1.2. A morning newspaper lists the following used-car prices for a foreign compact with age x_1 measured in years and selling price x_2 measured in thousands of dollars:

x_1	1	2	3	3	4	5	6	8	9	11
x_2	18.95	19.00	17.95	15.54	14.00	12.95	8.94	7.49	6.00	3.99

(a) Construct a scatter plot of the data and marginal dot diagrams.

(b) Infer the sign of the sample covariance s_{12} from the scatter plot.

(c) Compute the sample means $\bar{x}_1$ and $\bar{x}_2$ and the sample variances s_{11} and s_{22}. Compute the sample covariance s_{12} and the sample correlation coefficient r_{12}. Interpret these quantities.

(d) Display the sample mean array $\bar{\mathbf{x}}$, the sample variance-covariance array $\mathbf{S}_n$, and the sample correlation array $\mathbf{R}$ using (1-8).

1.3. The following are five measurements on the variables x_1, x_2, and x_3:

x_1	9	2	6	5	8
x_2	12	8	6	4	10
x_3	3	4	0	2	1

Find the arrays $\bar{\mathbf{x}}$, $\mathbf{S}_n$, and $\mathbf{R}$.

1.4. The world's 10 largest companies yield the following data:

The World's 10 Largest Companies[1]

Company	x_1 = sales (billions)	x_2 = profits (billions)	x_3 = assets (billions)
Citigroup	108.28	17.05	1,484.10
General Electric	152.36	16.59	750.33
American Intl Group	95.04	10.91	766.42
Bank of America	65.45	14.14	1,110.46
HSBC Group	62.97	9.52	1,031.29
ExxonMobil	263.99	25.33	195.26
Royal Dutch/Shell	265.19	18.54	193.83
BP	285.06	15.73	191.11
ING Group	92.01	8.10	1,175.16
Toyota Motor	165.68	11.13	211.15

[1]From www.Forbes.com partially based on *Forbes* The Forbes Global 2000, April 18, 2005.

(a) Plot the scatter diagram and marginal dot diagrams for variables x_1 and x_2. Comment on the appearance of the diagrams.

(b) Compute $\bar{x}_1$, $\bar{x}_2$, s_{11}, s_{22}, s_{12}, and r_{12}. Interpret r_{12}.

1.5. Use the data in Exercise 1.4.

(a) Plot the scatter diagrams and dot diagrams for (x_2, x_3) and (x_1, x_3). Comment on the patterns.

(b) Compute the $\bar{\mathbf{x}}$, $\mathbf{S}_n$, and $\mathbf{R}$ arrays for (x_1, x_2, x_3).

1.7. You are given the following $n = 3$ observations on $p = 2$ variables:

$$\text{Variable 1:} \quad x_{11} = 2 \quad x_{21} = 3 \quad x_{31} = 4$$

$$\text{Variable 2:} \quad x_{12} = 1 \quad x_{22} = 2 \quad x_{32} = 4$$

(a) Plot the pairs of observations in the two-dimensional "variable space." That is, construct a two-dimensional scatter plot of the data.

(b) Plot the data as two points in the three-dimensional "item space."

1.8. Evaluate the distance of the point $P = (-1, -1)$ to the point $Q = (1, 0)$ using the Euclidean distance formula in (1-12) with $p = 2$ and using the statistical distance in (1-20) with $a_{11} = 1/3$, $a_{22} = 4/27$, and $a_{12} = 1/9$. Sketch the locus of points that are a constant squared statistical distance 1 from the point Q.

1.9. Consider the following eight pairs of measurements on two variables x_1 and x_2:

x_1	-6	-3	-2	1	2	5	6	8
x_2	-2	-3	1	-1	2	1	5	3

(a) Plot the data as a scatter diagram, and compute s_{11}, s_{22}, and s_{12}.

(b) Using (1-18), calculate the corresponding measurements on variables $\tilde{x}_1$ and $\tilde{x}_2$, assuming that the original coordinate axes are rotated through an angle of $\theta = 26°$ [given $\cos(26°) = .899$ and $\sin(26°) = .438$].

(c) Using the $\tilde{x}_1$ and $\tilde{x}_2$ measurements from (b), compute the sample variances $\tilde{s}_{11}$ and $\tilde{s}_{22}$.

(d) Consider the *new* pair of measurements $(x_1, x_2) = (4, -2)$. Transform these to measurements on $\tilde{x}_1$ and $\tilde{x}_2$ using (1-18), and calculate the distance $d(O, P)$ of the new point $P = (\tilde{x}_1, \tilde{x}_2)$ from the origin $O = (0, 0)$ using (1-17).
Note: You will need $\tilde{s}_{11}$ and $\tilde{s}_{22}$ from (c).

