The corresponding univariate Student's t-test statistics for testing for no difference in the individual means have p-values of .46 and .08, respectively. Clearly, from a univariate perspective, we cannot detect a difference in mass means or a difference in snout-vent length means for the two genera of lizards.

However, consistent with the scatter diagram in Figure 6.7, a bivariate analysis strongly supports a difference in size between the two groups of lizards. Using Result 6.4 (also see Example 6.5), the T^2-statistic has an approximate χ_2^2 distribution. For this example, $T^2 = 225.4$ with a p-value less than .0001. A multivariate method is essential in this case. ■

Examples 6.16 and 6.17 demonstrate the efficacy of a multivariate test relative to its univariate counterparts. We encountered exactly this situation with the effluent data in Example 6.1.

In the context of random samples from several populations (recall the one-way MANOVA in Section 6.4), multivariate tests are based on the matrices

$$\mathbf{W} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)' \quad \text{and} \quad \mathbf{B} = \sum_{\ell=1}^{g} n_\ell (\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})'$$

Throughout this chapter, we have used

$$\text{Wilks' lambda statistic } \Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{B} + \mathbf{W}|}$$

which is equivalent to the likelihood ratio test. Three other multivariate test statistics are regularly included in the output of statistical packages.

$$\text{Lawley–Hotelling trace} = \text{tr}[\mathbf{B}\mathbf{W}^{-1}]$$

$$\text{Pillai trace} = \text{tr}[\mathbf{B}(\mathbf{B} + \mathbf{W})^{-1}]$$

$$\text{Roy's largest root} = \text{maximum eigenvalue of } \mathbf{W}(\mathbf{B} + \mathbf{W})^{-1}$$

All four of these tests appear to be nearly equivalent for extremely large samples. For moderate sample sizes, all comparisons are based on what is necessarily a limited number of cases studied by simulation. From the simulations reported to date, the first three tests have similar power, while the last, Roy's test, behaves differently. Its power is best only when there is a single nonzero eigenvalue and, at the same time, the power is large. This may approximate situations where a large difference exists in just one characteristic and it is between one group and all of the others. There is also some suggestion that Pillai's trace is slightly more robust against nonnormality. However, we suggest trying transformations on the original data when the residuals are nonnormal.

All four statistics apply in the two-way setting and in even more complicated MANOVA. More discussion is given in terms of the multivariate regression model in Chapter 7.

When, and only when, the multivariate tests signals a difference, or departure from the null hypothesis, do we probe deeper. We recommend calculating the Bonferroni intervals for all pairs of groups and all characteristics. The simultaneous confidence statements determined from the shadows of the confidence ellipse are, typically, too large. The one-at-a-time intervals may be suggestive of differences that

merit further study but, with the current data, cannot be taken as conclusive evidence for the existence of differences. We summarize the procedure developed in this chapter for comparing treatments. The first step is to check the data for outliers using visual displays and other calculations.

A Strategy for the Multivariate Comparison of Treatments

1. *Try to identify outliers.* Check the data group by group for outliers. Also check the collection of residual vectors from any fitted model for outliers. Be aware of any outliers so calculations can be performed with and without them.
2. *Perform a multivariate test of hypothesis.* Our choice is the likelihood ratio test, which is equivalent to Wilks' lambda test.
3. *Calculate the Bonferroni simultaneous confidence intervals.* If the multivariate test reveals a difference, then proceed to calculate the Bonferroni confidence intervals for all pairs of groups or treatments, and all characteristics. If no differences are significant, try looking at Bonferroni intervals for the larger set of responses that includes the differences and sums of pairs of responses.

We must issue one caution concerning the proposed strategy. It may be the case that differences would appear in only one of the many characteristics and, further, the differences hold for only a few treatment combinations. Then, these few active differences may become lost among all the inactive ones. That is, the overall test may not show significance whereas a univariate test restricted to the specific active variable would detect the difference. The best preventative is a good experimental design. To design an effective experiment when one specific variable is expected to produce differences, do not include too many other variables that are not expected to show differences among the treatments.

Exercises

6.1. Construct and sketch a joint 95% confidence region for the mean difference vector $\boldsymbol{\delta}$ using the effluent data and results in Example 6.1. Note that the point $\boldsymbol{\delta} = \mathbf{0}$ falls outside the 95% contour. Is this result consistent with the test of H_0: $\boldsymbol{\delta} = \mathbf{0}$ considered in Example 6.1? Explain.

6.2. Using the information in Example 6.1. construct the 95% Bonferroni simultaneous intervals for the components of the mean difference vector $\boldsymbol{\delta}$. Compare the lengths of these intervals with those of the simultaneous intervals constructed in the example.

