Comparison of Several Multivariate Means

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Outline

- Introduction
- Comparison of Univariate Mean
- Paired Comparisons and a Repeated Measures Design
- Comparing Mean Vectors from Two Populations
- Comparison of Several Univariate Population Mean (One-Way ANOVA)

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Outline

- Comparing Several Multivariate Population Means (One-Way Manova)
- Simultaneous Confidence Intervals for Treatment Effects
- Testing for Equality of Covariance Matrices
- Two-Way ANOVA
- Two-Way Multivariate Analysis of Variance

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Outline

- Profile Analysis
- ANOVA for Repeated Measures
- Repeated Measures Designs and Growth Curves
- Perspectives and Strategy for Analyzing Multivariate Models

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Introduction

- Extend previous ideas to handle problems involving the comparison of several mean vectors

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Questions

- What is the paired comparison?
- How to design experiments for paired comparison?
- How to test if the population means of paired groups are different?
- How to compute the confidence interval for the difference of population means of paired groups?

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Questions

- How to compare population means of two populations without paired experiments?
- In such a case, how to estimate the common variance?

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Scenarios

- To test if the differences are significant between
  - Teaching using Power Point vs. using chalks and blackboard only
  - Drug vs. placebos
  - Processing speed of MP3 player model I of brand A vs. model G of brand B
  - Performance of students going to cram schools vs. those not

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Paired Comparisons

- Measurements are recorded under different sets of conditions
- See if the responses differ significantly over these sets
- Two or more treatments can be administered to the same or similar experimental units
- Compare responses to assess the effects of the treatments

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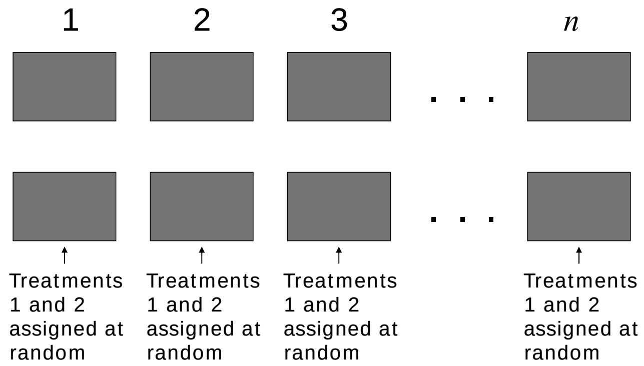
Experiment Design for Paired Comparisons



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Single Response (Univariate) Case

$D_j = X_{j1} - X_{j2},\ j = 1, 2, \cdots, n$

$D_j : N(\delta, \sigma_d^2)$

$t = \dfrac{\overline{D} - \delta}{s_d / \sqrt{n}} : t_{n-1}$

Reject $H_0 : \delta = 0$ in favor of $H_1 : \delta \neq 0$ if $|t| > t_{n-1}(\alpha/2)$

$100(1-\alpha)\%$ confidence interval for δ

$\overline{d} - t_{n-1}(\alpha/2)\dfrac{s_d}{\sqrt{n}} \le \delta \le \overline{d} + t_{n-1}(\alpha/2)\dfrac{s_d}{\sqrt{n}}$

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Comparing Means from Two Populations

- Without explicitly controlling for unit-to-unit variability, as in the paired comparison case
- Experimental units are randomly assigned to populations
- Applicable to a more general collection of experimental units

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Assumptions Concerning the Structure of Data

$X_{11}, X_{12}, \cdots, X_{1n_1}$: random sample from univariate population with mean μ_1 and variance σ_1^2

$X_{21}, X_{22}, \cdots, X_{2n_2}$: random sample from univariate population with mean μ_2 and variance σ_2^2

$X_{11}, X_{12}, \cdots, X_{1n_1}$ are independent of $X_{21}, X_{22}, \cdots, X_{2n_2}$

Further assumptions when n_1 and n_2 small :

