Multivariate Statistical Analysis Final Exam

Reference Solution

1. (10%) Eq. (6-16), p. 280, Textbook

$\mathbf{C}\bar{\mathbf{x}} = [-1 \quad -1]'$, $\mathbf{CSC}' = \begin{bmatrix} 10 & -6 \\ -6 & 9 \end{bmatrix}$,

$$(\mathbf{CSC}')^{-1} = \frac{1}{54}\begin{bmatrix} 9 & 6 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/9 \\ 1/9 & 5/27 \end{bmatrix} \approx \begin{bmatrix} 0.1667 & 0.1111 \\ 0.1111 & 0.1852 \end{bmatrix}$$

$T^2 = n(\mathbf{C}\bar{\mathbf{x}})'(\mathbf{CSC}')^{-1}\mathbf{C}\bar{\mathbf{x}} \approx 17.222$, larger than the critical value $\frac{(n-1)(q-1)}{(n-q+1)}F_{q-1,n-q+1}(\alpha) = \frac{29\times 2}{28}F_{2,28}(0.1) \approx \frac{58}{28}\times 2.50 \approx 5.1786$. Thus the hypothesis $H_0 : \mathbf{C}\boldsymbol{\mu} = 0$ is rejected at 10% significance level.

2. (10%) Eq. (6-21), p. 285, Textbook

$$\mathbf{S}_{pooled} = \frac{n_1 - 1}{n_1 + n_2 - 2}\mathbf{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\mathbf{S}_2 = \frac{1}{2}\left\{\begin{bmatrix} 6 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 10 & 3 \\ 3 & 6 \end{bmatrix}\right\} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

Eq. (6-51)~(6-53), p. 311, Textbook

$$u = \left[\sum_\ell \frac{1}{n_\ell - 1} - \frac{1}{\sum_\ell (n_\ell - 1)}\right]\left[\frac{2p^2 + 3p - 1}{6(p+1)(g-1)}\right] = \left[\frac{2}{15} - \frac{1}{30}\right]\left[\frac{2\times 4 + 3\times 2 - 1}{6\times 3\times 1}\right] = \frac{13}{180} \approx 0.0722$$

$$M = \left[\sum_\ell (n_\ell - 1)\right]\ln|\mathbf{S}_{pooled}| - \sum_\ell \left[(n_\ell - 1)\ln|\mathbf{S}_\ell|\right] = 30\times\ln 36 - 15\times(\ln 23 + \ln 51) \approx 1.4958$$

$C = (1-u)M = 1.388$, $v = \frac{1}{2}p(p+1)(g-1) = 3$, C is smaller than the critical value $\chi_v^2(\alpha) = \chi_3^2(0.1) = 6.25$. Thus H_0 is not rejected at 10% significance level.

3. (10%) slide 5, "Repeated Measures ANOVA", $g = 4, b = 3$

Subject\System	A	B	C	Average
1	5	3	2	10/3
2	5	6	3	14/3
3	4	5	5	14/3
4	6	6	6	6
Average	5	5	4	14/3

$$SS(System) = g\sum_{k=1}^{b}(\bar{x}_{\bullet k} - \bar{x})^2 = 4\left[(5-14/3)^2 + (5-14/3)^2 + (4-14/3)^2\right] \approx 2.667$$

$$SS(Subject) = b\sum_{\ell=1}^{g}(\bar{x}_{\ell\bullet} - \bar{x})^2 = 3\left[(10/3-14/3)^2 + 0 + 0 + (6-14/3)^2\right] = 10.667$$

$$SS(interaction) = \sum_{\ell=1}^{g} \sum_{k=1}^{b} \left(\bar{x}_{\ell k} - \bar{x}_{\ell \bullet} - \bar{x}_{\bullet k} + \bar{x}\right)^2$$

$$= \left(\frac{4}{3}\right)^2 + 0 + 1 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1 + 0 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1 + 1 + \left(\frac{2}{3}\right)^2 = \frac{22}{3} \approx 7.333$$

$df(System) = b - 1 = 2$, $df(Subject) = g - 1 = 3$,

$df(interaction) = (b-1)(g-1) = 6$, $df(total) = bg - 1 = 12 - 1 = 11 = 2 + 3 + 1$

Source	Sum of Squares	df	Mean Square	F
System	2.667	2	1.334	1.092
Subject	10.667	3	3.557	
Interaction	7.333	6	1.222	
Total	20.667	11		

$F_{system} = 1.092$ is smaller than the critical value: $F_{2,6}(0.05) \approx 5.14$. Thus the hypothesis can not be rejected.