6.3. The data corresponding to sample 8 in Table 6.1 seem unusually large. Remove sample 8. Construct a joint 95% confidence region for the mean difference vector $\boldsymbol{\delta}$ and the 95% Bonferroni simultaneous intervals for the components of the mean difference vector. Are the results consistent with a test of H_0: $\boldsymbol{\delta} = \mathbf{0}$? Discuss. Does the "outlier" make a difference in the analysis of these data?

6.4. Refer to Example 6.1.

(a) Redo the analysis in Example 6.1 after transforming the pairs of observations to ln(BOD) and ln(SS).

(b) Construct the 95% Bonferroni simultaneous intervals for the components of the mean vector $\boldsymbol{\delta}$ of transformed variables.

(c) Discuss any possible violation of the assumption of a bivariate normal distribution for the difference vectors of transformed observations.

6.5. A researcher considered three indices measuring the severity of heart attacks. The values of these indices for $n = 40$ heart-attack patients arriving at a hospital emergency room produced the summary statistics

$$\bar{\mathbf{x}} = \begin{bmatrix} 46.1 \\ 57.3 \\ 50.4 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 101.3 & 63.0 & 71.0 \\ 63.0 & 80.2 & 55.6 \\ 71.0 & 55.6 & 97.4 \end{bmatrix}$$

(a) All three indices are evaluated for each patient. Test for the equality of mean indices using (6-16) with $\alpha = .05$.

(b) Judge the differences in pairs of mean indices using 95% simultaneous confidence intervals. [See (6-18).]

6.6. Use the data for treatments 2 and 3 in Exercise 6.8.

(a) Calculate $\mathbf{S}_{\text{pooled}}$.

(b) Test $H_0\colon \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3 = \mathbf{0}$ employing a two-sample approach with $\alpha = .01$.

(c) Construct 99% simultaneous confidence intervals for the differences $\mu_{2i} - \mu_{3i}$, $i = 1, 2$.

6.7. Using the summary statistics for the electricity-demand data given in Example 6.4, compute T^2 and test the hypothesis $H_0\colon \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \mathbf{0}$, assuming that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$. Set $\alpha = .05$. Also, determine the linear combination of mean components most responsible for the rejection of H_0.

6.8. Observations on two responses are collected for three treatments. The observation vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are

$$\text{Treatment 1:} \quad \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$\text{Treatment 2:} \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{Treatment 3:} \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(a) Break up the observations into mean, treatment, and residual components, as in (6-39). Construct the corresponding arrays for each variable. (See Example 6.9.)

(b) Using the information in Part a, construct the one-way MANOVA table.

(c) Evaluate Wilks' lambda, Λ^*, and use Table 6.3 to test for treatment effects. Set $\alpha = .01$. Repeat the test using the chi-square approximation with Bartlett's correction. [See (6-43).] Compare the conclusions.

6.4. Refer to Example 6.1.

(a) Redo the analysis in Example 6.1 after transforming the pairs of observations to ln(BOD) and ln(SS).

(b) Construct the 95% Bonferroni simultaneous intervals for the components of the mean vector $\boldsymbol{\delta}$ of transformed variables.

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Treatment 1: $\begin{bmatrix} 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \end{bmatrix}$

Treatment 2: $\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Treatment 3: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(a) Break up the observations into mean, treatment, and residual components, as in (6-39). Construct the corresponding arrays for each variable. (See Example 6.9.)

(b) Using the information in Part a, construct the one-way MANOVA table.

(c) Evaluate Wilks' lambda, Λ^*, and use Table 6.3 to test for treatment effects. Set $\alpha = .01$. Repeat the test using the chi-square approximation with Bartlett's correction. [See (6-43).] Compare the conclusions.

6.9. Using the contrast matrix $\mathbf{C}$ in (6-13), verify the relationships $\mathbf{d}_j = \mathbf{C}\mathbf{x}_j$, $\bar{\mathbf{d}} = \mathbf{C}\bar{\mathbf{x}}$, and $\mathbf{S}_d = \mathbf{C}\mathbf{S}\mathbf{C}'$ in (6-14).