(e) Calculate the distance from $P = (4, -2)$ to the origin $O = (0, 0)$ using (1-19) and the expressions for a_{11}, a_{22}, and a_{12} in footnote 2.
Note: You will need s_{11}, s_{22}, and s_{12} from (a).
Compare the distance calculated here with the distance calculated using the $\tilde{x}_1$ and $\tilde{x}_2$ values in (d). (Within rounding error, the numbers should be the same.)

1.10. Are the following distance functions valid for distance from the origin? Explain.

(a) $x_1^2 + 4x_2^2 + x_1x_2 = (\text{distance})^2$

(b) $x_1^2 - 2x_2^2 = (\text{distance})^2$

1.11. Verify that distance defined by (1-20) with $a_{11} = 4$, $a_{22} = 1$, and $a_{12} = -1$ satisfies the first three conditions in (1-25). (The triangle inequality is more difficult to verify.)

1.12. Define the distance from the point $P = (x_1, x_2)$ to the origin $O = (0, 0)$ as

$$d(O, P) = \max(|x_1|, |x_2|)$$

(a) Compute the distance from $P = (-3, 4)$ to the origin.

(b) Plot the locus of points whose squared distance from the origin is 1.

(c) Generalize the foregoing distance expression to points in p dimensions.

√ **1.13.** A large city has major roads laid out in a grid pattern, as indicated in the following diagram. Streets 1 through 5 run north–south (NS), and streets A through E run east–west (EW). Suppose there are retail stores located at intersections $(A, 2)$, $(E, 3)$, and $(C, 5)$.

1.7. You are given the following $n = 3$ observations on $p = 2$ variables:

Variable 1: $x_{11} = 2 \quad x_{21} = 3 \quad x_{31} = 4$

Variable 2: $x_{12} = 1 \quad x_{22} = 2 \quad x_{32} = 4$

(a) Plot the pairs of observations in the two-dimensional "variable space." That is, construct a two-dimensional scatter plot of the data.

(b) Plot the data as two points in the three-dimensional "item space."

1.8. Evaluate the distance of the point $P = (-1, -1)$ to the point $Q = (1, 0)$ using the Euclidean distance formula in (1-12) with $p = 2$ and using the statistical distance in (1-20) with $a_{11} = 1/3$, $a_{22} = 4/27$, and $a_{12} = 1/9$. Sketch the locus of points that are a constant squared statistical distance 1 from the point Q.

1.9. Consider the following eight pairs of measurements on two variables x_1 and x_2:

x_1	−6	−3	−2	1	2	5	6	8
x_2	−2	−3	1	−1	2	1	5	3

(a) Plot the data as a scatter diagram, and compute s_{11}, s_{22}, and s_{12}.

(b) Using (1-18), calculate the corresponding measurements on variables $\tilde{x}_1$ and $\tilde{x}_2$, assuming that the original coordinate axes are rotated through an angle of $\theta = 26°$ [given $\cos(26°) = .899$ and $\sin(26°) = .438$].

(c) Using the $\tilde{x}_1$ and $\tilde{x}_2$ measurements from (b), compute the sample variances $\tilde{s}_{11}$ and $\tilde{s}_{22}$.

(d) Consider the *new* pair of measurements $(x_1, x_2) = (4, -2)$. Transform these to measurements on $\tilde{x}_1$ and $\tilde{x}_2$ using (1-18), and calculate the distance $d(O, P)$ of the new point $P = (\tilde{x}_1, \tilde{x}_2)$ from the origin $O = (0, 0)$ using (1-17).
Note: You will need $\tilde{s}_{11}$ and $\tilde{s}_{22}$ from (c).

(e) Calculate the distance from $P = (4, -2)$ to the origin $O = (0, 0)$ using (1-19) and the expressions for a_{11}, a_{22}, and a_{12} in footnote 2.
Note: You will need s_{11}, s_{22}, and s_{12} from (a).
Compare the distance calculated here with the distance calculated using the $\tilde{x}_1$ and $\tilde{x}_2$ values in (d). (Within rounding error, the numbers should be the same.)

1.10. Are the following distance functions valid for distance from the origin? Explain.

(a) $x_1^2 + 4x_2^2 + x_1x_2 = (\text{distance})^2$

(b) $x_1^2 - 2x_2^2 = (\text{distance})^2$

1.11. Verify that distance defined by (1-20) with $a_{11} = 4$, $a_{22} = 1$, and $a_{12} = -1$ satisfies the first three conditions in (1-25). (The triangle inequality is more difficult to verify.)

