Multivariate Statistical Analysis Final Solution
January 19, 2007

1. Follow Result 6.1 (p. 274),

$$T^2 = n\bar{\mathbf{d}}'\mathbf{S}_d^{-1}\bar{\mathbf{d}} = 30[-1 \quad 1]\frac{1}{36}\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{5\times 17}{6} = 14.167$$

$$\frac{(n-1)p}{n-p}F_{p,n-p}(\alpha) = \frac{(30-1)\times 2}{30-2}F_{2,28}(0.1) = \frac{29\times 2}{28}\times 2.50 = 5.179$$

$$T^2 > \frac{(n-1)p}{n-p}F_{p,n-p}(\alpha)$$

$\therefore$ Treatment difference does exist for 90% significance level

90% Bonferroni individual mean differences :

$$\delta_i = \bar{d}_i \pm t_{n-1}(\frac{\alpha}{2p})\sqrt{\frac{s_{di}^2}{n}} = \bar{d}_i \pm t_{29}(0.025)\sqrt{\frac{s_{di}^2}{n}}$$

$$\delta_1 = -1 \pm 2.045\sqrt{\frac{8}{30}} = -1 \pm 1.056, \text{ i.e., } (-2.056, 0.056)$$

$$\delta_2 = 1 \pm 2.045\sqrt{\frac{5}{30}} = 1 \pm 0.835, \text{ i.e., } (0.165, 1.835)$$

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2. Start from Equation (6-21) in p. 284,

$$\mathbf{S}_{\text{pooled}} = \frac{n_1 - 1}{n_1 + n_2 - 2}\mathbf{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\mathbf{S}_2 = \frac{15}{30}\begin{bmatrix} 6 & 1 \\ 1 & 4 \end{bmatrix} + \frac{15}{30}\begin{bmatrix} 10 & 3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

Equation (6 - 23) in p. 284,

$$T^2 = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}\mathbf{S}_{\text{pooled}}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = [-1 \quad 1]\frac{16\times 16}{32}\frac{1}{36}\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{2}{9}\times 17 = 3.778$$

Result 6.2 in p. 285,

$$\frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1}F_{p,n_1+n_2-p-1}(\alpha) = \frac{30\times 2}{29}F_{2,29}(0.1) = \frac{30\times 2}{29}\times 2.50 = 5.172$$

$$T^2 < \frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1}F_{p,n_1+n_2-p-1}(\alpha)$$

$\therefore$ accept $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ at the 90% significance level

90% Bonferroni simultaneous confidence intervals for individual mean differences:

Follow the equation in p. 290,

$$\mu_{1i}-\mu_{2i}: \quad (\bar{x}_{1i}-\bar{x}_{2i}) \pm t_{n_1+n_2-2}(\frac{\alpha}{2p})\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)s_{ii,\text{pooled}}}$$

$$\mu_{11}-\mu_{21}: \quad -1 \pm t_{30}(0.025)\sqrt{\frac{1}{8}\times 8} = -1 \pm 2.042, \text{i.e., (-3.042, 1.042)}$$

$$\mu_{12}-\mu_{22}: \quad 1 \pm t_{30}(0.025)\sqrt{\frac{1}{8}\times 5} = 1 \pm 1.614, \text{i.e., (-0.614, 2.614)}$$

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3. Use the approach of Example 6.8 in pp. 300-302. Arrange the observation pairs in rows to have

$$\begin{pmatrix} \begin{bmatrix}9\\3\end{bmatrix} \begin{bmatrix}5\\1\end{bmatrix} & \\ \begin{bmatrix}9\\7\end{bmatrix} \begin{bmatrix}0\\8\end{bmatrix} \begin{bmatrix}3\\0\end{bmatrix} \\ \begin{bmatrix}3\\8\end{bmatrix} \begin{bmatrix}1\\6\end{bmatrix} \begin{bmatrix}2\\7\end{bmatrix} \end{pmatrix} \quad \text{with} \quad \bar{\mathbf{x}}_1 = \begin{bmatrix}7\\2\end{bmatrix}, \bar{\mathbf{x}}_2 = \begin{bmatrix}4\\5\end{bmatrix}, \bar{\mathbf{x}}_3 = \begin{bmatrix}2\\7\end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix}4\\5\end{bmatrix}$$

Sum of squares for variable 1:

$$\begin{pmatrix} 9 & 5 & \\ 9 & 0 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 3 & \\ 0 & 0 & 0 \\ -2 & -2 & -2 \end{pmatrix} + \begin{pmatrix} 2 & -2 & \\ 5 & -4 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