Both populations are univariate normal

$\sigma_1^2 = \sigma_2^2$

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Pooled Estimate of Population Variance

$$\sum_{j=1}^{n_1}(x_{j1} - \overline{x}_1)(x_{j1} - \overline{x}_1)' \approx (n_1 - 1)\sigma^2$$

$$\sum_{j=1}^{n_2}(x_{j2} - \overline{x}_2)(x_{j2} - \overline{x}_2)' \approx (n_2 - 1)\sigma^2$$

$$s_{pooled}^2 = \frac{\sum_{j=1}^{n_1}(x_{j1} - \overline{x}_1)(x_{j1} - \overline{x}_1)' + \sum_{j=1}^{n_2}(x_{j2} - \overline{x}_2)(x_{j2} - \overline{x}_2)'}{n_1 + n_2 - 2}$$

$$= \frac{n_1 - 1}{n_1 + n_2 - 2}s_1^2 + \frac{n_2 - 1}{n_1 + n_2 - 2}s_2^2$$

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t-Statistics for Comparing Two Populations

$$X_{11}, X_{12}, \cdots, X_{1n_1} : N(\mu_1, \sigma^2)$$

$$X_{21}, X_{22}, \cdots, X_{2n_2} : N(\mu_2, \sigma^2)$$

$$\overline{X}_1 - \overline{X}_2 = \frac{1}{n_1} X_{11} + \cdots + \frac{1}{n_1} X_{1n_1} - \frac{1}{n_2} X_{21} + \cdots - \frac{1}{n_2} X_{2n_2}$$

$$: N(\mu_1 - \mu_2, \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\sigma^2)$$

$$\Rightarrow t = \left(\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)\right) / \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) s^2_{pooled}}$$

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Test of Hypothesis

Reject $H_0 : \mu_1 - \mu_2 = \delta_0$ in favor of $H_1 : \mu_1 - \mu_2 \neq \delta_0$

if $\left| \dfrac{\bar{x}_1 - \bar{x}_2 - \delta_0}{s_{pooled} \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \right| > t_{n_1+n_2-2}(\frac{\alpha}{2})$

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Questions

- How to make paired comparison for multivariate data?
- How to use the contrast matrix to carry out paired comparison for multivariate data?
- What is the repeated measures?
- How to test for equality of treatments in a repeated measures?

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Example 6.1: Effluent Data from Two Labs

Sample j	Commercial lab x_{1j1} (BOD)	x_{1j2} (SS)	State lab of hygiene x_{2j1} (BOD)	x_{2j2} (SS)
1	6	27	25	15
2	6	23	28	13
3	18	64	36	22
4	8	44	35	29
5	11	30	15	31
6	34	75	44	64
7	28	26	42	30
8	71	124	54	64
9	43	54	34	56
10	33	30	29	20
11	20	14	39	21

Source: Data courtesy of S. Weber.

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Multivariate Extension: Notations

X_{1j1} = variable 1 under treatment 1

X_{1j2} = variable 2 under treatment 1

$\vdots \qquad \vdots$

X_{1jp} = variable p under treatment 1

X_{2j1} = variable 1 under treatment 2

X_{2j2} = variable 2 under treatment 2

$\vdots \qquad \vdots$

X_{2jp} = variable p under treatment 2

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Result 6.1

$$D_{j1} = X_{1j1} - X_{2j1}$$
$$D_{j2} = X_{1j2} - X_{2j2}$$
$$\vdots \qquad \vdots$$
$$D_{jp} = X_{1jp} - X_{2jp}$$
$$\mathbf{D}_j = [D_{j1}, D_{j2}, \cdots, D_{jp}]'$$
$$\mathbf{D}_j : N_p(\boldsymbol{\delta}, \boldsymbol{\Sigma}_d), \quad j = 1, 2, \cdots, n$$
$$T^2 = n(\overline{\mathbf{D}} - \boldsymbol{\delta})'\mathbf{S}_d^{-1}(\overline{\mathbf{D}} - \boldsymbol{\delta}) : \frac{(n-1)p}{(n-p)} F_{p,n-p}$$
$$\overline{\mathbf{D}} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{D}_j, \quad \mathbf{S}_d = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{D}_j - \overline{\mathbf{D}})(\mathbf{D}_j - \overline{\mathbf{D}})'$$

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Test of Hypotheses and Confidence Regions