1.12. Define the distance from the point $P = (x_1, x_2)$ to the origin $O = (0, 0)$ as

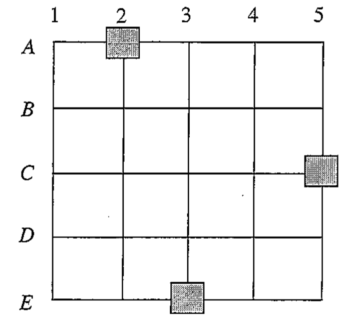
$$d(O, P) = \max(|x_1|, |x_2|)$$

(a) Compute the distance from $P = (-3, 4)$ to the origin.

(b) Plot the locus of points whose squared distance from the origin is 1.

(c) Generalize the foregoing distance expression to points in p dimensions.

√ **1.13.** A large city has major roads laid out in a grid pattern, as indicated in the following diagram. Streets 1 through 5 run north–south (NS), and streets A through E run east–west (EW). Suppose there are retail stores located at intersections $(A, 2)$, $(E, 3)$, and $(C, 5)$. Assume the distance along a street between two intersections in either the NS or EW direction is 1 unit. Define the distance between any two intersections (points) on the grid to be the "city block" distance. [For example, the distance between intersections $(D, 1)$ and $(C, 2)$, which we might call $d((D, 1), (C, 2))$, is given by $d((D, 1), (C, 2)) = d((D, 1), (D, 2)) + d((D, 2), (C, 2)) = 1 + 1 = 2$. Also, $d((D, 1), (C, 2)) = d((D, 1), (C, 1)) + d((C, 1), (C, 2)) = 1 + 1 = 2$.]



Locate a supply facility (warehouse) at an intersection such that the sum of the distances from the warehouse to the three retail stores is minimized.

The following exercises contain fairly extensive data sets. A computer may be necessary for the required calculations.

√ **1.14.** Table 1.6 contains some of the raw data discussed in Section 1.2. (See also the multiple-sclerosis data on the web at www.prenhall.com/statistics.) Two different visual stimuli ($S1$ and $S2$) produced responses in both the left eye (L) and the right eye (R) of subjects in the study groups. The values recorded in the table include x_1 (subject's age); x_2 (total response of both eyes to stimulus $S1$, that is, $S1L + S1R$); x_3 (difference between responses of eyes to stimulus $S1$, $|S1L - S1R|$); and so forth.

(a) Plot the two-dimensional scatter diagram for the variables x_2 and x_4 for the multiple-sclerosis group. Comment on the appearance of the diagram.

(b) Compute the $\bar{\mathbf{x}}$, $\mathbf{S}_n$, and $\mathbf{R}$ arrays for the non-multiple-sclerosis and multiple-sclerosis groups separately.

1.15. Some of the 98 measurements described in Section 1.2 are listed in Table 1.7 (See also the radiotherapy data on the web at www.prenhall.com/statistics.) The data consist of average ratings over the course of treatment for patients undergoing radiotherapy. Variables measured include x_1 (number of symptoms, such as sore throat or nausea); x_2 (amount of activity, on a 1–5 scale); x_3 (amount of sleep, on a 1–5 scale); x_4 (amount of food consumed, on a 1–3 scale); x_5 (appetite, on a 1–5 scale); and x_6 (skin reaction, on a 0–3 scale).

(a) Construct the two-dimensional scatter plot for variables x_2 and x_3 and the marginal dot diagrams (or histograms). Do there appear to be any errors in the x_3 data?

(b) Compute the $\bar{\mathbf{x}}$, $\mathbf{S}_n$, and $\mathbf{R}$ arrays. Interpret the pairwise correlations.

1.16. At the start of a study to determine whether exercise or dietary supplements would slow bone loss in older women, an investigator measured the mineral content of bones by photon absorptiometry. Measurements were recorded for three bones on the dominant and nondominant sides and are shown in Table 1.8. (See also the mineral-content data on the web at www.prenhall.com/statistics.)

Compute the $\bar{\mathbf{x}}$, $\mathbf{S}_n$, and $\mathbf{R}$ arrays. Interpret the pairwise correlations.

Table 1.6 Multiple-Sclerosis Data

Non-Multiple-Sclerosis Group Data

Subject number	x_1 (Age)	x_2 $(S1L + S1R)$	x_3 $\|S1L - S1R\|$	x_4 $(S2L + S2R)$	x_5 $\|S2L - S2R\|$
1	18	152.0	1.6	198.4	.0
2	19	138.0	.4	180.8	1.6
3	20	144.0	.0	186.4	.8
4	20	143.6	3.2	194.8	.0
5	20	148.8	.0	217.6	.0
⋮	⋮	⋮	⋮	⋮	⋮
65	67	154.4	2.4	205.2	6.0
66	69	171.2	1.6	210.4	.8
67	73	157.2	.4	204.8	.0
68	74	175.2	5.6	235.6	.4
69	79	155.0	1.4	204.4	.0