Try another solution, Eq. (6-16), p. 280, Textbook,

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \bar{\mathbf{x}} = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{C}\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$$\mathbf{S} = \frac{1}{3}\begin{bmatrix} 0 & 0 & -1 & 1 \\ -2 & 1 & 0 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -2 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2 & 5/3 \\ 1/3 & 5/3 & 10/3 \end{bmatrix}$$

$$\mathbf{CSC'} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2 & 5/3 \\ 1/3 & 5/3 & 10/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 & -1/3 \\ -5/3 & 1/3 & 4/3 \\ -4/3 & -5/3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1/3 & -5/3 \\ -1/3 & 2 & -5/3 \\ -5/3 & -5/3 & 10/3 \end{bmatrix}$$

$$T^2 = 4\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1/3 & -5/3 \\ -1/3 & 2 & -5/3 \\ -5/3 & -5/3 & 10/3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Since $\mathbf{CSC'}$ is singular, we can not find its inverse, and this approach does not work.

The choice of $\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ or similar form is inappropriate, either, since it can check only $\mu_A = \mu_B$ and $\mu_B = \mu_C$, but not $\mu_C = \mu_A$. It has been known that in statistics that the acceptance of hypotheses $\mu_A = \mu_B$ and $\mu_B = \mu_C$ does not always imply the acceptance of $\mu_C = \mu_A$, because the acceptance of these hypotheses are based on probability, not on strict logic.

4. (10%)

Use the approach of Example 6.8 in pp. 300-302, Textbook. Arrange the observation pairs in rows to have

$$\begin{pmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} & \begin{bmatrix} 5 \\ 4 \end{bmatrix} & \begin{bmatrix} 7 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 8 \\ 3 \end{bmatrix} & \begin{bmatrix} 7 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 4 \\ 4 \end{bmatrix} & \begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{pmatrix} \text{ with } \bar{\mathbf{x}}_1 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \bar{\mathbf{x}}_2 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \bar{\mathbf{x}}_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Sum of squares for variable 1:

$$\begin{pmatrix} 6 & 5 & 7 \\ 8 & 7 & 3 \\ 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

(observation) (mean) (treatment effect) (residual)

$$\mathrm{SS_{obs}} = \mathrm{SS_{mean}} + \mathrm{SS_{tr}} + \mathrm{SS_{res}}$$
$$261 = 225 + (1\times 6 + 4\times 3) + (1\times 5 + 4 + 9) = 225 + 18 + 18$$
$$\text{Total SS (corrected)} = \mathrm{SS_{obs}} - \mathrm{SS_{mean}} = 261 - 225 = 36$$

Sum of squares for variable 2:

$$\begin{pmatrix} 2 & 4 & 3 \\ 3 & 3 & 3 \\ 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

(observation) (mean) (treatment effect) (residual)

$$\mathrm{SS_{obs}} = \mathrm{SS_{mean}} + \mathrm{SS_{tr}} + \mathrm{SS_{res}}$$
$$85 = 81 + 0 + 1\times 4 = 81 + 0 + 4$$
$$\text{Total SS (corrected)} = \mathrm{SS_{obs}} - \mathrm{SS_{mean}} = 85 - 81 = 4$$

Sum of cross products

$\text{SCP}_{\text{obs}} = 6\times 2 + 5\times 4 + 7\times 3 + 8\times 3 + 7\times 3 + 3\times 3 + 4\times 4 + 3\times 2 + 2\times 3 = 135$

$\text{SCP}_{\text{mean}} = (5\times 3)\times 9 = 135$

$\text{SCP}_{\text{tr}} = 0$

$\text{SCP}_{\text{res}} = (-1)\times 1 + 1 = 0$

$\text{SCP}_{\text{obs}} = \text{SCP}_{\text{mean}} + \text{SCP}_{\text{tr}} + \text{SCP}_{\text{res}}$

