These can be expressed in matrix notation as

$$\begin{bmatrix} a_{11}X_1 & + & a_{12}X_2 & + \cdots + & a_{1p}X_p \\ a_{21}X_1 & + & a_{22}X_2 & + \cdots + & a_{2p}X_p \\ \vdots & & \vdots & & \vdots \\ a_{q1}X_1 & + & a_{q2}X_2 & + \cdots + & a_{qp}X_p \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qp} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = \mathbf{AX} \tag{3-38}$$

Taking the ith row of $\mathbf{A}$, $\mathbf{a}_i'$, to be $\mathbf{b}'$ and the kth row of $\mathbf{A}$, $\mathbf{a}_k'$, to be $\mathbf{c}'$, we see that Equations (3-36) imply that the ith row of $\mathbf{AX}$ has sample mean $\mathbf{a}_i'\bar{\mathbf{x}}$ and the ith and kth rows of $\mathbf{AX}$ have sample covariance $\mathbf{a}_i'\mathbf{S}\,\mathbf{a}_k$. Note that $\mathbf{a}_i'\mathbf{S}\,\mathbf{a}_k$ is the (i, k)th element of $\mathbf{ASA}'$.

Result 3.6. The q linear combinations $\mathbf{AX}$ in (3-38) have sample mean vector $\mathbf{A}\bar{\mathbf{x}}$ and sample covariance matrix $\mathbf{ASA}'$. ■

Exercises

3.1. Given the data matrix

$$\mathbf{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

(a) Graph the scatter plot in $p = 2$ dimensions. Locate the sample mean on your diagram.

(b) Sketch the $n = 3$-dimensional representation of the data, and plot the deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$.

(c) Sketch the deviation vectors in (b) emanating from the origin. Calculate the lengths of these vectors and the cosine of the angle between them. Relate these quantities to $\mathbf{S}_n$ and $\mathbf{R}$.

3.2. Given the data matrix

$$\mathbf{X} = \begin{bmatrix} 3 & 4 \\ 6 & -2 \\ 3 & 1 \end{bmatrix}$$

(a) Graph the scatter plot in $p = 2$ dimensions, and locate the sample mean on your diagram.

(b) Sketch the $n = 3$-space representation of the data, and plot the deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$.

(c) Sketch the deviation vectors in (b) emanating from the origin. Calculate their lengths and the cosine of the angle between them. Relate these quantities to $\mathbf{S}_n$ and $\mathbf{R}$.

3.3. Perform the decomposition of $\mathbf{y}_1$ into $\bar{x}_1\mathbf{1}$ and $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ using the first column of the data matrix in Example 3.9.

3.4. Use the six observations on the variable X_1, in units of millions, from Table 1.1.

(a) Find the projection on $\mathbf{1}' = [1, 1, 1, 1, 1, 1]$.

(b) Calculate the deviation vector $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$. Relate its length to the sample standard deviation.

These can be expressed in matrix notation as

$$\begin{bmatrix} a_{11}X_1 & + & a_{12}X_2 & + \cdots + & a_{1p}X_p \\ a_{21}X_1 & + & a_{22}X_2 & + \cdots + & a_{2p}X_p \\ \vdots & & \vdots & & \vdots \\ a_{q1}X_1 & + & a_{q2}X_2 & + \cdots + & a_{qp}X_p \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qp} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = \mathbf{AX} \tag{3-38}$$

Taking the ith row of $\mathbf{A}$, $\mathbf{a}_i'$, to be $\mathbf{b}'$ and the kth row of $\mathbf{A}$, $\mathbf{a}_k'$, to be $\mathbf{c}'$, we see that Equations (3-36) imply that the ith row of $\mathbf{AX}$ has sample mean $\mathbf{a}_i'\bar{\mathbf{x}}$ and the ith and kth rows of $\mathbf{AX}$ have sample covariance $\mathbf{a}_i'\mathbf{S}\,\mathbf{a}_k$. Note that $\mathbf{a}_i'\mathbf{S}\,\mathbf{a}_k$ is the (i, k)th element of $\mathbf{ASA}'$.

Result 3.6. The q linear combinations $\mathbf{AX}$ in (3-38) have sample mean vector $\mathbf{A}\bar{\mathbf{x}}$ and sample covariance matrix $\mathbf{ASA}'$. ■

Exercises

3.1. Given the data matrix

$$\mathbf{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

(a) Graph the scatter plot in $p = 2$ dimensions. Locate the sample mean on your diagram.
(b) Sketch the $n = 3$-dimensional representation of the data, and plot the deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$.
(c) Sketch the deviation vectors in (b) emanating from the origin. Calculate the lengths of these vectors and the cosine of the angle between them. Relate these quantities to $\mathbf{S}_n$ and $\mathbf{R}$.

3.2. Given the data matrix

$$\mathbf{X} = \begin{bmatrix} 3 & 4 \\ 6 & -2 \\ 3 & 1 \end{bmatrix}$$

(a) Graph the scatter plot in $p = 2$ dimensions, and locate the sample mean on your diagram.
(b) Sketch the $n = 3$-space representation of the data, and plot the deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$.
(c) Sketch the deviation vectors in (b) emanating from the origin. Calculate their lengths and the cosine of the angle between them. Relate these quantities to $\mathbf{S}_n$ and $\mathbf{R}$.

