Enhanced skin effect for partial-element equivalent-circuit (PEEC) models,” IEEE T-MTT, vol. 48, pp. 1435-1442, Sept. 2000.

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Did you learn

- TEM mode expression for general tx-line.
- Equivalent tx-line circuits for V & I, the R, L, C, G meaning, and general solution for V and I.
- Governing PDE and BC for electric potentials, and how it relates to TEM mode solution.
- Field theory for lossless tx-line, e.g., surface current, charge, stored e/m energy, power, ..
- Approximate e/m/J field in lossy conductors.
- Properties and relation among R, L, C, G.
- Wheeler's incremental inductance and resistance formula

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