$135 = 135 + 0 + 0$

$\text{Total SCP (corrected)} = \text{SCP}_{\text{obs}} - \text{SCP}_{\text{mean}} = 135 - 135 = 0$

MANOVA Table:

Treatment $\mathbf{B} = \begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix}$ d.f. = 3 – 1 = 2

Residual $\mathbf{W} = \begin{bmatrix} 36 & 0 \\ 0 & 4 \end{bmatrix}$ d.f. = 9 -3 = 6

Total (corrected) $\mathbf{B} + \mathbf{W} = \begin{bmatrix} 54 & 0 \\ 0 & 4 \end{bmatrix}$ d.f. = 9-1 = 8

$$\Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|} = \frac{36\times 4}{54\times 4} \approx 0.667$$

Comparing (Table 6.3, p=2, g = 3)

$$\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right)\left(\frac{\sum n_\ell - g - 1}{g-1}\right) \approx \frac{1-\sqrt{0.667}}{\sqrt{0.667}} \times \frac{9-3-1}{3-1} \approx 0.561$$

with $F_{2(g-1),2(\sum n_\ell - g - 1)}(\alpha) = F_{4,10}(0.1) = 2.61$,

we can not reject the hypothesis that the mean vectors of those three populations are the same at 90% significance level

5. (10%) Result 7.1, p. 364, Textbook

$\mathbf{Z}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix}$, $\mathbf{y}_1' = [2 \quad 1 \quad 3 \quad 4]$, $\mathbf{y}_2' = [8 \quad 6 \quad 5 \quad 2]$

$\mathbf{Z}'\mathbf{y}_1 = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$, $\mathbf{Z}'\mathbf{y}_2 = \begin{bmatrix} 21 \\ -20 \end{bmatrix}$, $\mathbf{Z}'\mathbf{Z} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$, $(\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$

$\hat{\boldsymbol{\beta}}_{(1)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_1 = \begin{bmatrix} 2.9 \\ 0.8 \end{bmatrix}$, $\hat{\boldsymbol{\beta}}_{(2)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_2 = \begin{bmatrix} 4.3 \\ -1.9 \end{bmatrix}$

$\hat{\mathbf{y}}_1 = \mathbf{Z}\hat{\boldsymbol{\beta}}_{(1)} = [1.3 \quad 2.1 \quad 2.9 \quad 3.7]'$, $\hat{\mathbf{y}}_2 = \mathbf{Z}\hat{\boldsymbol{\beta}}_{(2)} = [8.1 \quad 6.2 \quad 4.3 \quad 2.4]'$

$\hat{\boldsymbol{\varepsilon}}_{(1)} = \mathbf{y}_1 - \hat{\mathbf{y}}_1 = [0.7 \quad -1.1 \quad 0.1 \quad 0.3]'$, $\hat{\boldsymbol{\varepsilon}}_{(2)} = \mathbf{y}_2 - \hat{\mathbf{y}}_2 = [-0.1 \quad -0.2 \quad 0.7 \quad -0.4]'$

$\hat{\boldsymbol{\varepsilon}}_{(1)}'\hat{\boldsymbol{\varepsilon}}_{(1)} = 1.8$, $\hat{\boldsymbol{\varepsilon}}_{(2)}'\hat{\boldsymbol{\varepsilon}}_{(2)} = 0.7$

6. (10%) Eq. (7.-40), p. 399, Textbook

   100(1-α)% confidence region for $E([Y_1, Y_2]' \mid \mathbf{z}_0) = \boldsymbol{\beta}'\mathbf{z}_0$ is provided by the inequality

$$(\boldsymbol{\beta}'\mathbf{z}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0)'\left(\frac{n}{n-r-1}\hat{\boldsymbol{\Sigma}}\right)^{-1}(\boldsymbol{\beta}'\mathbf{z}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0) \le \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\left[\left(\frac{m(n-r-1)}{n-r-m}\right)F_{m,n-r-m}(\alpha)\right]$$

$$\hat{\boldsymbol{\beta}} = [\hat{\boldsymbol{\beta}}_{(1)} \quad \hat{\boldsymbol{\beta}}_{(2)}] = \begin{bmatrix} 2.9 & 4.3 \\ 0.8 & -1.9 \end{bmatrix}, \qquad \mathbf{z}_0 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},$$