(observation) (mean) (treatment effect) (residual)

$$\text{SS}_{\text{obs}} = \text{SS}_{\text{mean}} + \text{SS}_{\text{tr}} + \text{SS}_{\text{res}}$$
$$210 = 128 + (9\times 2 + 4\times 3) + (4\times 2 + 25 + 16 + 1\times 3) = 128 + 30 + 52$$
$$\text{Total SS (corrected)} = \text{SS}_{\text{obs}} - \text{SS}_{\text{mean}} = 210 - 128 = 82$$

Sum of squares for variable 2:

$$\begin{pmatrix} 3 & 1 & \\ 7 & 8 & 0 \\ 8 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 5 & \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} + \begin{pmatrix} -3 & -3 & \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 & \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{pmatrix}$$

(observation) (mean) (treatment effect) (residual)

$$\text{SS}_{\text{obs}} = \text{SS}_{\text{mean}} + \text{SS}_{\text{tr}} + \text{SS}_{\text{res}}$$
$$272 = 200 + (9\times 2 + 4\times 3) + (1\times 4 + 25 + 4 + 9) = 200 + 30 + 42$$
$$\text{Total SS (corrected)} = \text{SS}_{\text{obs}} - \text{SS}_{\text{mean}} = 272 - 200 = 72$$

Sum of cross products

$$\text{SCP}_{\text{obs}} = 9\times 3 + 5\times 1 + 9\times 7 + 3\times 8 + 1\times 6 + 2\times 7 = 139$$
$$\text{SCP}_{\text{mean}} = (4\times 5)\times 8 = 160$$
$$\text{SCP}_{\text{tr}} = (3\times(-3))\times 2 + ((-2)\times 2)\times 3 = -30$$
$$\text{SCP}_{\text{res}} = 2\times 1 + (-2)\times(-1) + 5\times 2 + (-4)\times 3 + (-1)\times(-5) + 1\times 1 + (-1)\times(-1) = 9$$
$$\text{SCP}_{\text{obs}} = \text{SCP}_{\text{mean}} + \text{SCP}_{\text{tr}} + \text{SCP}_{\text{res}}$$
$$139 = 160 - 30 + 9$$
$$\text{Total SCP (corrected)} = \text{SCP}_{\text{obs}} - \text{SCP}_{\text{mean}} = 139 - 160 = -21$$

MANOVA Table:

Treatment $\mathbf{B} = \begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix}$ d.f. = 3 – 1 = 2

Residual $\mathbf{W} = \begin{bmatrix} 52 & 9 \\ 9 & 42 \end{bmatrix}$ d.f. = 8 -3 = 5

Total (corrected) $\mathbf{B} + \mathbf{W} = \begin{bmatrix} 82 & -21 \\ -21 & 72 \end{bmatrix}$ d.f. = 8-1 = 7

$$\Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|} = \frac{52\times 42 - 9\times 9}{82\times 72 - 21\times 21} = 0.385$$

Comparing (Table 6.3, p=2, g = 3)

$$\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right)\left(\frac{\sum n_\ell - g - 1}{g-1}\right) = \frac{1-\sqrt{0.385}}{\sqrt{0.385}} \times \frac{8-3-1}{3-1} = 1.223$$

with $F_{2(g-1),2(\sum n_\ell - g - 1)}(\alpha) = F_{4,8}(0.1) = 2.81$,

we conclude that the mean vectors of those three populations are the same at 90% significance level

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4. Use Result 7.1 (p. 358) and follow Example 7.3 (pp. 359-360),

$$\mathbf{Z}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix},\ \mathbf{y}' = \begin{bmatrix} 2 & 1 & 3 & 4 \end{bmatrix},\ \mathbf{Z}'\mathbf{Z} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}.$$

$$(\mathbf{Z}'\mathbf{Z})^{-1} = \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.2 \end{bmatrix}$$

$$\mathbf{Z}'\mathbf{y} = \begin{bmatrix} 10 \\ 19 \end{bmatrix},\quad \hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y} = \begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.2 \end{bmatrix}\begin{bmatrix} 10 \\ 19 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0.8 \end{bmatrix}$$

$\hat{y} = 1.3 + 0.8z$

$$\hat{\mathbf{y}} = \mathbf{Z}\hat{\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1.3 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 2.1 \\ 2.9 \\ 3.7 \end{bmatrix}, \hat{\boldsymbol{\varepsilon}} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1.3 \\ 2.1 \\ 2.9 \\ 3.7 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -1.1 \\ 0.1 \\ 0.3 \end{bmatrix}$$

$\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = 0.7^2 + 1.1^2 + 0.1^2 + 0.3^2 = 1.8$