Multiple-Sclerosis Group Data

Subject number	x_1	x_2	x_3	x_4	x_5
1	23	148.0	.8	205.4	.6
2	25	195.2	3.2	262.8	.4
3	25	158.0	8.0	209.8	12.2
4	28	134.4	.0	198.4	3.2
5	29	190.2	14.2	243.8	10.6
⋮	⋮	⋮	⋮	⋮	⋮
25	57	165.6	16.8	229.2	15.6
26	58	238.4	8.0	304.4	6.0
27	58	164.0	.8	216.8	.8
28	58	169.8	.0	219.2	1.6
29	59	199.8	4.6	250.2	1.0

Source: Data courtesy of Dr. G. G. Celesia.

Table 1.7 Radiotherapy Data

x_1 Symptoms	x_2 Activity	x_3 Sleep	x_4 Eat	x_5 Appetite	x_6 Skin reaction
.889	1.389	1.555	2.222	1.945	1.000
2.813	1.437	.999	2.312	2.312	2.000
1.454	1.091	2.364	2.455	2.909	3.000
.294	.941	1.059	2.000	1.000	1.000
2.727	2.545	2.819	2.727	4.091	.000
⋮	⋮	⋮	⋮	⋮	⋮
4.100	1.900	2.800	2.000	2.600	2.000
.125	1.062	1.437	1.875	1.563	.000
6.231	2.769	1.462	2.385	4.000	2.000
3.000	1.455	2.090	2.273	3.272	2.000
.889	1.000	1.000	2.000	1.000	2.000

Source: Data courtesy of Mrs. Annette Tealey, R.N. Values of x_2 and x_3 less than 1.0 are due to errors in the data-collection process. Rows containing values of x_2 and x_3 less than 1.0 may be omitted.

2.7. Let $\mathbf{A}$ be as given in Exercise 2.6.

(a) Determine the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Write the spectral decomposition of $\mathbf{A}$.

(c) Find $\mathbf{A}^{-1}$.

(d) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

2.8. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

find the eigenvalues λ_1 and λ_2 and the associated normalized eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Determine the spectral decomposition (2-16) of $\mathbf{A}$.

2.9. Let $\mathbf{A}$ be as in Exercise 2.8.

(a) Find $\mathbf{A}^{-1}$.

(b) Compute the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

(c) Write the spectral decomposition of $\mathbf{A}^{-1}$, and compare it with that of $\mathbf{A}$ from Exercise 2.8.

2.10. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

These matrices are identical except for a small difference in the $(2, 2)$ position. Moreover, the columns of $\mathbf{A}$ (and $\mathbf{B}$) are nearly linearly dependent. Show that $\mathbf{A}^{-1} \doteq (-3)\mathbf{B}^{-1}$. Consequently, small changes—perhaps caused by rounding—can give substantially different inverses.

2.11. Show that the determinant of the $p \times p$ diagonal matrix $\mathbf{A} = \{a_{ij}\}$ with $a_{ij} = 0, i \neq j$, is given by the product of the diagonal elements; thus, $|\mathbf{A}| = a_{11}a_{22}\cdots a_{pp}$.
Hint: By Definition 2A.24, $|\mathbf{A}| = a_{11}\mathbf{A}_{11} + 0 + \cdots + 0$. Repeat for the submatrix $\mathbf{A}_{11}$ obtained by deleting the first row and first column of $\mathbf{A}$.

2.12. Show that the determinant of a square symmetric $p \times p$ matrix $\mathbf{A}$ can be expressed as the product of its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$; that is, $|\mathbf{A}| = \prod_{i=1}^{p} \lambda_i$.
Hint: From (2-16) and (2-20), $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}'$ with $\mathbf{P}'\mathbf{P} = \mathbf{I}$. From Result 2A.11(e), $|\mathbf{A}| = |\mathbf{P}\mathbf{\Lambda}\mathbf{P}'| = |\mathbf{P}||\mathbf{\Lambda}\mathbf{P}'| = |\mathbf{P}||\mathbf{\Lambda}||\mathbf{P}'| = |\mathbf{\Lambda}||\mathbf{I}|$, since $|\mathbf{I}| = |\mathbf{P}'\mathbf{P}| = |\mathbf{P}'||\mathbf{P}|$. Apply Exercise 2.11.

2.13. Show that $|\mathbf{Q}| = +1$ or -1 if $\mathbf{Q}$ is a $p \times p$ orthogonal matrix.
Hint: $|\mathbf{Q}\mathbf{Q}'| = |\mathbf{I}|$. Also, from Result 2A.11, $|\mathbf{Q}||\mathbf{Q}'| = |\mathbf{Q}|^2$. Thus, $|\mathbf{Q}|^2 = |\mathbf{I}|$. Now use Exercise 2.11.