$\mathbf{d}_j' = [d_{j1}, d_{j2}, \cdots, d_{jp}]$: observed differences

Reject $H_0 : \boldsymbol{\delta} = 0$ in favor of $H_1 : \boldsymbol{\delta} \neq 0$ if

$$T^2 = n\overline{\mathbf{d}}'\mathbf{S}_d^{-1}\overline{\mathbf{d}} > \frac{(n-1)p}{n-p} F_{p,n-p}(\alpha)$$

Confidence regions: $(\overline{\mathbf{d}} - \boldsymbol{\delta})'\mathbf{S}_d^{-1}(\overline{\mathbf{d}} - \boldsymbol{\delta}) \le \frac{(n-1)p}{n-p} F_{p,n-p}(\alpha)$

$$\delta_i : \overline{d}_i \pm \sqrt{\frac{(n-1)p}{n-p} F_{p,n-p}(\alpha)}\sqrt{\frac{s_{d_i}^2}{n}}, \quad \delta_i : \overline{d}_i \pm t_{n-1}\left(\frac{\alpha}{2p}\right)\sqrt{\frac{s_{d_i}^2}{n}}$$

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Example 6.1: Check Measurements from Two Labs

$$\bar{\mathbf{d}} = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix} = \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix}, \quad \mathbf{S}_d = \begin{bmatrix} 199.26 & 88.38 \\ 88.38 & 418.61 \end{bmatrix}$$

$$T^2 = 11\begin{bmatrix} -9.36 & 13.27 \end{bmatrix} \begin{bmatrix} 0.0055 & -0.0012 \\ -0.0012 & 0.0026 \end{bmatrix} \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix}$$

$$= 13.6 > \frac{2\times 10}{9} F_{2,9}(0.05) = 9.47$$

Reject $H_0 : \boldsymbol{\delta} = 0$

$\delta_1 : -9.36 \pm \sqrt{9.47}\sqrt{199.26/11}$ or $(-22.46, 3.74)$

$\delta_2 : 13.27 \pm \sqrt{9.47}\sqrt{418.61/11}$ or $(-5.71, 32.25)$

Both includes zero

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Alternative View

$$\bar{\mathbf{x}}' = \left[\bar{x}_{11}, \bar{x}_{12}, \cdots, \bar{x}_{1p}, \bar{x}_{21}, \bar{x}_{22}, \cdots, \bar{x}_{2p}\right]$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}$$

$$\underset{(p\times 2p)}{\mathbf{C}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & | & -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & | & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & | & 0 & 0 & \cdots & -1 \end{bmatrix}$$

$$\mathbf{d}_j = \mathbf{C}\mathbf{x}_j, \quad \bar{\mathbf{d}} = \mathbf{C}\bar{\mathbf{x}}, \quad \mathbf{S}_d = \mathbf{CSC}', \quad T^2 = n\bar{\mathbf{x}}'\mathbf{C}'\left(\mathbf{CSC}'\right)^{-1}\mathbf{C}\bar{\mathbf{x}}$$

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Repeated Measures Design for Comparing Measurements

- q treatments are compared with respect to a single response variable
- Each subject or experimental unit receives each treatment once over successive periods of time

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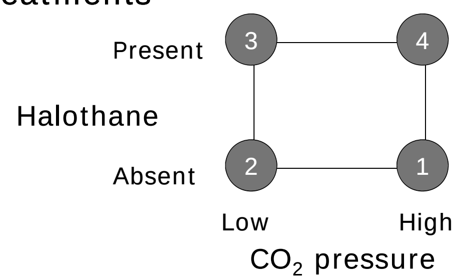
Example 6.2: Treatments in an Anesthetics Experiment

- 19 dogs were initially given the drug pentobarbitol followed by four treatments



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Example 6.2: Sleeping-Dog Data

Dog	Treatment 1	2	3	4
1	426	609	556	600
2	253	236	392	395
3	359	433	349	357
4	432	431	522	600
5	405	426	513	513
6	324	438	507	539
7	310	312	410	456
8	326	326	350	504
9	375	447	547	548
10	286	286	403	422
11	349	382	473	497
12	429	410	488	547
13	348	377	447	514
14	412	473	472	446
15	347	326	455	468
16	434	458	637	524
17	364	367	432	469
18	420	395	508	531
19	397	556	645	625

Source: Data courtesy of Dr. J. Atlee.