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5. Use Result 7.7 (p. 374),

   $100(1-\alpha)\%$ confidence interval for $E(Y \mid z = 1.5)$ is

   $$\mathbf{z}_0'\hat{\boldsymbol{\beta}} \pm t_{n-r-1}(\frac{\alpha}{2})\sqrt{\left(\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)s^2} \quad .$$

   $$\mathbf{z}_0'\hat{\boldsymbol{\beta}} = [1 \quad 1.5]\begin{bmatrix} 1.3 \\ 0.8 \end{bmatrix} = 2.5,\ r = 1,\ t_{n-r-1}(\frac{\alpha}{2}) = t_2(0.05) = 2.920$$

   With Result 7.2 in p.363,

   $$s^2 = \frac{\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}}{n-r-1} = \frac{1.8}{4-1-1} = 0.9$$

   $$\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0 = [1 \quad 1.5]\begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.2 \end{bmatrix}\begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = 0.25$$

   $\therefore$ 90% confidence interval for $E(Y \mid z = 1.5)$ is

   $2.5 \pm 2.920\sqrt{0.25 * 0.9} = 2.5 \pm 1.385$, i.e., $(1.115, 3.885)$

   The prediction interval for Y can be obtained by Result 7.8 (p. 375)

   $$\mathbf{z}_0'\hat{\boldsymbol{\beta}} \pm t_{n-r-1}(\frac{\alpha}{2})\sqrt{s^2\left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)} = 2.5 \pm 2.920\sqrt{0.9 \times (1 + 0.25)}$$

   $= 2.5 \pm 3.097$, i.e., $(-0.597, 5.597)$

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6. For use of Result 8.1 (p. 428), eigenvalues and the first eigenvector have to be found.

   $$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 8-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda) = 0 \text{ or } \begin{vmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = \lambda^2 - 13\lambda + 36 = 0, \quad , \quad ,$$

   $\therefore \lambda_1 = 9, \lambda_2 = 4, \lambda_3 = 2$

$$\lambda_1 = 9, \begin{bmatrix} -7 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = 0 \Rightarrow \mathbf{e}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$Y_1 = \mathbf{e}_1^{'}\mathbf{X} = \frac{1}{\sqrt{5}}(2X_2 + X_3) = 0.894X_2 + 0.447X_3$$

Proportion of total variance can be computed via Equation (8-7) in p. 429:

$$\text{proportion of the total population variance due to } Y_1 = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} = \frac{9}{15} = 0.6$$

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7. From Equations (9-12) and (9-13) in p. 485

$$\mathbf{L} = \sqrt{\lambda_1}\mathbf{e}_1 = \sqrt{\frac{9}{5}}\begin{bmatrix}0\\2\\1\end{bmatrix}$$

$$\mathbf{\Sigma} \approx \mathbf{LL}'+\mathbf{\Psi} = \frac{9}{5}\begin{bmatrix}0\\2\\1\end{bmatrix}\begin{bmatrix}0 & 2 & 1\end{bmatrix} + \begin{bmatrix}\psi_1 & 0 & 0\\0 & \psi_2 & 0\\0 & 0 & \psi_3\end{bmatrix} = \begin{bmatrix}\psi_1 & 0 & 0\\0 & 7.2+\psi_2 & 3.6\\0 & 3.6 & 1.8+\psi_3\end{bmatrix},.$$

$$\therefore \psi_1 = 2, \psi_2 = 8-7.2 = 0.8, \psi_3 = 5-1.8 = 3.2$$

$$\mathbf{\Psi} = \begin{bmatrix}2 & 0 & 0\\0 & 0.8 & 0\\0 & 0 & 3.2\end{bmatrix}, \text{residual} = \mathbf{\Sigma} - \mathbf{LL}' - \mathbf{\Psi} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1.6\\0 & -1.6 & 0\end{bmatrix}$$

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8. According to the hint, elements of $\mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' - \tilde{\mathbf{\Psi}}$ are the same as those of $\mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}'$, except that the diagonal elements of $\mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' - \tilde{\mathbf{\Psi}}$ are all zero. Hence we will have

$$(\text{sum of squared entries of } \mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' - \tilde{\mathbf{\Psi}}) \le (\text{sum of squared entries of } \mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}')$$

Since $\mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' = \hat{\lambda}_{m+1}\hat{\mathbf{e}}_{m+1}\hat{\mathbf{e}}_{m+1}^{'} + \cdots + \hat{\lambda}_p\hat{\mathbf{e}}_p\hat{\mathbf{e}}_p^{'}$ according to the spectral decomposition of $\mathbf{S}$ and the definition of $\tilde{\mathbf{L}}$, $\mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' = \hat{\mathbf{P}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\hat{\mathbf{P}}_{(2)}^{'}$, where

$$\hat{\mathbf{P}}_{(2)} = \begin{bmatrix}\hat{\mathbf{e}}_{m+1} & \cdots & \hat{\mathbf{e}}_p\end{bmatrix}, \quad \hat{\mathbf{\Lambda}}_{(2)} = diag\{\hat{\lambda}_{m+1} \quad \cdots \quad \hat{\lambda}_p\}$$