3.3. Perform the decomposition of $\mathbf{y}_1$ into $\bar{x}_1\mathbf{1}$ and $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ using the first column of the data matrix in Example 3.9.

3.4. Use the six observations on the variable X_1, in units of millions, from Table 1.1.
(a) Find the projection on $\mathbf{1}' = [1, 1, 1, 1, 1, 1]$.
(b) Calculate the deviation vector $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$. Relate its length to the sample standard deviation.

(c) Graph (to scale) the triangle formed by $\mathbf{y}_1$, $\bar{x}_1\mathbf{1}$, and $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$. Identify the length of each component in your graph.
(d) Repeat Parts a–c for the variable X_2 in Table 1.1.
(e) Graph (to scale) the two deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$. Calculate the value of the angle between them.

3.5. Calculate the generalized sample variance $|\mathbf{S}|$ for (a) the data matrix $\mathbf{X}$ in Exercise 3.1 and (b) the data matrix $\mathbf{X}$ in Exercise 3.2.

3.6. Consider the data matrix

$$\mathbf{X} = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

(a) Calculate the matrix of deviations (residuals), $\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}'$. Is this matrix of full rank? Explain.
(b) Determine $\mathbf{S}$ and calculate the generalized sample variance $|\mathbf{S}|$. Interpret the latter geometrically.
(c) Using the results in (b), calculate the total sample variance. [See (3-23).]

3.7. Sketch the solid ellipsoids $(\mathbf{x} - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) \le 1$ [see (3-16)] for the three matrices

$$\mathbf{S} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

(Note that these matrices have the *same* generalized variance $|\mathbf{S}|$.)

3.8. Given

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

(a) Calculate the total sample variance for each $\mathbf{S}$. Compare the results.
(b) Calculate the generalized sample variance for each $\mathbf{S}$, and compare the results. Comment on the discrepancies, if any, found between Parts a and b.

3.9. The following data matrix contains data on test scores, with x_1 = score on first test, x_2 = score on second test, and x_3 = total score on the two tests:

$$\mathbf{X} = \begin{bmatrix} 12 & 17 & 29 \\ 18 & 20 & 38 \\ 14 & 16 & 30 \\ 20 & 18 & 38 \\ 16 & 19 & 35 \end{bmatrix}$$

(a) Obtain the mean corrected data matrix, and verify that the columns are linearly dependent. Specify an $\mathbf{a}' = [a_1, a_2, a_3]$ vector that establishes the linear dependence.
(b) Obtain the sample covariance matrix $\mathbf{S}$, and verify that the generalized variance is zero. Also, show that $\mathbf{Sa} = \mathbf{0}$, so $\mathbf{a}$ can be rescaled to be an eigenvector corresponding to eigenvalue zero.
(c) Verify that the third column of the data matrix is the sum of the first two columns. That is, show that there is linear dependence, with $a_1 = 1$, $a_2 = 1$, and $a_3 = -1$.

3.10. When the generalized variance is zero, it is the columns of the mean corrected data matrix $\mathbf{X}_c = \mathbf{X} - \mathbf{1}\bar{\mathbf{x}}'$ that are linearly dependent, not necessarily those of the data matrix itself. Given the data

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 6 \\ 4 & 2 & 2 \\ 7 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

(a) Obtain the mean corrected data matrix, and verify that the columns are linearly dependent. Specify an $\mathbf{a}' = [a_1, a_2, a_3]$ vector that establishes the dependence.

(b) Obtain the sample covariance matrix $\mathbf{S}$, and verify that the generalized variance is zero.

(c) Show that the columns of the data matrix are linearly independent in this case.

3.11. Use the sample covariance obtained in Example 3.7 to verify (3-29) and (3-30), which state that $\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2}$ and $\mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2} = \mathbf{S}$.

√**3.12.** Show that $|\mathbf{S}| = (s_{11}s_{22}\cdots s_{pp})|\mathbf{R}|$.

Hint: From Equation (3-30), $\mathbf{S} = \mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2}$. Taking determinants gives $|\mathbf{S}| = |\mathbf{D}^{1/2}||\mathbf{R}||\mathbf{D}^{1/2}|$. (See Result 2A.11.) Now examine $|\mathbf{D}^{1/2}|$.

3.13. Given a data matrix $\mathbf{X}$ and the resulting sample correlation matrix $\mathbf{R}$, consider the standardized observations $(x_{jk} - \bar{x}_k)/\sqrt{s_{kk}}$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$. Show that these standardized quantities have sample covariance matrix $\mathbf{R}$.

3.14. Consider the data matrix $\mathbf{X}$ in Exercise 3.1. We have $n = 3$ observations on $p = 2$ variables X_1 and X_2. Form the linear combinations

$$\mathbf{c}'\mathbf{X} = [-1 \quad 2]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -X_1 + 2X_2$$

$$\mathbf{b}'\mathbf{X} = [2 \quad 3]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 2X_1 + 3X_2$$

(a) Evaluate the sample means, variances, and covariance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ from first principles. That is, calculate the observed values of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$, and then use the sample mean, variance, and covariance formulas.

(b) Calculate the sample means, variances, and covariance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ using (3-36). Compare the results in (a) and (b).

3.15. Repeat Exercise 3.14 using the data matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

3.10. When the generalized variance is zero, it is the columns of the mean corrected data matrix $\mathbf{X}_c = \mathbf{X} - \mathbf{1}\bar{\mathbf{x}}'$ that are linearly dependent, not necessarily those of the data matrix itself. Given the data

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 6 \\ 4 & 2 & 2 \\ 7 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

(a) Obtain the mean corrected data matrix, and verify that the columns are linearly dependent. Specify an $\mathbf{a}' = [a_1, a_2, a_3]$ vector that establishes the dependence.