6.10. Consider the univariate one-way decomposition of the observation $x_{\ell j}$ given by (6-34). Show that the mean vector $\bar{x}\,\mathbf{1}$ is always perpendicular to the treatment effect vector $(\bar{x}_1 - \bar{x})\mathbf{u}_1 + (\bar{x}_2 - \bar{x})\mathbf{u}_2 + \cdots + (\bar{x}_g - \bar{x})\mathbf{u}_g$ where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Big\}{n_1}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Big\}{n_2}, \quad \ldots, \quad \mathbf{u}_g = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Big\}{n_g}$$

6.11. A likelihood argument provides additional support for pooling the two independent sample covariance matrices to estimate a common covariance matrix in the case of two normal populations. Give the likelihood function, $L(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma})$, for two independent samples of sizes n_1 and n_2 from $N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$ populations, respectively. Show that this likelihood is maximized by the choices $\hat{\boldsymbol{\mu}}_1 = \bar{\mathbf{x}}_1$, $\hat{\boldsymbol{\mu}}_2 = \bar{\mathbf{x}}_2$ and

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n_1 + n_2}[(n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2] = \left(\frac{n_1 + n_2 - 2}{n_1 + n_2}\right)\mathbf{S}_{\text{pooled}}$$

Hint: Use (4-16) and the maximization Result 4.10.

6.12. *(Test for linear profiles, given that the profiles are parallel.)* Let $\boldsymbol{\mu}_1' = [\mu_{11}, \mu_{12}, \ldots, \mu_{1p}]$ and $\boldsymbol{\mu}_2' = [\mu_{21}, \mu_{22}, \ldots, \mu_{2p}]$ be the mean responses to p treatments for populations 1 and 2, respectively. Assume that the profiles given by the two mean vectors are parallel.

(a) Show that the hypothesis that the profiles are linear can be written as $H_0\colon (\mu_{1i} + \mu_{2i}) - (\mu_{1i-1} + \mu_{2i-1}) = (\mu_{1i-1} + \mu_{2i-1}) - (\mu_{1i-2} + \mu_{2i-2})$, $i = 3, \ldots, p$ or as $H_0\colon \mathbf{C}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) = \mathbf{0}$, where the $(p - 2) \times p$ matrix

$$\mathbf{C} = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix}$$

(b) Following an argument similar to the one leading to (6-73), we reject $H_0\colon \mathbf{C}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) = \mathbf{0}$ at level α if

$$T^2 = (\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)'\mathbf{C}'\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{C}\mathbf{S}_{\text{pooled}}\mathbf{C}'\right]^{-1}\mathbf{C}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2) > c^2$$

where

$$c^2 = \frac{(n_1 + n_2 - 2)(p - 2)}{n_1 + n_2 - p + 1}F_{p-2,\,n_1+n_2-p+1}(\alpha)$$

Let $n_1 = 30$, $n_2 = 30$, $\bar{\mathbf{x}}_1' = [6.4, 6.8, 7.3, 7.0]$, $\bar{\mathbf{x}}_2' = [4.3, 4.9, 5.3, 5.1]$, and

$$\mathbf{S}_{\text{pooled}} = \begin{bmatrix} .61 & .26 & .07 & .16 \\ .26 & .64 & .17 & .14 \\ .07 & .17 & .81 & .03 \\ .16 & .14 & .03 & .31 \end{bmatrix}$$

Test for linear profiles, assuming that the profiles are parallel. Use $\alpha = .05$.

6.13. *(Two-way MANOVA without replications.)* Consider the observations on two responses, x_1 and x_2, displayed in the form of the following two-way table (note that there is a *single* observation vector at each combination of factor levels):

		Factor 2 Level 1	Level 2	Level 3	Level 4
	Level 1	$\begin{bmatrix} 6 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 12 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 6 \end{bmatrix}$
Factor 1	Level 2	$\begin{bmatrix} 3 \\ 8 \end{bmatrix}$	$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -4 \\ 3 \end{bmatrix}$
	Level 3	$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -4 \\ -5 \end{bmatrix}$	$\begin{bmatrix} 3 \\ -3 \end{bmatrix}$	$\begin{bmatrix} -4 \\ -6 \end{bmatrix}$

With no replications, the two-way MANOVA model is

$$\mathbf{X}_{\ell k} = \boldsymbol{\mu} + \boldsymbol{\tau}_\ell + \boldsymbol{\beta}_k + \mathbf{e}_{\ell k}; \qquad \sum_{\ell=1}^{g} \boldsymbol{\tau}_\ell = \sum_{k=1}^{b} \boldsymbol{\beta}_k = \mathbf{0}$$

where the $\mathbf{e}_{\ell k}$ are independent $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ random vectors.