Because the sum of squared entries of an arbitrary matrix **A** equals tr{**AA'**},

$$\text{sum of squared entries of } \mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' = \text{tr}\{\hat{\mathbf{P}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\hat{\mathbf{P}}_{(2)}^{'}\hat{\mathbf{P}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\hat{\mathbf{P}}_{(2)}^{'}\} = \text{tr}\{\hat{\mathbf{P}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\hat{\mathbf{P}}_{(2)}^{'}\}$$

$$= \text{tr}\{\hat{\mathbf{P}}_{(2)}^{'}\hat{\mathbf{P}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\} = \text{tr}\{\hat{\mathbf{\Lambda}}_{(2)}\hat{\mathbf{\Lambda}}_{(2)}\} = \hat{\lambda}_{m+1}^2 + \cdots + \hat{\lambda}_p^2$$

thus we achieve

$$(\text{sum of squared entries of } \mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' - \tilde{\mathbf{\Psi}}) \le \hat{\lambda}_{m+1}^2 + \cdots + \hat{\lambda}_p^2, \text{ Q. E. D.}$$

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9. We will use Result 10.1 in pp. 545-546, with

$$\mathbf{\Sigma}_{11}=\begin{bmatrix}64&0\\0&1\end{bmatrix},\ \mathbf{\Sigma}_{12}=\begin{bmatrix}0&0\\0.6&0\end{bmatrix},\ \mathbf{\Sigma}_{21}=\begin{bmatrix}0&0.6\\0&0\end{bmatrix},\ \mathbf{\Sigma}_{22}=\begin{bmatrix}1&0\\0&64\end{bmatrix},\ \mathbf{\Sigma}_{22}^{-1}=\begin{bmatrix}1&0\\0&1/64\end{bmatrix}$$

$$\mathbf{\Sigma}_{11}^{1/2}=\begin{bmatrix}8&0\\0&1\end{bmatrix},\ \mathbf{\Sigma}_{11}^{-1/2}=\begin{bmatrix}0.125&0\\0&1\end{bmatrix},\ \mathbf{\Sigma}_{22}^{1/2}=\begin{bmatrix}1&0\\0&8\end{bmatrix},\ \mathbf{\Sigma}_{22}^{-1/2}=\begin{bmatrix}1&0\\0&0.125\end{bmatrix}$$

$$\mathbf{\Sigma}_{11}^{-1/2}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1/2}=\begin{bmatrix}0.125&0\\0&1\end{bmatrix}\begin{bmatrix}0&0\\0.6&0\end{bmatrix}\begin{bmatrix}1&0\\0&1/64\end{bmatrix}\begin{bmatrix}0&0.6\\0&0\end{bmatrix}\begin{bmatrix}0.125&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&(0.6)^2\end{bmatrix}$$

Its eigenvalues are $(0.6)^2$ and 0. $\therefore \rho_1^*=0.6$. The corresponding eigenvector is found through

$$\begin{bmatrix}-0.36&0\\0&0\end{bmatrix}\begin{bmatrix}e_{11}\\e_{12}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\text{ i.e., }\mathbf{e}_1=\begin{bmatrix}0\\1\end{bmatrix},\ U_1=\mathbf{e}_1'\mathbf{\Sigma}_{11}^{-1/2}\mathbf{X}^{(1)}=\begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}0.125&0\\0&1\end{bmatrix},\begin{bmatrix}X_1^{(1)}\\X_2^{(1)}\end{bmatrix}=X_2^{(1)}$$

$$\mathbf{f}_1\propto\mathbf{\Sigma}_{22}^{-1/2}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1/2}\mathbf{e}_1=\begin{bmatrix}1&0\\0&0.125\end{bmatrix}\begin{bmatrix}0&0.6\\0&0\end{bmatrix}\begin{bmatrix}0.125&0\\0&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0.6\\0\end{bmatrix}$$

After normalization, $\mathbf{f}_1=\begin{bmatrix}1\\0\end{bmatrix},\ V_1=\mathbf{f}_1'\mathbf{\Sigma}_{22}^{-1/2}\mathbf{X}^{(2)}=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}1&0\\0&0.125\end{bmatrix}\begin{bmatrix}X_1^{(2)}\\X_2^{(2)}\end{bmatrix}=X_1^{(2)}$

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10. Dependent on your opinions. (10%)