2.14. Show that $\underset{(p\times p)}{\mathbf{Q}'}\underset{(p\times p)}{\mathbf{A}}\underset{(p\times p)}{\mathbf{Q}}$ and $\underset{(p\times p)}{\mathbf{A}}$ have the same eigenvalues if $\mathbf{Q}$ is orthogonal.
Hint: Let λ be an eigenvalue of $\mathbf{A}$. Then $0 = |\mathbf{A} - \lambda\mathbf{I}|$. By Exercise 2.13 and Result 2A.11(e), we can write $0 = |\mathbf{Q}'||\mathbf{A} - \lambda\mathbf{I}||\mathbf{Q}| = |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{I}|$, since $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$.

2.15. A quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is said to be positive definite if the matrix $\mathbf{A}$ is positive definite. Is the quadratic form $3x_1^2 + 3x_2^2 - 2x_1x_2$ positive definite?

2.16. Consider an arbitrary $n \times p$ matrix $\mathbf{A}$. Then $\mathbf{A}'\mathbf{A}$ is a symmetric $p \times p$ matrix. Show that $\mathbf{A}'\mathbf{A}$ is necessarily nonnegative definite.
Hint: Set $\mathbf{y} = \mathbf{A}\mathbf{x}$ so that $\mathbf{y}'\mathbf{y} = \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x}$.

2.7. Let $\mathbf{A}$ be as given in Exercise 2.6.

(a) Determine the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Write the spectral decomposition of $\mathbf{A}$.

(c) Find $\mathbf{A}^{-1}$.

(d) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

2.8. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

find the eigenvalues λ_1 and λ_2 and the associated normalized eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Determine the spectral decomposition (2-16) of $\mathbf{A}$.

2.9. Let $\mathbf{A}$ be as in Exercise 2.8.

(a) Find $\mathbf{A}^{-1}$.

(b) Compute the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

(c) Write the spectral decomposition of $\mathbf{A}^{-1}$, and compare it with that of $\mathbf{A}$ from Exercise 2.8.

2.10. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

These matrices are identical except for a small difference in the (2, 2) position. Moreover, the columns of $\mathbf{A}$ (and $\mathbf{B}$) are nearly linearly dependent. Show that $\mathbf{A}^{-1} \doteq (-3)\mathbf{B}^{-1}$. Consequently, small changes—perhaps caused by rounding—can give substantially different inverses.

2.11. Show that the determinant of the $p \times p$ diagonal matrix $\mathbf{A} = \{a_{ij}\}$ with $a_{ij} = 0, i \neq j$, is given by the product of the diagonal elements; thus, $|\mathbf{A}| = a_{11}a_{22}\cdots a_{pp}$.
Hint: By Definition 2A.24, $|\mathbf{A}| = a_{11}\mathbf{A}_{11} + 0 + \cdots + 0$. Repeat for the submatrix $\mathbf{A}_{11}$ obtained by deleting the first row and first column of $\mathbf{A}$.

2.12. Show that the determinant of a square symmetric $p \times p$ matrix $\mathbf{A}$ can be expressed as the product of its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$; that is, $|\mathbf{A}| = \prod_{i=1}^{p} \lambda_i$.
Hint: From (2-16) and (2-20), $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}'$ with $\mathbf{P}'\mathbf{P} = \mathbf{I}$. From Result 2A.11(e), $|\mathbf{A}| = |\mathbf{P}\mathbf{\Lambda}\mathbf{P}'| = |\mathbf{P}||\mathbf{\Lambda}\mathbf{P}'| = |\mathbf{P}||\mathbf{\Lambda}||\mathbf{P}'| = |\mathbf{\Lambda}||\mathbf{I}|$, since $|\mathbf{I}| = |\mathbf{P}'\mathbf{P}| = |\mathbf{P}'||\mathbf{P}|$. Apply Exercise 2.11.

2.13. Show that $|\mathbf{Q}| = +1$ or -1 if $\mathbf{Q}$ is a $p \times p$ orthogonal matrix.
Hint: $|\mathbf{Q}\mathbf{Q}'| = |\mathbf{I}|$. Also, from Result 2A.11, $|\mathbf{Q}||\mathbf{Q}'| = |\mathbf{Q}|^2$. Thus, $|\mathbf{Q}|^2 = |\mathbf{I}|$. Now use Exercise 2.11.