(a) Decompose the observations for each of the two variables as

$$x_{\ell k} = \bar{x} + (\bar{x}_{\ell\cdot} - \bar{x}) + (\bar{x}_{\cdot k} - \bar{x}) + (x_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})$$

similar to the arrays in Example 6.9. *For each response*, this decomposition will result in several 3×4 matrices. Here $\bar{x}$ is the overall average, $\bar{x}_{\ell\cdot}$ is the average for the ℓth level of factor 1, and $\bar{x}_{\cdot k}$ is the average for the kth level of factor 2.

(b) Regard the rows of the matrices in Part a as strung out in a single "long" vector, and compute the sums of squares

$$SS_{\text{tot}} = SS_{\text{mean}} + SS_{\text{fac1}} + SS_{\text{fac2}} + SS_{\text{res}}$$

and sums of cross products

$$SCP_{\text{tot}} = SCP_{\text{mean}} + SCP_{\text{fac1}} + SCP_{\text{fac2}} + SCP_{\text{res}}$$

Consequently, obtain the matrices $\mathbf{SSP}_{\text{cor}}$, $\mathbf{SSP}_{\text{fac1}}$, $\mathbf{SSP}_{\text{fac2}}$, and $\mathbf{SSP}_{\text{res}}$ with degrees of freedom $gb - 1$, $g - 1$, $b - 1$, and $(g - 1)(b - 1)$, respectively.

(c) Summarize the calculations in Part b in a MANOVA table.

Let $n_1 = 30$, $n_2 = 30$, $\bar{\mathbf{x}}_1' = [6.4, 6.8, 7.3, 7.0]$, $\bar{\mathbf{x}}_2' = [4.3, 4.9, 5.3, 5.1]$, and

$$\mathbf{S}_{\text{pooled}} = \begin{bmatrix} .61 & .26 & .07 & .16 \\ .26 & .64 & .17 & .14 \\ .07 & .17 & .81 & .03 \\ .16 & .14 & .03 & .31 \end{bmatrix}$$

Test for linear profiles, assuming that the profiles are parallel. Use $\alpha = .05$.

6.13. *(Two-way MANOVA without replications.)* Consider the observations on two responses, x_1 and x_2, displayed in the form of the following two-way table (note that there is a *single* observation vector at each combination of factor levels):

		Factor 2 Level 1	Level 2	Level 3	Level 4
Factor 1	Level 1	$\begin{bmatrix}6\\8\end{bmatrix}$	$\begin{bmatrix}4\\6\end{bmatrix}$	$\begin{bmatrix}8\\12\end{bmatrix}$	$\begin{bmatrix}2\\6\end{bmatrix}$
	Level 2	$\begin{bmatrix}3\\8\end{bmatrix}$	$\begin{bmatrix}-3\\2\end{bmatrix}$	$\begin{bmatrix}4\\3\end{bmatrix}$	$\begin{bmatrix}-4\\3\end{bmatrix}$
	Level 3	$\begin{bmatrix}-3\\2\end{bmatrix}$	$\begin{bmatrix}-4\\-5\end{bmatrix}$	$\begin{bmatrix}3\\-3\end{bmatrix}$	$\begin{bmatrix}-4\\-6\end{bmatrix}$

With no replications, the two-way MANOVA model is

$$\mathbf{X}_{\ell k} = \boldsymbol{\mu} + \boldsymbol{\tau}_\ell + \boldsymbol{\beta}_k + \mathbf{e}_{\ell k}; \qquad \sum_{\ell=1}^{g} \boldsymbol{\tau}_\ell = \sum_{k=1}^{b} \boldsymbol{\beta}_k = \mathbf{0}$$

where the $\mathbf{e}_{\ell k}$ are independent $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ random vectors.

(a) Decompose the observations for each of the two variables as

$$x_{\ell k} = \bar{x} + (\bar{x}_{\ell\cdot} - \bar{x}) + (\bar{x}_{\cdot k} - \bar{x}) + (x_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})$$

similar to the arrays in Example 6.9. *For each response*, this decomposition will result in several 3×4 matrices. Here $\bar{x}$ is the overall average, $\bar{x}_{\ell\cdot}$ is the average for the ℓth level of factor 1, and $\bar{x}_{\cdot k}$ is the average for the kth level of factor 2.