(b) Obtain the sample covariance matrix $\mathbf{S}$, and verify that the generalized variance is zero.

(c) Show that the columns of the data matrix are linearly independent in this case.

3.11. Use the sample covariance obtained in Example 3.7 to verify (3-29) and (3-30), which state that $\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2}$ and $\mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2} = \mathbf{S}$.

√**3.12.** Show that $|\mathbf{S}| = (s_{11}s_{22}\cdots s_{pp})|\mathbf{R}|$.

Hint: From Equation (3-30), $\mathbf{S} = \mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2}$. Taking determinants gives $|\mathbf{S}| = |\mathbf{D}^{1/2}||\mathbf{R}||\mathbf{D}^{1/2}|$. (See Result 2A.11.) Now examine $|\mathbf{D}^{1/2}|$.

3.13. Given a data matrix $\mathbf{X}$ and the resulting sample correlation matrix $\mathbf{R}$, consider the standardized observations $(x_{jk} - \bar{x}_k)/\sqrt{s_{kk}}$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$. Show that these standardized quantities have sample covariance matrix $\mathbf{R}$.

3.14. Consider the data matrix $\mathbf{X}$ in Exercise 3.1. We have $n = 3$ observations on $p = 2$ variables X_1 and X_2. Form the linear combinations

$$\mathbf{c}'\mathbf{X} = [-1 \quad 2]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -X_1 + 2X_2$$

$$\mathbf{b}'\mathbf{X} = [2 \quad 3]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 2X_1 + 3X_2$$

(a) Evaluate the sample means, variances, and covariance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ from first principles. That is, calculate the observed values of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$, and then use the sample mean, variance, and covariance formulas.

(b) Calculate the sample means, variances, and covariance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ using (3-36). Compare the results in (a) and (b).

3.15. Repeat Exercise 3.14 using the data matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

and the linear combinations

$$\mathbf{b}'\mathbf{X} = [1 \quad 1 \quad 1]\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

and

$$\mathbf{c}'\mathbf{X} = [1 \quad 2 \quad -3]\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

3.16. Let $\mathbf{V}$ be a vector random variable with mean vector $E(\mathbf{V}) = \boldsymbol{\mu}_V$ and covariance matrix $E(\mathbf{V} - \boldsymbol{\mu}_V)(\mathbf{V} - \boldsymbol{\mu}_V)' = \boldsymbol{\Sigma}_V$. Show that $E(\mathbf{V}\mathbf{V}') = \boldsymbol{\Sigma}_V + \boldsymbol{\mu}_V\boldsymbol{\mu}_V'$.

3.17. Show that, if $\underset{(p\times 1)}{\mathbf{X}}$ and $\underset{(q\times 1)}{\mathbf{Z}}$ are independent, then each component of $\mathbf{X}$ is independent of each component of $\mathbf{Z}$.

Hint: $P[X_1 \le x_1, X_2 \le x_2, \ldots, X_p \le x_p \text{ and } Z_1 \le z_1, \ldots, Z_q \le z_q]$

$$= P[X_1 \le x_1, X_2 \le x_2, \ldots, X_p \le x_p]\cdot P[Z_1 \le z_1, \ldots, Z_q \le z_q]$$

by independence. Let $x_2, \ldots, x_p$ and $z_2, \ldots, z_q$ tend to infinity, to obtain

$$P[X_1 \le x_1 \text{ and } Z_1 \le z_1] = P[X_1 \le x_1]\cdot P[Z_1 \le z_1]$$

for all x_1, z_1. So X_1 and Z_1 are independent. Repeat for other pairs.

√**3.18.** Energy consumption in 2001, by state, from the major sources

x_1 = petroleum $\qquad$ x_2 = natural gas

x_3 = hydroelectric power $\qquad$ x_4 = nuclear electric power

is recorded in quadrillions (10^{15}) of BTUs (Source: *Statistical Abstract of the United States 2006*).

The resulting mean and covariance matrix are

$$\bar{\mathbf{x}} = \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix}$$

(a) Using the summary statistics, determine the sample mean and variance of a state's total energy consumption for these major sources.

(b) Determine the sample mean and variance of the excess of petroleum consumption over natural gas consumption. Also find the sample covariance of this variable with the total variable in part a.