2.14. Show that $\underset{(p\times p)}{\mathbf{Q}'}\underset{(p\times p)}{\mathbf{A}}\underset{(p\times p)}{\mathbf{Q}}$ and $\underset{(p\times p)}{\mathbf{A}}$ have the same eigenvalues if $\mathbf{Q}$ is orthogonal.
Hint: Let λ be an eigenvalue of $\mathbf{A}$. Then $0 = |\mathbf{A} - \lambda\mathbf{I}|$. By Exercise 2.13 and Result 2A.11(e), we can write $0 = |\mathbf{Q}'||\mathbf{A} - \lambda\mathbf{I}||\mathbf{Q}| = |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{I}|$, since $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$.

2.15. A quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is said to be positive definite if the matrix $\mathbf{A}$ is positive definite. Is the quadratic form $3x_1^2 + 3x_2^2 - 2x_1x_2$ positive definite?

2.16. Consider an arbitrary $n \times p$ matrix $\mathbf{A}$. Then $\mathbf{A}'\mathbf{A}$ is a symmetric $p \times p$ matrix. Show that $\mathbf{A}'\mathbf{A}$ is necessarily nonnegative definite.
Hint: Set $\mathbf{y} = \mathbf{A}\mathbf{x}$ so that $\mathbf{y}'\mathbf{y} = \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x}$.

2.17. Prove that every eigenvalue of a $k \times k$ positive definite matrix $\mathbf{A}$ is positive.
Hint: Consider the definition of an eigenvalue, where $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$. Multiply on the left by $\mathbf{e}'$ so that $\mathbf{e}'\mathbf{A}\mathbf{e} = \lambda\mathbf{e}'\mathbf{e}$.

2.18. Consider the sets of points (x_1, x_2) whose "distances" from the origin are given by

$$c^2 = 4x_1^2 + 3x_2^2 - 2\sqrt{2}x_1x_2$$

for $c^2 = 1$ and for $c^2 = 4$. Determine the major and minor axes of the ellipses of constant distances and their associated lengths. Sketch the ellipses of constant distances and comment on their positions. What will happen as c^2 increases?

2.19. Let $\underset{(m\times m)}{\mathbf{A}^{1/2}} = \sum_{i=1}^{m} \sqrt{\lambda_i}\,\mathbf{e}_i\mathbf{e}_i' = \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}'$, where $\mathbf{P}\mathbf{P}' = \mathbf{P}'\mathbf{P} = \mathbf{I}$. (The λ_i's and the $\mathbf{e}_i$'s are the eigenvalues and associated normalized eigenvectors of the matrix $\mathbf{A}$.) Show Properties (1)–(4) of the square-root matrix in (2-22).

2.20. Determine the square-root matrix $\mathbf{A}^{1/2}$, using the matrix $\mathbf{A}$ in Exercise 2.3. Also, determine $\mathbf{A}^{-1/2}$, and show that $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}^{1/2} = \mathbf{I}$.

2.21. (See Result 2A.15) Using the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

(a) Calculate $\mathbf{A}'\mathbf{A}$ and obtain its eigenvalues and eigenvectors.

(b) Calculate $\mathbf{A}\mathbf{A}'$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in part a.

(c) Obtain the singular-value decomposition of $\mathbf{A}$.

2.22. (See Result 2A.15) Using the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

(a) Calculate $\mathbf{A}\mathbf{A}'$ and obtain its eigenvalues and eigenvectors.

(b) Calculate $\mathbf{A}'\mathbf{A}$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in part a.

(c) Obtain the singular-value decomposition of $\mathbf{A}$.

2.23. Verify the relationships $\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \mathbf{\Sigma}$ and $\boldsymbol{\rho} = (\mathbf{V}^{1/2})^{-1}\mathbf{\Sigma}(\mathbf{V}^{1/2})^{-1}$, where $\mathbf{\Sigma}$ is the $p \times p$ population covariance matrix [Equation (2-32)], $\boldsymbol{\rho}$ is the $p \times p$ population correlation matrix [Equation (2-34)], and $\mathbf{V}^{1/2}$ is the population standard deviation matrix [Equation (2-35)].

2.24. Let $\mathbf{X}$ have covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find

(a) $\mathbf{\Sigma}^{-1}$

(b) The eigenvalues and eigenvectors of $\mathbf{\Sigma}$.

(c) The eigenvalues and eigenvectors of $\mathbf{\Sigma}^{-1}$.

2.25. Let $\mathbf{X}$ have covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

(a) Determine $\boldsymbol{\rho}$ and $\mathbf{V}^{1/2}$.

(b) Multiply your matrices to check the relation $\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \boldsymbol{\Sigma}$.

2.26. Use $\boldsymbol{\Sigma}$ as given in Exercise 2.25.

(a) Find ρ_{13}.

(b) Find the correlation between X_1 and $\frac{1}{2}X_2 + \frac{1}{2}X_3$.

2.27. Derive expressions for the mean and variances of the following linear combinations in terms of the means and covariances of the random variables X_1, X_2, and X_3.