(b) Regard the rows of the matrices in Part a as strung out in a single "long" vector, and compute the sums of squares

$$\text{SS}_{\text{tot}} = \text{SS}_{\text{mean}} + \text{SS}_{\text{fac1}} + \text{SS}_{\text{fac2}} + \text{SS}_{\text{res}}$$

and sums of cross products

$$\text{SCP}_{\text{tot}} = \text{SCP}_{\text{mean}} + \text{SCP}_{\text{fac1}} + \text{SCP}_{\text{fac2}} + \text{SCP}_{\text{res}}$$

Consequently, obtain the matrices $\mathbf{SSP}_{\text{cor}}$, $\mathbf{SSP}_{\text{fac1}}$, $\mathbf{SSP}_{\text{fac2}}$, and $\mathbf{SSP}_{\text{res}}$ with degrees of freedom $gb - 1$, $g - 1$, $b - 1$, and $(g-1)(b-1)$, respectively.

(c) Summarize the calculations in Part b in a MANOVA table.

Hint: This MANOVA table is consistent with the two-way MANOVA table for comparing factors and their interactions where $n = 1$. Note that, with $n = 1$, $\mathbf{SSP}_{\text{res}}$ in the general two-way MANOVA table is a zero matrix with zero degrees of freedom. The matrix of interaction sum of squares and cross products now becomes the *residual* sum of squares and cross products matrix.

(d) Given the summary in Part c, test for factor 1 and factor 2 main effects at the $\alpha = .05$ level.

Hint: Use the results in (6-67) and (6-69) with $gb(n-1)$ replaced by $(g-1)(b-1)$.

Note: The tests require that $p \le (g-1)(b-1)$ so that $\mathbf{SSP}_{\text{res}}$ will be positive definite (with probability 1).

6.14. A *replicate* of the experiment in Exercise 6.13 yields the following data:

		Factor 2 Level 1	Level 2	Level 3	Level 4
Factor 1	Level 1	$\begin{bmatrix}14\\8\end{bmatrix}$	$\begin{bmatrix}6\\2\end{bmatrix}$	$\begin{bmatrix}8\\2\end{bmatrix}$	$\begin{bmatrix}16\\-4\end{bmatrix}$
	Level 2	$\begin{bmatrix}1\\6\end{bmatrix}$	$\begin{bmatrix}5\\12\end{bmatrix}$	$\begin{bmatrix}0\\15\end{bmatrix}$	$\begin{bmatrix}2\\7\end{bmatrix}$
	Level 3	$\begin{bmatrix}3\\-2\end{bmatrix}$	$\begin{bmatrix}-2\\7\end{bmatrix}$	$\begin{bmatrix}-11\\1\end{bmatrix}$	$\begin{bmatrix}-6\\6\end{bmatrix}$

(a) Use these data to decompose each of the two measurements in the observation vector as

$$x_{\ell k} = \bar{x} + (\bar{x}_{\ell\cdot} - \bar{x}) + (\bar{x}_{\cdot k} - \bar{x}) + (x_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})$$

where $\bar{x}$ is the overall average, $\bar{x}_{\ell\cdot}$ is the average for the ℓth level of factor 1, and $\bar{x}_{\cdot k}$ is the average for the kth level of factor 2. Form the corresponding arrays for each of the two responses.

(b) Combine the preceding data with the data in Exercise 6.13 and carry out the necessary calculations to complete the general two-way MANOVA table.

(c) Given the results in Part b, test for interactions, and if the interactions do not exist, test for factor 1 and factor 2 main effects. Use the likelihood ratio test with $\alpha = .05$.

(d) If main effects, but no interactions, exist, examine the nature of the main effects by constructing Bonferroni simultaneous 95% confidence intervals for differences of the components of the factor effect parameters.

6.15. Refer to Example 6.13.

(a) Carry out approximate chi-square (likelihood ratio) tests for the factor 1 and factor 2 effects. Set $\alpha = .05$. Compare these results with the results for the exact F-tests given in the example. Explain any differences.

(b) Using (6-70), construct simultaneous 95% confidence intervals for differences in the factor 1 effect parameters for *pairs* of the three responses. Interpret these intervals. Repeat these calculations for factor 2 effect parameters.

The following exercises may require the use of a computer.

6.16. Four measures of the response *stiffness* on each of 30 boards are listed in Table 4.3 (see Example 4.14). The measures, on a given board, are repeated in the sense that they were made one after another. Assuming that the measures of stiffness arise from four treatments, test for the equality of treatments in a *repeated measures design* context. Set $\alpha = .05$. Construct a 95% (simultaneous) confidence interval for a contrast in the mean levels representing a comparison of the dynamic measurements with the static measurements.