3.19. Using the summary statistics for the first three variables in Exercise 3.18, verify the relation

$$|\mathbf{S}| = (s_{11}\, s_{22}\, s_{33})\, |\mathbf{R}|$$

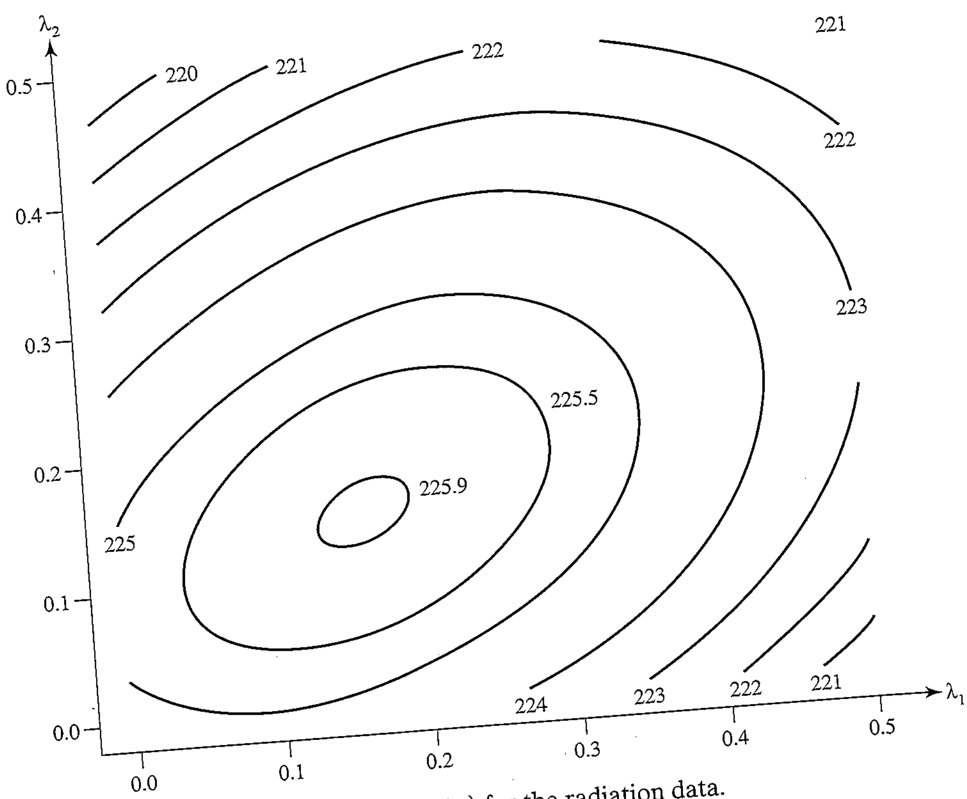


Figure 4.15 Contour plot of $\ell(\lambda_1, \lambda_2)$ for the radiation data.

If the data includes some large negative values and have a single long tail, a more general transformation (see Yeo and Johnson [14]) should be applied.

$$x^{(\lambda)} = \begin{cases} \{(x+1)^{\lambda} - 1\}/\lambda & x \geq 0, \lambda \neq 0 \\ \ln(x+1) & x \geq 0, \lambda = 0 \\ -\{(-x+1)^{2-\lambda} - 1\}/(2-\lambda) & x < 0, \lambda \neq 2 \\ -\ln(-x+1) & x < 0, \lambda = 2 \end{cases}$$

Exercises

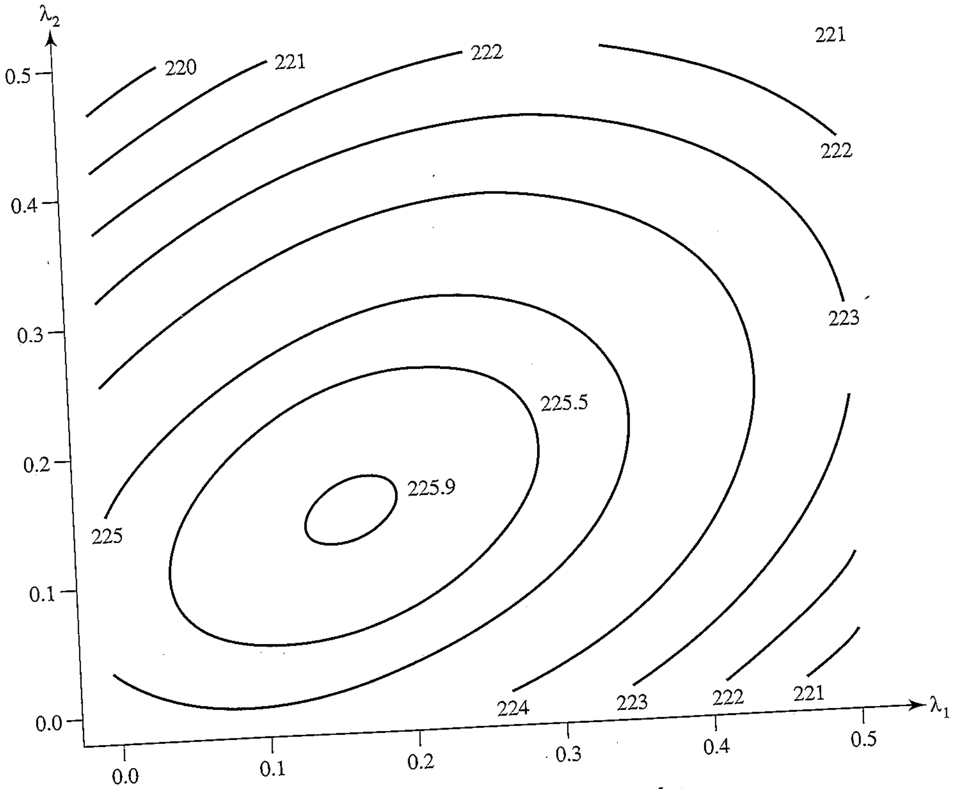
4.1. Consider a bivariate normal distribution with $\mu_1 = 1$, $\mu_2 = 3$, $\sigma_{11} = 2$, $\sigma_{22} = 1$ and $\rho_{12} = -.8$.

(a) Write out the bivariate normal density.