$$\hat{\boldsymbol{\beta}}'\mathbf{z}_0 = \begin{bmatrix} 2.9 & 0.8 \\ 4.3 & -1.9 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 5.25 \end{bmatrix}$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \frac{1}{4}\begin{bmatrix} 0.7 & -1.1 & 0.1 & 0.3 \\ -0.1 & -0.2 & 0.7 & -0.4 \end{bmatrix}\begin{bmatrix} 0.7 & -0.1 \\ -1.1 & -0.2 \\ 0.1 & 0.7 \\ 0.3 & -0.4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1.8 & 0.1 \\ 0.1 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.45 & 0.025 \\ 0.025 & 0.175 \end{bmatrix}$$

$$\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0 = [1 \quad -0.5]\begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = [1 \quad -0.5]\begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = 0.25$$

   90% confidence region for $E([Y_1, Y_2] \mid \mathbf{z}_0) = \boldsymbol{\beta}'\mathbf{z}_0$ is

$$(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix} 2.5 \\ 5.25 \end{bmatrix})'\left(\frac{4}{4-1-1}\begin{bmatrix} 0.45 & 0.025 \\ 0.025 & 0.175 \end{bmatrix}\right)^{-1}(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix} 2.5 \\ 5.25 \end{bmatrix}) \le 0.25\left[\left(\frac{2(4-1-1)}{4-1-2}\right)F_{2,1}(0.1)\right]$$

   i.e.,

$$(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix} 2.5 \\ 5.25 \end{bmatrix})'\begin{bmatrix} 1.12 & -0.16 \\ -0.16 & 2.88 \end{bmatrix}(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix} 2.5 \\ 5.25 \end{bmatrix}) \le 49.50$$

   Eq. (7-42), p. 399, Textbook

   100(1-α)% prediction region for $\mathbf{Y}_0 = [Y_1, Y_2]'$ is provided by the inequality

$$(\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0)'\left(\frac{n}{n-r-1}\hat{\boldsymbol{\Sigma}}\right)^{-1}(\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0) \le (1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0)\left[\left(\frac{m(n-r-1)}{n-r-m}\right)F_{m,n-r-m}(\alpha)\right]$$

   Thus, 90% prediction region for $\mathbf{Y}_0 = [Y_1, Y_2]'$ is

$$(\mathbf{Y}_0 - \begin{bmatrix} 2.5 \\ 5.25 \end{bmatrix})'\begin{bmatrix} 1.12 & -0.16 \\ -0.16 & 2.88 \end{bmatrix}(\mathbf{Y}_0 - \begin{bmatrix} 2.5 \\ 5.25 \end{bmatrix}) \le 247.5$$

7. (10%) Eq. (7-49), Textbook

$$1-\rho_{Y(\mathbf{Z})}^2 = \frac{\sigma_{YY} - \sigma_{\mathbf{Z}Y}^{'}\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\sigma_{\mathbf{Z}Y}}{\sigma_{YY}} = \frac{|\mathbf{\Sigma}_{\mathbf{ZZ}}|}{|\mathbf{\Sigma}_{\mathbf{ZZ}}|}\frac{(\sigma_{YY} - \sigma_{\mathbf{Z}Y}^{'}\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\sigma_{\mathbf{Z}Y})}{\sigma_{YY}}$$

Exercise 4.11, $|\mathbf{A}| = |\mathbf{A}_{22}||\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}|$, $\therefore |\mathbf{\Sigma}_{\mathbf{zz}}|(\sigma_{YY} - \sigma_{\mathbf{Z}Y}^{'}\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\sigma_{\mathbf{Z}Y}) = |\mathbf{\Sigma}|$

$$1-\rho_{Y(\mathbf{Z})}^2 = \frac{|\mathbf{\Sigma}|}{|\mathbf{\Sigma}_{\mathbf{ZZ}}|\sigma_{YY}}$$