6.17. The data in Table 6.8 were collected to test two psychological models of numerical cognition. Does the processing of numbers depend on the way the numbers are presented (words, Arabic digits)? Thirty-two subjects were required to make a series of

Table 6.8 Number Parity Data (Median Times in Milliseconds)

WordDiff (x_1)	WordSame (x_2)	ArabicDiff (x_3)	ArabicSame (x_4)
869.0	860.5	691.0	601.0
995.0	875.0	678.0	659.0
1056.0	930.5	833.0	826.0
1126.0	954.0	888.0	728.0
1044.0	909.0	865.0	839.0
925.0	856.5	1059.5	797.0
1172.5	896.5	926.0	766.0
1408.5	1311.0	854.0	986.0
1028.0	887.0	915.0	735.0
1011.0	863.0	761.0	657.0
726.0	674.0	663.0	583.0
982.0	894.0	831.0	640.0
1225.0	1179.0	1037.0	905.5
731.0	662.0	662.5	624.0
975.5	872.5	814.0	735.0
1130.5	811.0	843.0	657.0
945.0	909.0	867.5	754.0
747.0	752.5	777.0	687.5
656.5	659.5	572.0	539.0
919.0	833.0	752.0	611.0
751.0	744.0	683.0	553.0
774.0	735.0	671.0	612.0
941.0	931.0	901.5	700.0
751.0	785.0	789.0	735.0
767.0	737.5	724.0	639.0
813.5	750.5	711.0	625.0
1289.5	1140.0	904.5	784.5
1096.5	1009.0	1076.0	983.0
1083.0	958.0	918.0	746.5
1114.0	1046.0	1081.0	796.0
708.0	669.0	657.0	572.5
1201.0	925.0	1004.5	673.5

Source: Data courtesy of J. Carr.

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6.16. Four measures of the response *stiffness* on each of 30 boards are listed in Table 4.3 (see Example 4.14). The measures, on a given board, are repeated in the sense that they were made one after another. Assuming that the measures of stiffness arise from four treatments, test for the equality of treatments in a *repeated measures design* context. Set $\alpha = .05$. Construct a 95% (simultaneous) confidence interval for a contrast in the mean levels representing a comparison of the dynamic measurements with the static measurements.

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1056.0	930.5	833.0	826.0
1126.0	954.0	888.0	728.0
1044.0	909.0	865.0	839.0
925.0	856.5	1059.5	797.0
1172.5	896.5	926.0	766.0
1408.5	1311.0	854.0	986.0
1028.0	887.0	915.0	735.0
1011.0	863.0	761.0	657.0
726.0	674.0	663.0	583.0
982.0	894.0	831.0	640.0
1225.0	1179.0	1037.0	905.5
731.0	662.0	662.5	624.0
975.5	872.5	814.0	735.0
1130.5	811.0	843.0	657.0
945.0	909.0	867.5	754.0
747.0	752.5	777.0	687.5
656.5	659.5	572.0	539.0
919.0	833.0	752.0	611.0
751.0	744.0	683.0	553.0
774.0	735.0	671.0	612.0
941.0	931.0	901.5	700.0
751.0	785.0	789.0	735.0
767.0	737.5	724.0	639.0
813.5	750.5	711.0	625.0
1289.5	1140.0	904.5	784.5
1096.5	1009.0	1076.0	983.0
1083.0	958.0	918.0	746.5
1114.0	1046.0	1081.0	796.0
708.0	669.0	657.0	572.5
1201.0	925.0	1004.5	673.5

Source: Data courtesy of J. Carr.

quick numerical judgments about two numbers presented as either two number words ("two," "four") or two single Arabic digits ("2," "4"). The subjects were asked to respond "same" if the two numbers had the same numerical parity (both even or both odd) and "different" if the two numbers had a different parity (one even, one odd). Half of the subjects were assigned a block of Arabic digit trials, followed by a block of number word trials, and half of the subjects received the blocks of trials in the reverse order. Within each block, the order of "same" and "different" parity trials was randomized for each subject. For each of the four combinations of parity and format, the median reaction times for correct responses were recorded for each subject. Here

X_1 = median reaction time for word format–different parity combination
X_2 = median reaction time for word format–same parity combination
X_3 = median reaction time for Arabic format–different parity combination
X_4 = median reaction time for Arabic format–same parity combination

(a) Test for treatment effects using a *repeated measures design*. Set $\alpha = .05$.

(b) Construct 95% (simultaneous) confidence intervals for the contrasts representing the number format effect, the parity type effect and the interaction effect. Interpret the resulting intervals.

(c) The absence of interaction supports the M model of numerical cognition, while the presence of interaction supports the C and C model of numerical cognition. Which model is supported in this experiment?

(d) For each subject, construct three difference scores corresponding to the number format contrast, the parity type contrast, and the interaction contrast. Is a multivariate normal distribution a reasonable population model for these data? Explain.