(a) $X_1 - 2X_2$

(b) $-X_1 + 3X_2$

(c) $X_1 + X_2 + X_3$

(e) $X_1 + 2X_2 - X_3$

(f) $3X_1 - 4X_2$ if X_1 and X_2 are independent random variables.

2.28. Show that

$$\text{Cov}(c_{11}X_1 + c_{12}X_2 + \cdots + c_{1p}X_p, c_{21}X_1 + c_{22}X_2 + \cdots + c_{2p}X_p) = \mathbf{c}_1'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{c}_2$$

where $\mathbf{c}_1' = [c_{11}, c_{12}, \ldots, c_{1p}]$ and $\mathbf{c}_2' = [c_{21}, c_{22}, \ldots, c_{2p}]$. This verifies the off-diagonal elements $\mathbf{C}\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{C}'$ in (2-45) or diagonal elements if $\mathbf{c}_1 = \mathbf{c}_2$.

Hint: By (2-43), $Z_1 - E(Z_1) = c_{11}(X_1 - \mu_1) + \cdots + c_{1p}(X_p - \mu_p)$ and $Z_2 - E(Z_2) = c_{21}(X_1 - \mu_1) + \cdots + c_{2p}(X_p - \mu_p)$. So $\text{Cov}(Z_1, Z_2) = E[(Z_1 - E(Z_1))(Z_2 - E(Z_2))] = E[(c_{11}(X_1 - \mu_1) + \cdots + c_{1p}(X_p - \mu_p))(c_{21}(X_1 - \mu_1) + c_{22}(X_2 - \mu_2) + \cdots + c_{2p}(X_p - \mu_p))]$.

The product

$$\begin{aligned} &(c_{11}(X_1 - \mu_1) + c_{12}(X_2 - \mu_2) + \cdots \\ &\quad + c_{1p}(X_p - \mu_p))(c_{21}(X_1 - \mu_1) + c_{22}(X_2 - \mu_2) + \cdots + c_{2p}(X_p - \mu_p)) \\ &= \left(\sum_{\ell=1}^{p} c_{1\ell}(X_\ell - \mu_\ell)\right)\left(\sum_{m=1}^{p} c_{2m}(X_m - \mu_m)\right) \\ &= \sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}(X_\ell - \mu_\ell)(X_m - \mu_m) \end{aligned}$$

has expected value

$$\sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}\sigma_{\ell m} = [c_{11}, \ldots, c_{1p}]\boldsymbol{\Sigma}[c_{21}, \ldots, c_{2p}]'.$$

Verify the last step by the definition of matrix multiplication. The same steps hold for all elements.

2.32. You are given the random vector $\mathbf{X}' = [X_1, X_2, \ldots, X_5]$ with mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = [2, 4, -1, 3, 0]$ and variance–covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 4 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 3 & 1 & -1 & 0 \\ \frac{1}{2} & 1 & 6 & 1 & -1 \\ -\frac{1}{2} & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and consider the linear combinations $\mathbf{AX}^{(1)}$ and $\mathbf{BX}^{(2)}$. Find

(a) $E(\mathbf{X}^{(1)})$
(b) $E(\mathbf{AX}^{(1)})$
(c) $\text{Cov}(\mathbf{X}^{(1)})$
(d) $\text{Cov}(\mathbf{AX}^{(1)})$
(e) $E(\mathbf{X}^{(2)})$
(f) $E(\mathbf{BX}^{(2)})$
(g) $\text{Cov}(\mathbf{X}^{(2)})$
(h) $\text{Cov}(\mathbf{BX}^{(2)})$
(i) $\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$
(j) $\text{Cov}(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)})$

2.33. Repeat Exercise 2.32, but with $\mathbf{X}$ partitioned as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \hdashline X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

and with $\mathbf{A}$ and $\mathbf{B}$ replaced by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

2.34. Consider the vectors $\mathbf{b}' = [2, -1, 4, 0]$ and $\mathbf{d}' = [-1, 3, -2, 1]$. Verify the Cauchy–Schwarz inequality $(\mathbf{b}'\mathbf{d})^2 \le (\mathbf{b}'\mathbf{b})(\mathbf{d}'\mathbf{d})$.

