Multivariate Statistical Analysis Final Exam

Reference Solution

1. (10%) Eq. (6-21), p. 285, Textbook

$$\boldsymbol{S}_{pooled} = \frac{n_1 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_2 = \frac{60}{121}\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} + \frac{61}{121}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2.4959 & 1 \\ 1 & 4.4959 \end{bmatrix}$$

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Eq. (6-24), p. 286, Textbook

$T^2 = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{S}_{pooled}\right]^{-1} (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2) \approx 73.6409$, larger than the critical value $\frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1 + n_2 - p - 1}(\alpha) = \frac{121 \times 2}{120} F_{2,120}(0.1) \approx \frac{242}{120} \times 2.35 \approx 4.7392$.

Thus the hypothesis $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ is rejected at 10% significance level.

2. (10%) Eq. (6-21), p. 285, Textbook

$$\boldsymbol{S}_{pooled} = \frac{n_1 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\boldsymbol{S}_2 = \frac{60}{121}\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} + \frac{61}{121}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2.4959 & 1 \\ 1 & 4.4959 \end{bmatrix}$$

Eq. (6-51)~(6-53), p. 311, Textbook

$$u = \left[\sum_\ell \frac{1}{n_\ell - 1} - \frac{1}{\sum_\ell (n_\ell - 1)}\right]\left[\frac{2p^2 + 3p - 1}{6(p+1)(g-1)}\right] = \left[\frac{1}{60} + \frac{1}{61} - \frac{1}{121}\right]\left[\frac{2 \times 4 + 3 \times 2 - 1}{6 \times 3 \times 1}\right] \approx 0.0179$$

$$M = \left[\sum_\ell (n_\ell - 1)\right] \ln\left|\boldsymbol{S}_{pooled}\right| - \sum_\ell \left[(n_\ell - 1)\ln\left|\boldsymbol{S}_\ell\right|\right] = 121 \times \ln 10.2211 - 60 \times \ln 14 - 61 \times \ln 7 \approx 4.2149$$

$C = (1 - u)M \approx 4.1394$, $\nu = \frac{1}{2} p(p+1)(g-1) = 3$, C is smaller than the critical value $\chi^2_\nu(\alpha) = \chi^2_3(0.1) = 6.25$. Thus H_0 is not rejected at 10% significance level.

3. (10%) slides 174~187, "Comparison of Several Multivariate Means", $g = 4, b = 3$

Subject\System	A	B	C	Average
1	7	6	5	6
2	5	4	5	14/3
3	6	4	5	5
4	4	5	6	5
Means	22/4	19/4	21/4	62/12

$$SS(System) = g\sum_{k=1}^{b}\left(\bar{x}_{\bullet k} - \bar{x}\right)^2 = 4\left[(22/4-62/12)^2 + (19/4-62/12)^2 + (21/4-62/12)^2\right]$$
$$\approx 1.1667$$

$$SS(Subject) = b\sum_{\ell=1}^{g}\left(\bar{x}_{\ell\bullet} - \bar{x}\right)^2 = 3\left[(6-62/12)^2 + (14/3-62/12)^2 + 2*(5-62/12)^2\right]$$
$$= 3$$

$$SS(interaction) = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\left(x_{\ell k} - \bar{x}_{\ell\bullet} - \bar{x}_{\bullet k} + \bar{x}\right)^2$$
$$= (7-22/4-6+62/12)^2 + (6-19/4-6+62/12)^2 + (5-21/4-6+62/12)^2 +$$
$$(5-22/4-14/3+62/12)^2 + (4-19/4-14/3+62/12)^2 + (5-21/4-14/3+62/12)^2 +$$
$$(6-22/4-5+62/12)^2 + (4-19/4-5+62/12)^2 + (5-21/4-5+62/12)^2 +$$
$$(4-22/4-5+62/12)^2 + (5-19/4-5+62/12)^2 + (6-21/4-5+62/12)^2$$
$$\approx 5.5$$

$df(System) = b-1 = 2$, $df(Subject) = g-1 = 3$,
$df(interaction) = (b-1)(g-1) = 6$, $df(total) = bg-1 = 12-1 = 11 = 2+3+1$

Source	Sum of Squares	df	Mean Square	F
System	1.1667	2	0.5834	0.6364
Subject	3	3	1	
Interaction	5.5	6	0.9167	
Total	9.6667	11		

$F_{system} = 0.6364$ is smaller than the critical value: $F_{2,6}(0.05) \approx 5.14$. Thus the hypothesis can not be rejected.