6.18. Jolicoeur and Mosimann [12] studied the relationship of size and shape for painted turtles. Table 6.9 contains their measurements on the carapaces of 24 female and 24 male turtles.

(a) Test for equality of the two population mean vectors using $\alpha = .05$.

(b) If the hypothesis in Part a is rejected, find the linear combination of mean components most responsible for rejecting H_0.

(c) Find simultaneous confidence intervals for the component mean differences. Compare with the Bonferroni intervals.

Hint: You may wish to consider logarithmic transformations of the observations.

6.19. In the first phase of a study of the cost of transporting milk from farms to dairy plants, a survey was taken of firms engaged in milk transportation. Cost data on X_1 = fuel, X_2 = repair, and X_3 = capital, all measured on a per-mile basis, are presented in Table 6.10 on page 345 for $n_1 = 36$ gasoline and $n_2 = 23$ diesel trucks.

(a) Test for differences in the mean cost vectors. Set $\alpha = .01$.

(b) If the hypothesis of equal cost vectors is rejected in Part a, find the linear combination of mean components most responsible for the rejection.

(c) Construct 99% simultaneous confidence intervals for the pairs of mean components. Which costs, if any, appear to be quite different?

(d) Comment on the validity of the assumptions used in your analysis. Note in particular that observations 9 and 21 for gasoline trucks have been identified as multivariate outliers. (See Exercise 5.22 and [2].) Repeat Part a with these observations deleted. Comment on the results.

Table 6.17 Peanut Data

Factor 1 Location	Factor 2 Variety	x_1 Yield	x_2 SdMatKer	x_3 SeedSize
1	5	195.3	153.1	51.4
1	5	194.3	167.7	53.7
2	5	189.7	139.5	55.5
2	5	180.4	121.1	44.4
1	6	203.0	156.8	49.8
1	6	195.9	166.0	45.8
2	6	202.7	166.1	60.4
2	6	197.6	161.8	54.1
1	8	193.5	164.5	57.8
1	8	187.0	165.1	58.6
2	8	201.5	166.8	65.0
2	8	200.0	173.8	67.2

Source: Data courtesy of Yolanda Lopez.

(d) Larger numbers correspond to better yield and grade–grain characteristics. Using location 2, can we conclude that one variety is better than the other two for each characteristic? Discuss your answer, using 95% Bonferroni simultaneous intervals for pairs of varieties.

6.32. In one experiment involving remote sensing, the spectral reflectance of three species of 1-year-old seedlings was measured at various wavelengths during the growing season. The seedlings were grown with two different levels of nutrient: the optimal level, coded +, and a suboptimal level, coded −. The species of seedlings used were sitka spruce (SS), Japanese larch (JL), and lodgepole pine (LP). Two of the variables measured were

$$X_1 = \text{percent spectral reflectance at wavelength 560 nm (green)}$$
$$X_2 = \text{percent spectral reflectance at wavelength 720 nm (near infrared)}$$

The cell means (CM) for Julian day 235 for each combination of species and nutrient level are as follows. These averages are based on four replications.

560CM	720CM	Species	Nutrient
10.35	25.93	SS	+
13.41	38.63	JL	+
7.78	25.15	LP	+
10.40	24.25	SS	−
17.78	41.45	JL	−
10.40	29.20	LP	−

(a) Treating the cell means as individual observations, perform a two-way MANOVA to test for a species effect and a nutrient effect. Use $\alpha = .05$.

(b) Construct a two-way ANOVA for the 560CM observations and another two-way ANOVA for the 720CM observations. Are these results consistent with the MANOVA results in Part a? If not, can you explain any differences?

6.33. Refer to Exercise 6.32. The data in Table 6.18 are measurements on the variables

$$X_1 = \text{percent spectral reflectance at wavelength 560 nm (green)}$$
$$X_2 = \text{percent spectral reflectance at wavelength 720 nm (near infrared)}$$

for three species (sitka spruce [SS], Japanese larch [JL], and lodgepole pine [LP]) of 1-year-old seedlings taken at three different times (Julian day 150 [1], Julian day 235 [2], and Julian day 320 [3]) during the growing season. The seedlings were all grown with the optimal level of nutrient.

(a) Perform a two-factor MANOVA using the data in Table 6.18. Test for a species effect, a time effect and species–time interaction. Use $\alpha = .05$.