(b) Write out the squared statistical distance expression $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ as a quadratic function of x_1 and x_2.

4.2. Consider a bivariate normal population with $\mu_1 = 0$, $\mu_2 = 2$, $\sigma_{11} = 2$, $\sigma_{22} = 1$, and $\rho_{12} = .5$.

(a) Write out the bivariate normal density.



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(a) Write out the bivariate normal density.

(b) Write out the squared statistical distance expression $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ as a quadratic function of x_1 and x_2.

4.2. Consider a bivariate normal population with $\mu_1 = 0$, $\mu_2 = 2$, $\sigma_{11} = 2$, $\sigma_{22} = 1$, and $\rho_{12} = .5$.

(a) Write out the bivariate normal density.

(b) Write out the squared generalized distance expression $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ as a function of x_1 and x_2.

(c) Determine (and sketch) the constant-density contour that contains 50% of the probability.

4.3. Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [-3, 1, 4]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a) X_1 and X_2

(b) X_2 and X_3

(c) (X_1, X_2) and X_3

(d) $\dfrac{X_1 + X_2}{2}$ and X_3

(e) X_2 and $X_2 - \frac{5}{2}X_1 - X_3$

4.4. Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [2, -3, 1]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(a) Find the distribution of $3X_1 - 2X_2 + X_3$.

(b) Relabel the variables if necessary, and find a 2×1 vector $\mathbf{a}$ such that X_2 and $X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

4.5. Specify each of the following.

(a) The conditional distribution of X_1, given that $X_2 = x_2$ for the joint distribution in Exercise 4.2.

(b) The conditional distribution of X_2, given that $X_1 = x_1$ and $X_3 = x_3$ for the joint distribution in Exercise 4.3.

(c) The conditional distribution of X_3, given that $X_1 = x_1$ and $X_2 = x_2$ for the joint distribution in Exercise 4.4.

4.6. Let $\mathbf{X}$ be distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}' = [1, -1, 2]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a) X_1 and X_2

(b) X_1 and X_3

(c) X_2 and X_3

(d) (X_1, X_3) and X_2

(e) X_1 and $X_1 + 3X_2 - 2X_3$

4.7. Refer to Exercise 4.6 and specify each of the following.

(a) The conditional distribution of X_1, given that $X_3 = x_3$.

(b) The conditional distribution of X_1, given that $X_2 = x_2$ and $X_3 = x_3$.

4.8. (Example of a nonnormal bivariate distribution with normal marginals.) Let X_1 be $N(0,1)$, and let

$$X_2 = \begin{cases} -X_1 & \text{if } -1 \le X_1 \le 1 \\ X_1 & \text{otherwise} \end{cases}$$

Show each of the following.

(a) X_2 also has an $N(0,1)$ distribution.

(b) X_1 and X_2 do *not* have a bivariate normal distribution.

Hint:

(a) Since X_1 is $N(0,1)$, $P[-1 < X_1 \le x] = P[-x \le X_1 < 1]$ for any x. When $-1 < x_2 < 1$, $P[X_2 \le x_2] = P[X_2 \le -1] + P[-1 < X_2 \le x_2] = P[X_1 \le -1] + P[-1 < -X_1 \le x_2] = P[X_1 \le -1] + P[-x_2 \le X_1 < 1]$. But $P[-x_2 \le X_1 < 1] = P[-1 < X_1 \le x_2]$ from the symmetry argument in the first line of this hint. Thus, $P[X_2 \le x_2] = P[X_1 \le -1] + P[-1 < X_1 \le x_2] = P[X_1 \le x_2]$, which is a standard normal probability.

(b) Consider the linear combination $X_1 - X_2$, which equals zero with probability $P[|X_1| > 1] = .3174$.

4.9. Refer to Exercise 4.8, but modify the construction by replacing the break point 1 by c so that

$$X_2 = \begin{cases} -X_1 & \text{if } -c \le X_1 \le c \\ X_1 & \text{elsewhere} \end{cases}$$

Show that c can be chosen so that $\text{Cov}(X_1, X_2) = 0$, but that the two random variables are not independent.

Hint:

For $c = 0$, evaluate $\text{Cov}(X_1, X_2) = E[X_1(X_1)]$.
For c very large, evaluate $\text{Cov}(X_1, X_2) \doteq E[X_1(-X_1)]$.

4.10. Show each of the following.

(a)

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}|$$

(b)

$$\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}| \quad \text{for} \quad |\mathbf{A}| \ne 0$$

Hint:

(a) $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$. Expanding the determinant $\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$ by the first row (see Definition 2A.24) gives 1 times a determinant of the same form, with the order of $\mathbf{I}$ reduced by one. This procedure is repeated until $1 \times |\mathbf{B}|$ is obtained. Similarly, expanding the determinant $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row gives $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = |\mathbf{A}|$.

4.7. Refer to Exercise 4.6 and specify each of the following.

(a) The conditional distribution of X_1, given that $X_3 = x_3$.

(b) The conditional distribution of X_1, given that $X_2 = x_2$ and $X_3 = x_3$.

4.8. (Example of a nonnormal bivariate distribution with normal marginals.) Let X_1 be $N(0,1)$, and let

$$X_2 = \begin{cases} -X_1 & \text{if } -1 \le X_1 \le 1 \\ X_1 & \text{otherwise} \end{cases}$$

Show each of the following.

(a) X_2 also has an $N(0,1)$ distribution.