4. (10%)

Use the approach of Example 6.9 in pp. 304-306, Textbook. Arrange the observation pairs in rows to have

$$\begin{pmatrix} \begin{bmatrix}5\\3\end{bmatrix} & \begin{bmatrix}6\\4\end{bmatrix} & \begin{bmatrix}4\\2\end{bmatrix} \\ \begin{bmatrix}7\\3\end{bmatrix} & \begin{bmatrix}8\\3\end{bmatrix} & \begin{bmatrix}6\\3\end{bmatrix} \\ \begin{bmatrix}7\\3\end{bmatrix} & \begin{bmatrix}6\\2\end{bmatrix} & \begin{bmatrix}5\\4\end{bmatrix} \end{pmatrix} \text{ with } \bar{\boldsymbol{x}}_1 = \begin{bmatrix}5\\3\end{bmatrix}, \bar{\boldsymbol{x}}_2 = \begin{bmatrix}7\\3\end{bmatrix}, \bar{\boldsymbol{x}}_3 = \begin{bmatrix}6\\3\end{bmatrix}, \bar{\boldsymbol{x}} = \begin{bmatrix}6\\3\end{bmatrix}$$

Sum of squares for variable 1:

$$\begin{pmatrix} 5 & 6 & 4 \\ 7 & 8 & 6 \\ 7 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{pmatrix} + \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

(observation) (mean) (treatment effect) (residual)

$$\text{SS}_{\text{obs}} = \text{SS}_{\text{mean}} + \text{SS}_{\text{tr}} + \text{SS}_{\text{res}}$$
$$336 = 324 + (1\times 6) + (1\times 6) = 336$$
$$\text{Total SS (corrected)} = \text{SS}_{\text{obs}} - \text{SS}_{\text{mean}} = 336 - 324 = 12$$

Sum of squares for variable 2:

$$\begin{pmatrix} 3 & 4 & 2 \\ 3 & 3 & 3 \\ 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

(observation) (mean) (treatment effect) (residual)

$$\text{SS}_{\text{obs}} = \text{SS}_{\text{mean}} + \text{SS}_{\text{tr}} + \text{SS}_{\text{res}}$$
$$85 = 81 + 0 + 1\times 4 = 81 + 0 + 4$$
$$\text{Total SS (corrected)} = \text{SS}_{\text{obs}} - \text{SS}_{\text{mean}} = 85 - 81 = 4$$

Sum of cross products

$$\text{SCP}_{\text{obs}} = 5\times 3 + 6\times 4 + 4\times 2 + 7\times 3 + 8\times 3 + 6\times 3 + 7\times 3 + 6\times 2 + 5\times 4 = 163$$
$$\text{SCP}_{\text{mean}} = (6\times 3)\times 9 = 162$$
$$\text{SCP}_{\text{tr}} = 0$$
$$\text{SCP}_{\text{res}} = 1 + 1 + (-1)\times 1 = 1$$
$$\text{SCP}_{\text{obs}} = \text{SCP}_{\text{mean}} + \text{SCP}_{\text{tr}} + \text{SCP}_{\text{res}}$$
$$163 = 162 + 0 + 1$$
$$\text{Total SCP (corrected)} = \text{SCP}_{\text{obs}} - \text{SCP}_{\text{mean}} = 163 - 162 = 1$$

MANOVA Table:

Treatment $\boldsymbol{B} = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$ d.f. = 3 – 1 = 2

Residual $\boldsymbol{W} = \begin{bmatrix} 6 & 1 \\ 1 & 4 \end{bmatrix}$ d.f. = 9 -3 = 6