Table 6.18 Spectral Reflectance Data

560 nm	720 nm	Species	Time	Replication
9.33	19.14	SS	1	1
8.74	19.55	SS	1	2
9.31	19.24	SS	1	3
8.27	16.37	SS	1	4
10.22	25.00	SS	2	1
10.13	25.32	SS	2	2
10.42	27.12	SS	2	3
10.62	26.28	SS	2	4
15.25	38.89	SS	3	1
16.22	36.67	SS	3	2
17.24	40.74	SS	3	3
12.77	67.50	SS	3	4
12.07	33.03	JL	1	1
11.03	32.37	JL	1	2
12.48	31.31	JL	1	3
12.12	33.33	JL	1	4
15.38	40.00	JL	2	1
14.21	40.48	JL	2	2
9.69	33.90	JL	2	3
14.35	40.15	JL	2	4
38.71	77.14	JL	3	1
44.74	78.57	JL	3	2
36.67	71.43	JL	3	3
37.21	45.00	JL	3	4
8.73	23.27	LP	1	1
7.94	20.87	LP	1	2
8.37	22.16	LP	1	3
7.86	21.78	LP	1	4
8.45	26.32	LP	2	1
6.79	22.73	LP	2	2
8.34	26.67	LP	2	3
7.54	24.87	LP	2	4
14.04	44.44	LP	3	1
13.51	37.93	LP	3	2
13.33	37.93	LP	3	3
12.77	60.87	LP	3	4

Source: Data courtesy of Mairtin Mac Siurtain.

(b) Do you think the usual MANOVA assumptions are satisfied for the these data? Discuss with reference to a residual analysis, and the possibility of correlated observations over time.

(c) Foresters are particularly interested in the interaction of species and time. Does interaction show up for one variable but not for the other? Check by running a univariate two-factor ANOVA for each of the two responses.

(d) Can you think of another method of analyzing these data (or a different experimental design) that would allow for a potential time trend in the spectral reflectance numbers?

6.34. Refer to Example 6.15.

(a) Plot the profiles, the components of $\bar{\mathbf{x}}_1$ versus time and those of $\bar{\mathbf{x}}_2$ versus time, on the same graph. Comment on the comparison.

(b) Test that linear growth is adequate. Take $\alpha = .01$.

6.35. Refer to Example 6.15 but treat all 31 subjects as a single group. The maximum likelihood estimate of the $(q + 1) \times 1$ $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = (\mathbf{B}'\mathbf{S}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{S}^{-1}\bar{\mathbf{x}}$$

where $\mathbf{S}$ is the sample covariance matrix.

The estimated covariances of the maximum likelihood estimators are

$$\widehat{\text{Cov}}(\hat{\boldsymbol{\beta}}) = \frac{(n-1)(n-2)}{(n-1-p+q)(n-p+q)n}(\mathbf{B}'\mathbf{S}^{-1}\mathbf{B})^{-1}$$

Fit a quadratic growth curve to this single group and comment on the fit.

6.36. Refer to Example 6.4. Given the summary information on electrical usage in this example, use Box's M-test to test the hypothesis $H_0: \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}$. Here $\boldsymbol{\Sigma}_1$ is the covariance matrix for the two measures of usage for the population of Wisconsin homeowners *with* air conditioning, and $\boldsymbol{\Sigma}_2$ is the electrical usage covariance matrix for the population of Wisconsin homeowners *without* air conditioning. Set $\alpha = .05$.

6.37. Table 6.9 page 344 contains the carapace measurements for 24 female and 24 male turtles. Use Box's M-test to test $H_0: \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}$. where $\boldsymbol{\Sigma}_1$ is the population covariance matrix for carapace measurements for female turtles, and $\boldsymbol{\Sigma}_2$ is the population covariance matrix for carapace measurements for male turtles. Set $\alpha = .05$.

6.38. Table 11.7 page 662 contains the values of three trace elements and two measures of hydrocarbons for crude oil samples taken from three groups (zones) of sandstone. Use Box's M-test to test equality of population covariance matrices for the three sandstone groups. Set $\alpha = .05$. Here there are $p = 5$ variables and you may wish to consider transformations of the measurements on these variables to make them more nearly normal.

6.39. Anacondas are some of the largest snakes in the world. Jesus Ravis and his fellow researchers capture a snake and measure its (i) snout vent length (cm) or the length from the snout of the snake to its vent where it evacuates waste and (ii) weight (kilograms). A sample of these measurements in shown in Table 6.19.

(a) Test for equality of means between males and females using $\alpha = .05$. Apply the large sample statistic.

(b) Is it reasonable to pool variances in this case? Explain.

(c) Find the 95% Boneferroni confidence intervals for the mean differences between males and females on both length and weight.

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