(b) X_1 and X_2 do *not* have a bivariate normal distribution.

Hint:

(a) Since X_1 is $N(0,1)$, $P[-1 < X_1 \le x] = P[-x \le X_1 < 1]$ for any x. When $-1 < x_2 < 1$, $P[X_2 \le x_2] = P[X_2 \le -1] + P[-1 < X_2 \le x_2] = P[X_1 \le -1]$ $+ P[-1 < -X_1 \le x_2] = P[X_1 \le -1] + P[-x_2 \le X_1 < 1]$. But $P[-x_2 \le X_1 < 1]$ $= P[-1 < X_1 \le x_2]$ from the symmetry argument in the first line of this hint. Thus, $P[X_2 \le x_2] = P[X_1 \le -1] + P[-1 < X_1 \le x_2] = P[X_1 \le x_2]$, which is a standard normal probability.

(b) Consider the linear combination $X_1 - X_2$, which equals zero with probability $P[|X_1| > 1] = .3174$.

4.9. Refer to Exercise 4.8, but modify the construction by replacing the break point 1 by c so that

$$X_2 = \begin{cases} -X_1 & \text{if } -c \le X_1 \le c \\ X_1 & \text{elsewhere} \end{cases}$$

Show that c can be chosen so that $\text{Cov}(X_1, X_2) = 0$, but that the two random variables are not independent.

Hint:

For $c = 0$, evaluate $\text{Cov}(X_1, X_2) = E[X_1(X_1)]$

For c very large, evaluate $\text{Cov}(X_1, X_2) \doteq E[X_1(-X_1)]$.

4.10. Show each of the following.

(a)

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}|$$

(b)

$$\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}| \quad \text{for} \quad |\mathbf{A}| \ne 0$$

Hint:

(a) $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$. Expanding the determinant $\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$ by the first row (see Definition 2A.24) gives 1 times a determinant of the same form, with the order of $\mathbf{I}$ reduced by one. This procedure is repeated until $1 \times |\mathbf{B}|$ is obtained. Similarly, expanding the determinant $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row gives $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = |\mathbf{A}|$.

(b) $\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$. But expanding the determinant $\begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row gives $\begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = 1$. Now use the result in Part a.

4.11. Show that, if $\mathbf{A}$ is square,

$$\begin{aligned} |\mathbf{A}| &= |\mathbf{A}_{22}||\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}| \quad \text{for } |\mathbf{A}_{22}| \ne 0 \\ &= |\mathbf{A}_{11}||\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}| \quad \text{for } |\mathbf{A}_{11}| \ne 0 \end{aligned}$$

Hint: Partition $\mathbf{A}$ and verify that

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} \end{bmatrix}$$

Take determinants on both sides of this equality. Use Exercise 4.10 for the first and third determinants on the left and for the determinant on the right. The second equality for $|\mathbf{A}|$ follows by considering

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix}$$

4.12. Show that, for $\mathbf{A}$ symmetric,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix}$$

Thus, $(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1}$ is the upper left-hand block of $\mathbf{A}^{-1}$.

Hint: Premultiply the expression in the hint to Exercise 4.11 by $\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix}^{-1}$ and postmultiply by $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix}^{-1}$. Take inverses of the resulting expression.

4.13. Show the following if $\mathbf{X}$ is $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}| \ne 0$.

(a) Check that $|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{22}||\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}|$. (Note that $|\boldsymbol{\Sigma}|$ can be factored into the product of contributions from the marginal and conditional distributions.)

(b) Check that

$$\begin{aligned} (\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) &= [\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)]' \\ &\quad \times (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1}[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)] \\ &\quad + (\mathbf{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \end{aligned}$$

(Thus, the joint density exponent can be written as the sum of two terms corresponding to contributions from the conditional and marginal distributions.)

(c) Given the results in Parts a and b, identify the marginal distribution of $\mathbf{X}_2$ and the conditional distribution of $\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2$.

Hint:

(a) Apply Exercise 4.11.

(b) Note from Exercise 4.12 that we can write $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ as

$$\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}'\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{I}\end{bmatrix}\begin{bmatrix}(\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1}\end{bmatrix}$$
$$\times\begin{bmatrix}\mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$

If we group the product so that

$$\begin{bmatrix}\mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}=\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$

the result follows.

4.14. If $\mathbf{X}$ is distributed as $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}|\neq 0$, show that the joint density can be written as the product of marginal densities for

$$\underset{(q\times 1)}{\mathbf{X}_1} \quad\text{and}\quad \underset{((p-q)\times 1)}{\mathbf{X}_2} \quad\text{if}\quad \boldsymbol{\Sigma}_{12}=\underset{(q\times(p-q))}{\mathbf{0}}$$

Hint: Show by block multiplication that

$$\begin{bmatrix}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1}\end{bmatrix}\text{ is the inverse of }\boldsymbol{\Sigma}=\begin{bmatrix}\boldsymbol{\Sigma}_{11} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}\end{bmatrix}$$

Then write

$$(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})=[(\mathbf{x}_1-\boldsymbol{\mu}_1)',(\mathbf{x}_2-\boldsymbol{\mu}_2)']\begin{bmatrix}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$
$$=(\mathbf{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1)+(\mathbf{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)$$

Note that $|\boldsymbol{\Sigma}|=|\boldsymbol{\Sigma}_{11}||\boldsymbol{\Sigma}_{22}|$ from Exercise 4.10(a). Now factor the joint density.