Result 2A.8(c): $\mathbf{A}^{-1}$ has *j*, *i*th entry $(-1)^{i+j}|\mathbf{A}_{ij}|/|\mathbf{A}|$. Thus, $\sigma^{YY} = |\mathbf{\Sigma}_{\mathbf{ZZ}}|/|\mathbf{\Sigma}|$ is the entry of $\mathbf{\Sigma}^{-1}$ in the first row and first column. $1-\rho_{Y(\mathbf{Z})}^2 = \dfrac{1}{\sigma^{YY}\sigma_{YY}}$

Exercise 2.23: $\boldsymbol{\rho} = (\mathbf{V}^{1/2})^{-1}\mathbf{\Sigma}(\mathbf{V}^{1/2})^{-1}$, $\boldsymbol{\rho}^{-1} = \mathbf{V}^{1/2}\mathbf{\Sigma}^{-1}\mathbf{V}^{1/2}$, the entry in the (1,1) Position of $\boldsymbol{\rho}^{-1}$ is $\rho^{YY} = \sigma^{YY}\sigma_{YY}$. Hence, $1-\rho_{Y(\mathbf{Z})}^2 = \dfrac{1}{\rho^{YY}}$.

Another proof:

Let $\mathbf{\Sigma}^{-1} = \begin{bmatrix} \sigma^{YY} & \boldsymbol{\sigma}^{\mathbf{Z}Y}{}' \\ \boldsymbol{\sigma}^{\mathbf{Z}Y} & \mathbf{\Sigma}^{\mathbf{ZZ}} \end{bmatrix}$

$$\mathbf{\Sigma}\mathbf{\Sigma}^{-1} = \begin{bmatrix} \sigma_{YY} & \boldsymbol{\sigma}_{\mathbf{Z}Y}{}' \\ \boldsymbol{\sigma}_{\mathbf{Z}Y} & \mathbf{\Sigma}_{\mathbf{ZZ}} \end{bmatrix}\begin{bmatrix} \sigma^{YY} & \boldsymbol{\sigma}^{\mathbf{Z}Y}{}' \\ \boldsymbol{\sigma}^{\mathbf{Z}Y} & \mathbf{\Sigma}^{\mathbf{ZZ}} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$\boldsymbol{\sigma}_{\mathbf{Z}Y}\boldsymbol{\sigma}^{\mathbf{Z}Y}{}' + \mathbf{\Sigma}_{\mathbf{ZZ}}\mathbf{\Sigma}^{\mathbf{ZZ}} = \mathbf{I}$, thus $\mathbf{\Sigma}^{\mathbf{ZZ}} = \mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{I} - \boldsymbol{\sigma}_{\mathbf{Z}Y}\boldsymbol{\sigma}^{\mathbf{Z}Y}{}')$.

$\sigma_{YY}\boldsymbol{\sigma}^{\mathbf{Z}Y}{}' + \sigma_{\mathbf{Z}Y}{}'\mathbf{\Sigma}^{\mathbf{ZZ}} = \sigma_{YY}\boldsymbol{\sigma}^{\mathbf{Z}Y}{}' + \boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{I} - \boldsymbol{\sigma}_{\mathbf{Z}Y}\boldsymbol{\sigma}^{\mathbf{Z}Y}{}') = \mathbf{0}'$

$\boldsymbol{\sigma}^{\mathbf{Z}Y}{}' = -(\sigma_{YY} - \sigma_{\mathbf{Z}Y}{}'\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y})^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}$

$\sigma_{YY}\sigma^{YY} + \boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\boldsymbol{\sigma}^{\mathbf{Z}Y} = 1$, thus

$$\sigma^{YY} = \sigma_{YY}{}^{-1}(1 - \boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\boldsymbol{\sigma}^{\mathbf{Z}Y}) = \frac{1}{\sigma_{YY}}\left(1 + \frac{\boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}{\sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}\right) = \frac{1}{\sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}$$

Because $\boldsymbol{\rho}^{-1} = \mathbf{V}^{1/2}\mathbf{\Sigma}^{-1}\mathbf{V}^{1/2}$, $\rho^{YY} = \sigma_{YY}\sigma^{YY} = \dfrac{\sigma_{YY}}{\sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}{}'\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}$

Thus $1-\rho_{Y(\mathbf{Z})}^2 = \dfrac{\sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}^{'}\mathbf{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}{\sigma_{YY}} = \dfrac{1}{\rho^{YY}}$.

