Multivariate Statistical Analysis Mid Term Solution

November 17, 2006

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1. The answer depends on who you are. (6%)

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2. $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (6%)

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3. $\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix} = 9\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix} = 4\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (6%)

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4. $$\begin{aligned} (\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) &= [x_1+1 \quad x_2-1]\frac{1}{36}\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x_1+1 \\ x_2-1 \end{bmatrix} \\ &= \frac{1}{36}\left\{5(x_1+1)^2 - 4(x_1+1)(x_2-1) + 8(x_2-1)^2\right\} \\ &= \frac{1}{36}\left\{5x_1^2 - 4x_1x_2 + 8x_2^2 + 14x_1 - 20x_2 + 17\right\} \end{aligned}$$. (6%)

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5. Eq.(4 − 7), *p*.153, Text book : major axes and directions $= \pm c\sqrt{\lambda_i}\,\mathbf{e}_i$

$$\therefore \text{ major axes and directions } = \pm 3c\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \pm 3\,c\begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix}$$

$$\text{and } \pm 2c\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \pm 2\,c\begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix}$$. (6%)

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6. From Result 4.7, p. 163, and Table 3, p. 751, in Appendix of the Textbook, $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \chi_2^2(0.1) = 4.61$, $c = \sqrt{4.61} = 2.1471$,

$$\text{major axes and directions} = \pm 3\sqrt{4.61}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \pm 6.4413\begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix}$$

$$\text{and } \pm 2\sqrt{4.61}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \pm 4.2942\begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix}$$ (6%)

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7. From discussions on p. 176 and Eq. (4-28), p. 177, of the Textbook, $n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \approx \chi_2^2(0.1) = 4.61$.

$$\therefore (\bar{\mathbf{x}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \approx \frac{1}{100}\chi_2^2(0.1) = 0.0461$$,

$$c = \sqrt{0.0461} = 0.1\times\sqrt{4.61} = 0.2147$$,

$$\text{major axes and directions} = \pm\, 0.3\sqrt{4.61}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \pm 0.6441\begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix}$$. (6%)

$$\text{and} \pm 0.2\sqrt{4.61}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \pm 0.4294\begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix}$$

Based on Eq. (5 - 6) or (5 - 7), p. 212, and Table 4, pp. 752 - 753, of the Textbook,

$$T^2 = n(\bar{\mathbf{x}}-\boldsymbol{\mu}_0)'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}_0) = 30[0.5 \quad -1.5]\frac{1}{36}\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 0.5 \\ -1.5 \end{bmatrix}$$

8. $$= \frac{5}{6}[0.5 \quad -1.5]\begin{bmatrix} 5.5 \\ -13 \end{bmatrix} = 22.25\times\frac{5}{6} = 18.5417$$

$$\text{Critical value} = \frac{(n-1)p}{n-p}F_{p,n-p}(0.1) = \frac{29\times 2}{28}F_{2,28}(0.1) = \frac{58}{28}\times 2.50 = 5.1786$$

$T^2 >$ critical value. $\therefore$ reject H_0

(6%)

From Eq. (5 - 18), p. 221, of the Textbook

$$n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \le \frac{p(n-1)}{n-p}F_{p,n-p}(0.1) = 5.1786$$

$$(\bar{\mathbf{x}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \le \frac{5.17857}{30} = 0.1726$$

9. By Eq. (5 - 19), p. 221, of the Textbook, (6%)

$$\text{the major axes and directions} = \pm\sqrt{\lambda_i}\sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(0.1)}\,\mathbf{e}_i$$

$$\text{namely,} \pm 3\sqrt{0.1726}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \pm 1.2464\begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix}$$

$$\text{and} \pm 2\sqrt{0.1726}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \pm 0.8309\begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix}$$

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10. Use Eq. (5.24) in p. 225 of the Textbook, ,

$$\bar{x}_1 - \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.1)}\sqrt{\frac{s_{11}}{n}} \le \mu_1 \le \bar{x}_1 + \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.1)}\sqrt{\frac{s_{11}}{n}}$$

$$\bar{x}_2 - \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.1)}\sqrt{\frac{s_{22}}{n}} \le \mu_2 \le \bar{x}_2 + \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.1)}\sqrt{\frac{s_{22}}{n}}$$

$$\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.1)} = \sqrt{5.1786} = 2.2757$$. (6%)

$$-0.5 - \sqrt{5.1786}\sqrt{\frac{8}{30}} \le \mu_1 \le -0.5 + \sqrt{5.1786}\sqrt{\frac{8}{30}}, \quad \mu_1 \in (-1.6751, 0.6751)$$

$$-0.5 - \sqrt{5.1786}\sqrt{\frac{5}{30}} \le \mu_2 \le -0.5 + \sqrt{5.1786}\sqrt{\frac{5}{30}}, \quad \mu_2 \in (-1.4290, 0.4290)$$