✓ **4.15.** Show that $\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})'$ and $\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})'$ are both $p\times p$ matrices of zeros. Here $\mathbf{x}_j'=[x_{j1},x_{j2},\ldots,x_{jp}]$, $j=1,2,\ldots,n$, and

$$\bar{\mathbf{x}}=\frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j$$

4.16. Let $\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3$, and $\mathbf{X}_4$ be independent $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ random vectors.

(a) Find the marginal distributions for each of the random vectors

$$\mathbf{V}_1=\tfrac{1}{4}\mathbf{X}_1-\tfrac{1}{4}\mathbf{X}_2+\tfrac{1}{4}\mathbf{X}_3-\tfrac{1}{4}\mathbf{X}_4$$

and

$$\mathbf{V}_2=\tfrac{1}{4}\mathbf{X}_1+\tfrac{1}{4}\mathbf{X}_2-\tfrac{1}{4}\mathbf{X}_3-\tfrac{1}{4}\mathbf{X}_4$$

(b) Find the joint density of the random vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ defined in (a).

4.17. Let $\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4$, and $\mathbf{X}_5$ be independent and identically distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Find the mean vector and covariance matrices for each of the two linear combinations of random vectors

$$\tfrac{1}{5}\mathbf{X}_1+\tfrac{1}{5}\mathbf{X}_2+\tfrac{1}{5}\mathbf{X}_3+\tfrac{1}{5}\mathbf{X}_4+\tfrac{1}{5}\mathbf{X}_5$$

Hint:

(a) Apply Exercise 4.11.

(b) Note from Exercise 4.12 that we can write $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ as

$$\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}'\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{I}\end{bmatrix}\begin{bmatrix}(\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1}\end{bmatrix}$$
$$\times\begin{bmatrix}\mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$

If we group the product so that

$$\begin{bmatrix}\mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}=\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$

the result follows.

4.14. If $\mathbf{X}$ is distributed as $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}|\neq 0$, show that the joint density can be written as the product of marginal densities for

$$\underset{(q\times 1)}{\mathbf{X}_1} \quad\text{and}\quad \underset{((p-q)\times 1)}{\mathbf{X}_2} \quad\text{if}\quad \boldsymbol{\Sigma}_{12}=\underset{(q\times(p-q))}{\mathbf{0}}$$

Hint: Show by block multiplication that

$$\begin{bmatrix}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1}\end{bmatrix}\text{ is the inverse of }\boldsymbol{\Sigma}=\begin{bmatrix}\boldsymbol{\Sigma}_{11} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}\end{bmatrix}$$

Then write

$$(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})=[(\mathbf{x}_1-\boldsymbol{\mu}_1)',(\mathbf{x}_2-\boldsymbol{\mu}_2)']\begin{bmatrix}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0}\\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1-\boldsymbol{\mu}_1\\ \mathbf{x}_2-\boldsymbol{\mu}_2\end{bmatrix}$$
$$=(\mathbf{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1)+(\mathbf{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)$$

Note that $|\boldsymbol{\Sigma}|=|\boldsymbol{\Sigma}_{11}||\boldsymbol{\Sigma}_{22}|$ from Exercise 4.10(a). Now factor the joint density.

4.15. Show that $\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})'$ and $\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})'$ are both $p\times p$ matrices of zeros. Here $\mathbf{x}_j'=[x_{j1},x_{j2},\ldots,x_{jp}]$, $j=1,2,\ldots,n$, and

$$\bar{\mathbf{x}}=\frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j$$

4.16. Let $\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3$, and $\mathbf{X}_4$ be independent $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ random vectors.

(a) Find the marginal distributions for each of the random vectors

$$\mathbf{V}_1=\tfrac{1}{4}\mathbf{X}_1-\tfrac{1}{4}\mathbf{X}_2+\tfrac{1}{4}\mathbf{X}_3-\tfrac{1}{4}\mathbf{X}_4$$

and

$$\mathbf{V}_2=\tfrac{1}{4}\mathbf{X}_1+\tfrac{1}{4}\mathbf{X}_2-\tfrac{1}{4}\mathbf{X}_3-\tfrac{1}{4}\mathbf{X}_4$$

(b) Find the joint density of the random vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ defined in (a).

4.17. Let $\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4$, and $\mathbf{X}_5$ be independent and identically distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Find the mean vector and covariance matrices for each of the two linear combinations of random vectors

$$\tfrac{1}{5}\mathbf{X}_1+\tfrac{1}{5}\mathbf{X}_2+\tfrac{1}{5}\mathbf{X}_3+\tfrac{1}{5}\mathbf{X}_4+\tfrac{1}{5}\mathbf{X}_5$$

and

$$\mathbf{X}_1-\mathbf{X}_2+\mathbf{X}_3-\mathbf{X}_4+\mathbf{X}_5$$

in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Also, obtain the covariance between the two linear combinations of random vectors.

4.18. Find the maximum likelihood estimates of the 2×1 mean vector $\boldsymbol{\mu}$ and the 2×2 covariance matrix $\boldsymbol{\Sigma}$ based on the random sample

$$\mathbf{X}=\begin{bmatrix}3 & 6\\ 4 & 4\\ 5 & 7\\ 4 & 7\end{bmatrix}$$

from a bivariate normal population.