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2.37. With **A**

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Let

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(b) Fi
(c) W

2.32. You are given the random vector $\mathbf{X}' = [X_1, X_2, \dots, X_5]$ with mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = [2, 4, -1, 3, 0]$ and variance–covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 4 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 3 & 1 & -1 & 0 \\ \frac{1}{2} & 1 & 6 & 1 & -1 \\ -\frac{1}{2} & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and consider the linear combinations $\mathbf{AX}^{(1)}$ and $\mathbf{BX}^{(2)}$. Find

(a) $E(\mathbf{X}^{(1)})$
(b) $E(\mathbf{AX}^{(1)})$
(c) $\text{Cov}(\mathbf{X}^{(1)})$
(d) $\text{Cov}(\mathbf{AX}^{(1)})$
(e) $E(\mathbf{X}^{(2)})$
(f) $E(\mathbf{BX}^{(2)})$
(g) $\text{Cov}(\mathbf{X}^{(2)})$
(h) $\text{Cov}(\mathbf{BX}^{(2)})$
(i) $\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$
(j) $\text{Cov}(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)})$

2.33. Repeat Exercise 2.32, but with $\mathbf{X}$ partitioned as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \hdashline X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

and with $\mathbf{A}$ and $\mathbf{B}$ replaced by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

2.34. Consider the vectors $\mathbf{b}' = [2, -1, 4, 0]$ and $\mathbf{d}' = [-1, 3, -2, 1]$. Verify the Cauchy-Schwarz inequality $(\mathbf{b}'\mathbf{d})^2 \le (\mathbf{b}'\mathbf{b})(\mathbf{d}'\mathbf{d})$.

2.35. Using the vectors $\mathbf{b}' = [-4, 3]$ and $\mathbf{d}' = [1, 1]$, verify the extended Cauchy–Schwarz inequality $(\mathbf{b}'\mathbf{d})^2 \le (\mathbf{b}'\mathbf{B}\mathbf{b})(\mathbf{d}'\mathbf{B}^{-1}\mathbf{d})$ if

$$\mathbf{B} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

2.36. Find the maximum and minimum values of the quadratic form $4x_1^2 + 4x_2^2 + 6x_1x_2$ for all points $\mathbf{x}' = [x_1, x_2]$ such that $\mathbf{x}'\mathbf{x} = 1$.

2.37. With $\mathbf{A}$ as given in Exercise 2.6, find the maximum value of $\mathbf{x}'\mathbf{A}\,\mathbf{x}$ for $\mathbf{x}'\mathbf{x} = 1$.

2.38. Find the maximum and minimum values of the ratio $\mathbf{x}'\mathbf{A}\,\mathbf{x}/\mathbf{x}'\mathbf{x}$ for any nonzero vectors $\mathbf{x}' = [x_1, x_2, x_3]$ if

$$\mathbf{A} = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix}$$

2.39. Show that

$$\underset{(r\times s)(s\times t)(t\times v)}{\mathbf{A}\;\;\mathbf{B}\;\;\mathbf{C}} \text{ has } (i, j)\text{th entry } \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{i\ell}b_{\ell k}c_{kj}$$

Hint: $\mathbf{BC}$ has (ℓ, j)th entry $\sum_{k=1}^{t} b_{\ell k}c_{kj} = d_{\ell j}$. So $\mathbf{A}(\mathbf{BC})$ has (i, j)th element

$$a_{i1}d_{1j} + a_{i2}d_{2j} + \cdots + a_{is}\,d_{sj} = \sum_{\ell=1}^{s} a_{i\ell}\left(\sum_{k=1}^{t} b_{\ell k}c_{kj}\right) = \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{i\ell}b_{\ell k}c_{kj}$$

2.40. Verify (2-24): $E(\mathbf{X} + \mathbf{Y}) = E(\mathbf{X}) + E(\mathbf{Y})$ and $E(\mathbf{AXB}) = \mathbf{A}E(\mathbf{X})\mathbf{B}$.
Hint: $\mathbf{X} + \mathbf{Y}$ has $X_{ij} + Y_{ij}$ as its (i, j)th element. Now, $E(X_{ij} + Y_{ij}) = E(X_{ij}) + E(Y_{ij})$ by a univariate property of expectation, and this last quantity is the (i, j)th element of $E(\mathbf{X}) + E(\mathbf{Y})$. Next (see Exercise 2.39), $\mathbf{AXB}$ has (i, j)th entry $\sum_\ell \sum_k a_{i\ell}X_{\ell k}b_{kj}$, and by the additive property of expectation,

$$E\left(\sum_\ell \sum_k a_{i\ell}X_{\ell k}b_{kj}\right) = \sum_\ell \sum_k a_{i\ell}E(X_{\ell k})b_{kj}$$

which is the (i, j)th element of $\mathbf{A}E(\mathbf{X})\mathbf{B}$.

2.41. You are given the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = [3, 2, -2, 0]$ and variance–covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

(a) Find $E(\mathbf{AX})$, the mean of $\mathbf{AX}$.
(b) Find $\text{Cov}(\mathbf{AX})$, the variances and covariances of $\mathbf{AX}$.
(c) Which pairs of linear combinations have zero covariances?