8. (10%) For use of the result in p. 442, Textbook, eigenvalues and the first eigenvector have to be found.

$$\begin{vmatrix} 12-\lambda & 6 & 0 \\ 6 & 7-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda) = 0 \text{ or } \begin{vmatrix} 12-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = \lambda^2 - 19\lambda + 48 = 0, \quad , \quad ,$$

$$\therefore \hat{\lambda}_1 = 16,\ \hat{\lambda}_2 = 5,\ \hat{\lambda}_3 = 3$$

$$\hat{\lambda}_1 = 16, \begin{bmatrix} -4 & 6 & 0 \\ 6 & -9 & 0 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} \hat{e}_{11} \\ \hat{e}_{12} \\ \hat{e}_{13} \end{bmatrix} = 0 \Rightarrow \hat{\mathbf{e}}_1 = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$Y_1 = \hat{\mathbf{e}}_1^{'}\mathbf{x} = \frac{1}{\sqrt{13}}(3x_1 + 2x_2) = 0.832x_1 + 0.555x_2$$

$$\text{proportion of the total sample variance due to } Y_1 = \frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3} = \frac{16}{24} \approx 0.667$$

Eq. (8-33) in p. 456, Textbook: $\dfrac{\hat{\lambda}_i}{1 + z(\alpha/2)\sqrt{2/n}} \le \lambda_i \le \dfrac{\hat{\lambda}_i}{1 - z(\alpha/2)\sqrt{2/n}}$.

Hnece the 95% confidence interval for the first eigenvalue is

$$\left(\frac{\hat{\lambda}_1}{1 + z(0.025)\sqrt{2/200}}, \frac{\hat{\lambda}_1}{1 - z(0.025)\sqrt{2/200}}\right) \approx \left(\frac{16}{1.196}, \frac{16}{0.804}\right) \approx (13.378, 19.900)$$

9. (10%) From Equations (9-15) and (9-16) in p. 490, Textbook

$$\tilde{\mathbf{L}} = \sqrt{\hat{\lambda}_1}\hat{\mathbf{e}}_1 = \sqrt{\frac{16}{13}} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{S} \approx \tilde{\mathbf{L}}\tilde{\mathbf{L}}' + \tilde{\mathbf{\Psi}} = \frac{16}{13} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{\psi}_1 & 0 & 0 \\ 0 & \tilde{\psi}_2 & 0 \\ 0 & 0 & \tilde{\psi}_3 \end{bmatrix} \approx \begin{bmatrix} 11.077 + \tilde{\psi}_1 & 7.385 & 0 \\ 7.385 & 4.923 + \tilde{\psi}_2 & 0 \\ 0 & 0 & \tilde{\psi}_3 \end{bmatrix}$$

$$\therefore \tilde{\psi}_1 \approx 12 - 11.077 = 0.923,\ \tilde{\psi}_2 \approx 7 - 4.923 = 2.077,\ \tilde{\psi}_3 = 5$$

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix} 0.923 & 0 & 0 \\ 0 & 2.077 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \text{residual} = \mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' - \tilde{\mathbf{\Psi}} = \begin{bmatrix} 0 & -1.385 & 0 \\ -1.385 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eq. (9-51) in p. 515, Textbook, factor score by unweighted least square procedur: $$\hat{\mathbf{f}}_j = \left[\frac{1}{\sqrt{\hat{\lambda}_1}} \hat{\mathbf{e}}_1' (\mathbf{x}_j - \overline{\mathbf{x}}) \quad \frac{1}{\sqrt{\hat{\lambda}_2}} \hat{\mathbf{e}}_2' (\mathbf{x}_j - \overline{\mathbf{x}}) \quad \cdots \quad \frac{1}{\sqrt{\hat{\lambda}_m}} \hat{\mathbf{e}}_m' (\mathbf{x}_j - \overline{\mathbf{x}}) \right]' .$$

Thus, $$\hat{f} = \frac{1}{\sqrt{16}\sqrt{13}} \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2-1 \\ 3-0 \\ 1-(-1) \end{bmatrix} \approx 0.624$$

10. (10%) Depend on your comments and suggestions..