4.19. Let $\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_{20}$ be a random sample of size $n=20$ from an $N_6(\boldsymbol{\mu},\boldsymbol{\Sigma})$ population. Specify each of the following completely.

(a) The distribution of $(\mathbf{X}_1-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1-\boldsymbol{\mu})$

(b) The distributions of $\bar{\mathbf{X}}$ and $\sqrt{n}(\bar{\mathbf{X}}-\boldsymbol{\mu})$

(c) The distribution of $(n-1)\,\mathbf{S}$

4.20. For the random variables $\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_{20}$ in Exercise 4.19, specify the distribution of $\mathbf{B}(19\mathbf{S})\mathbf{B}'$ in each case.

(a) $\mathbf{B}=\begin{bmatrix}1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1\end{bmatrix}$

(b) $\mathbf{B}=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}$

4.21. Let $\mathbf{X}_1,\ldots,\mathbf{X}_{60}$ be a random sample of size 60 from a four-variate normal distribution having mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. Specify each of the following completely.

(a) The distribution of $\bar{\mathbf{X}}$

(b) The distribution of $(\mathbf{X}_1-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1-\boldsymbol{\mu})$

(c) The distribution of $n(\bar{\mathbf{X}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu})$

(d) The approximate distribution of $n(\bar{\mathbf{X}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu})$

4.22. Let $\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_{75}$ be a random sample from a population distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. What is the approximate distribution of each of the following?

(a) $\bar{\mathbf{X}}$

(b) $n(\bar{\mathbf{X}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu})$

4.23. Consider the annual rates of return (including dividends) on the Dow-Jones industrial average for the years 1996–2005. These data, multiplied by 100, are

−0.6 3.1 25.3 −16.8 −7.1 −6.2 25.2 22.6 26.0.

Use these 10 observations to complete the following.

(a) Construct a Q–Q plot. Do the data seem to be normally distributed? Explain.

(b) Carry out a test of normality based on the correlation coefficient r_Q. [See (4-31).] Let the significance level be $\alpha=.10$.

4.24. Exercise 1.4 contains data on three variables for the world's 10 largest companies as of April 2005. For the sales (x_1) and profits (x_2) data:

(a) Construct Q–Q plots. Do these data appear to be normally distributed? Explain.

(b) Carry out a test of normality based on the correlation coefficient r_Q. [See (4–31).] Set the significance level at $\alpha = .10$. Do the results of these tests corroborate the results in Part a?

4.25. Refer to the data for the world's 10 largest companies in Exercise 1.4. Construct a chi-square plot using all *three* variables. The chi-square quantiles are

0.3518 0.7978 1.2125 1.6416 2.1095 2.6430 3.2831 4.1083 5.3170 7.8147

4.26. Exercise 1.2 gives the age x_1, measured in years, as well as the selling price x_2, measured in thousands of dollars, for $n = 10$ used cars. These data are reproduced as follows:

x_1	1	2	3	3	4	5	6	8	9	11
x_2	18.95	19.00	17.95	15.54	14.00	12.95	8.94	7.49	6.00	3.99

(a) Use the results of Exercise 1.2 to calculate the squared statistical distances $(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}})$, $j = 1, 2, \ldots, 10$, where $\mathbf{x}_j' = [x_{j1}, x_{j2}]$.

(b) Using the distances in Part a, determine the proportion of the observations falling within the estimated 50% probability contour of a bivariate normal distribution.

(c) Order the distances in Part a and construct a chi-square plot.

(d) Given the results in Parts b and c, are these data approximately bivariate normal? Explain.

4.27. Consider the radiation data (with door closed) in Example 4.10. Construct a Q–Q plot for the natural logarithms of these data. [Note that the natural logarithm transformation corresponds to the value $\lambda = 0$ in (4-34).] Do the natural logarithms appear to be normally distributed? Compare your results with Figure 4.13. Does the choice $\lambda = \frac{1}{4}$ or $\lambda = 0$ make much difference in this case?

The following exercises may require a computer.

4.28. Consider the air-pollution data given in Table 1.5. Construct a Q–Q plot for the solar radiation measurements and carry out a test for normality based on the correlation coefficient r_Q [see (4-31)]. Let $\alpha = .05$ and use the entry corresponding to $n = 40$ in Table 4.2.

4.29. Given the air-pollution data in Table 1.5, examine the pairs $X_5 = NO_2$ and $X_6 = O_3$ for bivariate normality.

(a) Calculate statistical distances $(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}})$, $j = 1, 2, \ldots, 42$, where $\mathbf{x}_j' = [x_{j5}, x_{j6}]$.

(b) Determine the proportion of observations $\mathbf{x}_j' = [x_{j5}, x_{j6}]$, $j = 1, 2, \ldots, 42$, falling within the approximate 50% probability contour of a bivariate normal distribution.

(c) Construct a chi-square plot of the ordered distances in Part a.

4.30. Consider the used-car data in Exercise 4.26.

(a) Determine the power transformation $\hat{\lambda}_1$ that makes the x_1 values approximately normal. Construct a Q–Q plot for the transformed data.

(b) Determine the power transformations $\hat{\lambda}_2$ that makes the x_2 values approximately normal. Construct a Q–Q plot for the transformed data.

(c) Determine the power transformations $\hat{\boldsymbol{\lambda}}' = [\hat{\lambda}_1, \hat{\lambda}_2]$ that make the $[x_1, x_2]$ values jointly normal using (4-40). Compare the results with those obtained in Parts a and b.