Total (corrected) $\boldsymbol{B} + \boldsymbol{W} = \begin{bmatrix} 12 & 1 \\ 1 & 4 \end{bmatrix}$ d.f. = 9-1 = 8

$$\Lambda^* = \frac{|\boldsymbol{W}|}{|\boldsymbol{B}+\boldsymbol{W}|} = \frac{24-1}{48-1} \approx 0.4894$$

Comparing (Table 6.3, p=2, g = 3)

$$\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right)\left(\frac{\sum n_\ell - g - 1}{g-1}\right) \approx \frac{1-\sqrt{0.4894}}{\sqrt{0.4894}} \times \frac{9-3-1}{3-1} \approx 1.0736$$

with $F_{2(g-1),2(\sum n_\ell - g - 1)}(\alpha) = F_{4,10}(0.1) = 2.61$,

we can not reject the hypothesis that the mean vectors of those three populations are the same at 90% significance level

5. (10%) Result 7.1, p. 364, Textbook

$$\mathbf{Z}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{y}_1' = \begin{bmatrix} 4 & 2 & 3 & 1 \end{bmatrix}, \quad \mathbf{y}_2' = \begin{bmatrix} 3 & 5 & 7 & 8 \end{bmatrix}$$

$$\mathbf{Z}'\mathbf{y}_1 = \begin{bmatrix} 10 \\ -9 \end{bmatrix}, \quad \mathbf{Z}'\mathbf{y}_2 = \begin{bmatrix} 23 \\ -3 \end{bmatrix}, \quad \mathbf{Z}'\mathbf{Z} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}, \quad (\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}}_{(1)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_1 = \begin{bmatrix} 2.1 \\ -0.8 \end{bmatrix}, \quad \hat{\boldsymbol{\beta}}_{(2)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_2 = \begin{bmatrix} 6.6 \\ 1.7 \end{bmatrix}$$

$$\hat{\mathbf{y}}_1 = \mathbf{Z}\hat{\boldsymbol{\beta}}_{(1)} = \begin{bmatrix} 3.7 & 2.9 & 2.1 & 1.3 \end{bmatrix}', \quad \hat{\mathbf{y}}_2 = \mathbf{Z}\hat{\boldsymbol{\beta}}_{(2)} = \begin{bmatrix} 3.2 & 4.9 & 6.6 & 8.3 \end{bmatrix}'$$

$$\hat{\boldsymbol{\varepsilon}}_{(1)} = \mathbf{y}_1 - \hat{\mathbf{y}}_1 = \begin{bmatrix} 0.3 & -0.9 & 0.9 & -0.3 \end{bmatrix}', \quad \hat{\boldsymbol{\varepsilon}}_{(2)} = \mathbf{y}_2 - \hat{\mathbf{y}}_2 = \begin{bmatrix} -0.2 & 0.1 & 0.4 & -0.3 \end{bmatrix}'$$

$$\hat{\boldsymbol{\varepsilon}}_{(1)}'\hat{\boldsymbol{\varepsilon}}_{(1)} = 1.8, \quad \hat{\boldsymbol{\varepsilon}}_{(2)}'\hat{\boldsymbol{\varepsilon}}_{(2)} = 0.3.$$

6. (10%) Eq. (7.-40), p. 399, Textbook

100(1-α)% confidence region for $E([Y_1, Y_2]' \mid \mathbf{z}_0) = \boldsymbol{\beta}'\mathbf{z}_0$ is provided by the inequality

$$(\boldsymbol{\beta}'\mathbf{z}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0)'\left(\frac{n}{n-r-1}\hat{\boldsymbol{\Sigma}}\right)^{-1}(\boldsymbol{\beta}'\mathbf{z}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0) \le \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\left[\left(\frac{m(n-r-1)}{n-r-m}\right)F_{m,n-r-m}(\alpha)\right]$$