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11. From Eq. (5 - 29) in p. 232, the Textbook,

$$\bar{x}_1 - t_{n-1}(\frac{0.1}{2p})\sqrt{\frac{s_{11}}{n}} \le \mu_1 \le \bar{x}_1 + t_{n-1}(\frac{0.1}{2p})\sqrt{\frac{s_{11}}{n}}$$

$$\bar{x}_2 - t_{n-1}(\frac{0.1}{2p})\sqrt{\frac{s_{22}}{n}} \le \mu_2 \le \bar{x}_2 + t_{n-1}(\frac{0.1}{2p})\sqrt{\frac{s_{22}}{n}}$$ (6%)

$t_{29}(0.025) = 2.045$ (Table 2, p. 750, Appendix of the Textbook)

$$-0.5 - 2.045\sqrt{\frac{8}{30}} \le \mu_1 \le -0.5 + 2.045\sqrt{\frac{8}{30}}, \quad \mu_1 \in (-1.5560, 0.5560)$$

$$-0.5 - 2.045\sqrt{\frac{5}{30}} \le \mu_2 \le -0.5 + 2.045\sqrt{\frac{5}{30}}, \quad \mu_2 \in (-1.3349, 0.3349)$$

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12. $\mu_{02} = 1$, because it is excluded in the 90% simultaneous T^2 and Bonferroni intervals

(6%)

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13. From the equation in pp. 249, LCL = 0, UCL = $\frac{(n-1)p}{n-p}F_{p,n-p}(0.1)$ (6%)

$$\therefore \text{UCL} = \frac{29 \times 2}{28}F_{2,28}(0.1) = \frac{58 \times 2.5}{28} = 5.1786$$

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14. From the equation in p. 248, $T^2 = \frac{n}{n+1}(\mathbf{x} - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x} - \bar{\mathbf{x}})$

$$\therefore T^2 = \frac{30}{31}\begin{bmatrix}0.5 & -1.5\end{bmatrix}\frac{1}{36}\begin{bmatrix}5 & -2 \\ -2 & 8\end{bmatrix}\begin{bmatrix}0.5 \\ -1.5\end{bmatrix} = \frac{1}{31} \times \frac{5}{6} \times 22.25 = 0.5981 < \text{UCL}$$

(6%)

15.

$$\frac{d\eta^b e^{-\eta/2}}{d\eta} = b\eta^{b-1}e^{-\eta/2} - \frac{1}{2}\eta^b e^{-\eta/2} = \left(b - \frac{1}{2}\eta\right)\eta^{b-1}e^{-\eta/2} = 0 \Rightarrow \eta = 2b$$

$$\frac{d^2\eta^b e^{-\eta/2}}{d\eta^2} = b(b-1)\eta^{b-2}e^{-\eta/2} - b\eta^{b-1}e^{-\eta/2} + \frac{1}{4}\eta^b e^{-\eta/2} = [b(b-1) - b\eta + \frac{1}{4}\eta^2]\eta^{b-2}e^{-\eta/2}$$

when $\eta = 2b$, $\frac{d^2\eta^b e^{-\eta/2}}{d\eta^2} = \left[b^2 - b - 2b^2 + b^2\right](2b)^{b-2}e^{-b} = -b(2b)^{b-2}e^{-b} < 0$

$\eta^b e^{-\eta/2} = (2b)^b e^{-b}$

(6%)

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16. Let the exact weight be w_0. Suppose that the error ε of the measured weight W is an additive random variable, i.e., $W = w_0 + \varepsilon$. Assume that there is no bias in measurement, we know that $E(\varepsilon) = 0$ and $E(W) = w_0$. By Result 4.13 (central limit theorem in p. 176 of the Textbook), when the size of sample n is large, $\sqrt{n}(\overline{W} - w_0)$ has an approximate normal distribution $N(0, \sigma^2)$. In other words, the sample mean has an approximate normal distribution $N(w_0, \sigma^2/n)$. When n is much larger than σ^2, the sample mean will be very close to the population mean w_0, because the spreading of the random variable $\overline{W}$, $\sigma/\sqrt{n}$, will be very small. (10%)