Multivariate Statistical Analysis Mid Term Solution
November 12, 2010

1. A population mean vector is the expectation computed by the random variables and their joint probability distribution to the population, which are independent of the samples.
   A sample mean vector is the arithmetic mean of a set of random vector samples sampled from identical populations independently. It depends on the values of the samples. Different set of samples will result in different sample mean vectors..
   (8%)

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2. $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (8%)

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3. Eq.(4 − 7), p.153, Text book : major axes and directions $= c\sqrt{\lambda_i}\mathbf{e}_i$, where e_i is a normalized eigenvector.

   $\therefore$ major axes and directions = $c2\sqrt{2}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $c\sqrt{3}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. (8%)

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4. From Result 4.7 and Table 3 in Appendix of the Textbook,

   $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \chi_2^2(0.05) \approx 5.99$, $c = \sqrt{5.99}$,

   major axes and directions $= 2\sqrt{2}\sqrt{5.99}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\sqrt{3}\sqrt{5.99}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ (8%)

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5. Based on Eq. (5 - 6) or (5 - 7),

   $T^2 = n(\bar{\mathbf{x}}-\boldsymbol{\mu}_0)'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}_0) = 30[-0.5 \quad 1.5]\frac{1}{24}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}$

   $= \frac{5}{4}[-0.5 \quad 1.5]\begin{bmatrix} -6.5 \\ 7 \end{bmatrix} = 13.75 \times \frac{5}{4} = 17.1875$

   Critical value $= \frac{(n-1)p}{n-p}F_{p,n-p}(0.05) = \frac{29 \times 2}{28}F_{2,28}(0.05) \approx \frac{58}{28} \times 3.34 \approx 6.92$

   $T^2 >$ critical value. $\therefore$ reject H_0

(8%)

From Eq. (5-18),

$$n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \le \frac{p(n-1)}{n-p}F_{p,n-p}(0.05) = 6.92$$

6. $(\bar{\mathbf{x}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \le \frac{6.92}{30} \approx 0.23$ (8%)

By Eq. (5-19), major axes and directions $= \sqrt{\lambda_i}\sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(0.05)}\mathbf{e}_i$

namely, $2\sqrt{2}\sqrt{0.23}\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$ and $\sqrt{3}\sqrt{0.23}\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$

Use Eq. (5.24),

$$\bar{x}_1 - \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{11}}{n}} \le \mu_1 \le \bar{x}_1 + \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{11}}{n}}$$

$$\bar{x}_2 - \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{22}}{n}} \le \mu_2 \le \bar{x}_2 + \sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{22}}{n}}$$

7. $\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)} \approx \sqrt{6.92} \approx 2.63$. (8%)

$$0.5 - \sqrt{6.92}\sqrt{\frac{4}{30}} \le \mu_1 \le 0.5 + \sqrt{6.92}\sqrt{\frac{4}{30}}, \quad \mu_1 \in (-0.46, 1.46)$$

$$0.5 - \sqrt{6.92}\sqrt{\frac{7}{30}} \le \mu_2 \le 0.5 + \sqrt{6.92}\sqrt{\frac{7}{30}}, \quad \mu_2 \in (-0.77, 1.77)$$

From Eq. (5-29),

$$\bar{x}_1 - t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{11}}{n}} \le \mu_1 \le \bar{x}_1 + t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{11}}{n}}$$

$$\bar{x}_2 - t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{22}}{n}} \le \mu_2 \le \bar{x}_2 + t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{22}}{n}}$$

8. $p = 2, \quad 0.05/(2p) = 0.0125, \quad t_{29}(0.01) \approx 2.462, \quad t_{29}(0.025) \approx 2.045$

$t_{29}(0.0125) \approx 2.462 + ((2.045 - 2.462)/(0.025 - 0.01)) * (0.0125 - 0.01) = 2.3925$

$$0.5 - 2.3925\sqrt{\frac{4}{30}} \le \mu_1 \le 0.5 + 2.3925\sqrt{\frac{4}{30}}, \quad \mu_1 \in (-0.37, 1.37)$$

$$0.5 - 2.3925\sqrt{\frac{7}{30}} \le \mu_2 \le 0.5 + 2.3925\sqrt{\frac{7}{30}}, \quad \mu_2 \in (-0.65, 1.65)$$

(8%)

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9. Suppose that the possibility that a component of the mean vector falls in the corresponding one-at-a-time confidence interval is 1-α. Since all one-at-a-time confidence interval must hold simultaneously, if there are p variables, the total possibility of the mean vector falls in the box formed by all p one-at-a-time confidence intervals will be $(1-\alpha)^p$. On the other hand, the same probability for the Bonferroni simultaneous confidence intervals will be 1-α, which is larger than $(1-\alpha)^p$. Thus the lengths of the one-at-a-time confidence intervals are less than those of the Bonferroni simultaneous confidence intervals. (8%)

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10. Let $\overline{C_i}$ be the set that C_i is false.

$$P[\text{at least one } C_i \text{ false}] = P[\overline{C}_1 \cup \overline{C}_2 \cup \overline{C}_3]$$
$$= P[\overline{C}_1] + P[\overline{C}_2] + P[\overline{C}_3] - P[\overline{C}_1 \cap \overline{C}_2] - P[\overline{C}_2 \cap \overline{C}_3] - P[\overline{C}_3 \cap \overline{C}_1] + P[\overline{C}_1 \cap \overline{C}_2 \cap \overline{C}_3]$$
$$\leq P[\overline{C}_1] + P[\overline{C}_2] + P[\overline{C}_3]$$

because $P[\overline{C}_1 \cap \overline{C}_2] + P[\overline{C}_2 \cap \overline{C}_3] + P[\overline{C}_3 \cap \overline{C}_1] \geq P[\overline{C}_1 \cap \overline{C}_2 \cap \overline{C}_3]$

Thus,

$$1 - P[\text{at least one } C_i \text{ false}] \geq 1 - (P[\overline{C}_1] + P[\overline{C}_2] + P[\overline{C}_3]) = 1 - \sum_{i=1}^{3} P[C_i \text{ false}]$$

(8%)

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11. (1) b (control group: before; experiment group: after; they are paired) (2%)
    (2) c (control group: treatment with a placebo; experiment group: treatment with the new medicine; they are unpaired) (2%)
    (3) a (two independent populations) (2%)
    (4) b (control group: younger brother/sister; experiment group: older brother/sister; they are paired for almost equal in family lives) (2%)
    (5) a (two independent populations) (2%)
    (6) a (two independent populations) (2%)

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12. This is an open problem. You can get partial or total points depending on how persuasive your answer is. My personal point of view is that when the determinant of the sample covariance matrix is zero, some variables can be expressed by other variables, and the number of variables could be reduced, such that the sample size is larger than the number of the remaining variables, and then some methods can be developed to get information about the population. (8%)