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\boldsymbol{\beta}}_{(1)} & \hat{\boldsymbol{\beta}}_{(2)} \end{bmatrix} = \begin{bmatrix} 2.1 & 6.6 \\ -0.8 & 1.7 \end{bmatrix}, \quad \mathbf{z}_0 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad \hat{\boldsymbol{\beta}}'\mathbf{z}_0 = \begin{bmatrix} 2.1 & -0.8 \\ 6.6 & 1.7 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 7.45 \end{bmatrix}$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \frac{1}{4}\begin{bmatrix} 0.3 & -0.9 & 0.9 & -0.3 \\ -0.2 & 0.1 & 0.4 & -0.3 \end{bmatrix}\begin{bmatrix} 0.3 & -0.2 \\ -0.9 & 0.1 \\ 0.9 & 0.4 \\ -0.3 & -0.3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1.8 & 0.3 \\ 0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.45 & 0.075 \\ 0.075 & 0.075 \end{bmatrix}$$

$$\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0 = [1 \quad 0.5]\begin{bmatrix}0.3 & 0.1\\0.1 & 0.2\end{bmatrix}\begin{bmatrix}1\\0.5\end{bmatrix} = [1 \quad 0.5]\begin{bmatrix}0.35\\0.2\end{bmatrix} = 0.45$$

90% confidence region for $E([Y_1, Y_2] \mid \mathbf{z}_0) = \boldsymbol{\beta}'\mathbf{z}_0$ is

$$(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix}1.7\\7.45\end{bmatrix})'\left(\frac{4}{4-1-1}\begin{bmatrix}0.45 & 0.075\\0.075 & 0.075\end{bmatrix}\right)^{-1}(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix}1.7\\7.45\end{bmatrix}) \le 0.45\left[\left(\frac{2(4-1-1)}{4-1-2}\right)F_{2,1}(0.1)\right]$$

i.e.,

$$(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix}1.7\\7.45\end{bmatrix})'\begin{bmatrix}1.3333 & -1.3333\\-1.3333 & 8\end{bmatrix}(\boldsymbol{\beta}'\mathbf{z}_0 - \begin{bmatrix}1.7\\7.45\end{bmatrix}) \le 0.45 * 4 * 49.5 = 89.1$$

Eq. (7-42), p. 399, Textbook

100(1-α)% prediction region for $\mathbf{Y}_0 = [Y_1, Y_2]'$ is provided by the inequality

$$(\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0)'\left(\frac{n}{n-r-1}\hat{\boldsymbol{\Sigma}}\right)^{-1}(\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0) \le \left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)\left[\left(\frac{m(n-r-1)}{n-r-m}\right)F_{m,n-r-m}(\alpha)\right]$$

Thus, 90% prediction region for $\mathbf{Y}_0 = [Y_1, Y_2]'$ is

$$(\mathbf{Y}_0 - \begin{bmatrix}1.7\\7.45\end{bmatrix})'\begin{bmatrix}1.3333 & -1.3333\\-1.3333 & 8\end{bmatrix}(\mathbf{Y}_0 - \begin{bmatrix}1.7\\7.45\end{bmatrix}) \le 1.45 * 4 * 49.5 = 287.1$$

7. (10%) Result 7.12, p. 402, Textbook

   Writing $b_0 + \mathbf{b}'\mathbf{Z} = b_0 + \mathbf{b}'\mathbf{Z} + (\mu_Y - \mathbf{b}'\boldsymbol{\mu}_\mathbf{Z}) - (\mu_Y - \mathbf{b}'\boldsymbol{\mu}_\mathbf{Z})$, we get

$$\begin{aligned}E(Y - b_0 - \mathbf{b}'\mathbf{Z})^2 &= E[Y - \mu_Y - (\mathbf{b}'\mathbf{Z} - \mathbf{b}'\boldsymbol{\mu}_\mathbf{Z}) + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_\mathbf{Z})]^2\\ &= E(Y - \mu_Y)^2 + E[\mathbf{b}'(\mathbf{Z} - \boldsymbol{\mu}_\mathbf{Z})]^2 + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_\mathbf{Z})^2 - 2E[\mathbf{b}'(\mathbf{Z} - \boldsymbol{\mu}_\mathbf{Z})(Y - \mu_Y)]\\ &= \sigma_{YY} + \mathbf{b}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}\mathbf{b} + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_\mathbf{Z})^2 - 2\mathbf{b}'\boldsymbol{\sigma}_{\mathbf{Z}Y}\end{aligned}$$

   Adding and subtracting $\boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}$, we obtain

$$E(Y - b_0 - \mathbf{b}'\mathbf{Z})^2 = \sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_\mathbf{Z})^2 + (\mathbf{b} - \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y})'\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\mathbf{b} - \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y})$$

   The mean square error is minimized by taking $\mathbf{b} = \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} = \boldsymbol{\beta}$, making the last term zero, and then choosing $b_0 = \mu_Y - (\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y})'\boldsymbol{\mu}_\mathbf{Z} = \beta_0$ to make the third term zero.

8. (10%) For use of the result in p. 442, Textbook, eigenvalues and the first eigenvector have to be found.

$$\begin{vmatrix} 13-\lambda & 0 & 5 \\ 0 & 7-\lambda & 0 \\ 5 & 0 & 13-\lambda \end{vmatrix} = 0 \Rightarrow (7-\lambda) = 0 \text{ or } \begin{vmatrix} 13-\lambda & 5 \\ 5 & 13-\lambda \end{vmatrix} = \lambda^2 - 26\lambda + 144 = 0, \quad , \ ,$$

$$\therefore \hat{\lambda}_1 = 18, \hat{\lambda}_2 = 8, \hat{\lambda}_3 = 7$$

$$\hat{\lambda}_1 = 18, \begin{bmatrix} -5 & 0 & 5 \\ 0 & -11 & 0 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} \hat{e}_{11} \\ \hat{e}_{12} \\ \hat{e}_{13} \end{bmatrix} = 0 \Rightarrow \hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$Y_1 = \hat{\mathbf{e}}_1' \mathbf{x} = \frac{1}{\sqrt{2}}(x_1 + x_2) \approx 0.707 x_1 + 0.707 x_2$$

$$\text{proportion of the total sample variance due to } Y_1 = \frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3} = \frac{18}{33} \approx 0.545$$

Eq. (8-33) in p. 456, Textbook: $\dfrac{\hat{\lambda}_i}{1 + z(\alpha/2)\sqrt{2/n}} \le \lambda_i \le \dfrac{\hat{\lambda}_i}{1 - z(\alpha/2)\sqrt{2/n}}$.

Hnece the 95% confidence interval for the first eigenvalue is

$$\left(\frac{\hat{\lambda}_1}{1 + z(0.025)\sqrt{2/200}}, \frac{\hat{\lambda}_1}{1 - z(0.025)\sqrt{2/200}}\right) \approx \left(\frac{18}{1.196}, \frac{18}{0.804}\right) \approx (15.05, 22.388)$$

9. (10%) From Equations (9-15) and (9-16) in p. 490, Textbook

$$\tilde{\mathbf{L}} = \sqrt{\hat{\lambda}_1} \hat{\mathbf{e}}_1 = \sqrt{\frac{18}{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

$$\mathbf{S} \approx \tilde{\mathbf{L}}\tilde{\mathbf{L}}' + \tilde{\mathbf{\Psi}} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \end{bmatrix} + \begin{bmatrix} \tilde{\psi}_1 & 0 & 0 \\ 0 & \tilde{\psi}_2 & 0 \\ 0 & 0 & \tilde{\psi}_3 \end{bmatrix} = \begin{bmatrix} 9 + \tilde{\psi}_1 & 0 & 9 \\ 0 & \tilde{\psi}_2 & 0 \\ 9 & 0 & 9 + \tilde{\psi}_3 \end{bmatrix}$$

$$\therefore \tilde{\psi}_1 = 13 - 9 = 4, \tilde{\psi}_2 = 7, \tilde{\psi}_3 = 13 - 9 = 4$$

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \text{residual} = \mathbf{S} - \tilde{\mathbf{L}}\tilde{\mathbf{L}}' - \tilde{\mathbf{\Psi}} = \begin{bmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

10. (10%) Depend on your comments and suggestions..