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Contrast Matrix

$$\mathbf{X}_j = \begin{bmatrix} X_{j1} \\ X_{j2} \\ \vdots \\ X_{jq} \end{bmatrix}, \quad j = 1, 2, \cdots, n \quad \boldsymbol{\mu} = E(\mathbf{X}_j)$$

$$\begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \\ \vdots \\ \mu_1 - \mu_q \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_q \end{bmatrix} = \mathbf{C}\boldsymbol{\mu}$$

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Test for Equality of Treatments in a Repeated Measures Design

$\mathbf{X} : N_q(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \quad \mathbf{C}$: contrast matrix

Test of $H_0 : \mathbf{C}\boldsymbol{\mu} = 0$ vs. $H_1 : \mathbf{C}\boldsymbol{\mu} \neq 0$

Reject H_0 if

$$T^2 = n(\mathbf{C}\overline{\mathbf{x}})'(\mathbf{CSC}')^{-1}\mathbf{C}\overline{\mathbf{x}} > \frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha)$$

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Example 6.2: Contrast Matrix

$$(\mu_3 + \mu_4) - (\mu_1 + \mu_2) = (\text{Halothane contrast})$$
$$(\mu_1 + \mu_3) - (\mu_2 + \mu_4) = (\text{CO}_2 \text{ contrast})$$
$$(\mu_1 + \mu_4) - (\mu_2 + \mu_3) = (\text{H} - \text{CO}_2 \text{ interaction})$$

$$\mathbf{C} = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

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Example 6.2: Test of Hypotheses

$$\bar{\mathbf{x}} = \begin{bmatrix} 368.21 \\ 404.63 \\ 479.26 \\ 502.89 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 2819.29 & & & \\ 3568.42 & 7963.14 & & \\ 2943.49 & 5303.98 & 6851.32 & \\ 2295.35 & 4065.44 & 4499.63 & 4878.99 \end{bmatrix}$$

$$\mathbf{C}\bar{\mathbf{x}} = \begin{bmatrix} 209.31 \\ -60.05 \\ -12.79 \end{bmatrix}, \quad \mathbf{CSC}' = \begin{bmatrix} 9432.32 & 1098.92 & 927.62 \\ 1098.92 & 5195.84 & 914.54 \\ 927.62 & 914.54 & 7557.44 \end{bmatrix}$$

$$T^2 = n(\mathbf{C}\bar{\mathbf{x}})'(\mathbf{CSC}')^{-1}(\mathbf{C}\bar{\mathbf{x}}) = 116$$

$$\frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(0.05) = 10.94$$

Reject $H_0 : \mathbf{C\mu} = 0$

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Example 6.2: Simultaneous Confidence Intervals

Contrast of halothane influence

$$(\bar{x}_3 + \bar{x}_4) - (\bar{x}_1 + \bar{x}_2) \pm \sqrt{\frac{18(3)}{16} F_{3,16}(0.05) \frac{\mathbf{c}_1'\mathbf{S}\mathbf{c}_1}{19}} = 209.31 \pm 73.70$$

CO_2 pressure influence

$$-60.05 \pm \sqrt{10.94}\sqrt{\frac{5195.84}{19}} = -60.5 \pm 54.70$$

H - CO_2 "interaction"

$$-12.79 \pm \sqrt{10.94}\sqrt{\frac{7557.44}{19}} = -12.79 \pm 65.97$$

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Questions

- How to compare mean vectors from two populations, not forming paired comparison groups?
- How to pool covariance matrices from two populations?
- How to find simultaneous confidence intervals for comparing mean vectors from two populations?

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Questions

- What is the multivariate Behrens-Fisher problem and how to solve it?

Comparing Mean Vectors from Two Populations

- Populations: Sets of experiment settings
- Without explicitly controlling for unit-to-unit variability, as in the paired comparison case
- Experimental units are randomly assigned to populations
- Applicable to a more general collection of experimental units

Assumptions Concerning the Structure of Data

$\mathbf{X}_{11}, \mathbf{X}_{12}, \cdots, \mathbf{X}_{1n_1}$: random sample from $p-$variate population with mean vector $\boldsymbol{\mu}_1$ and covariance $\boldsymbol{\Sigma}_1$

$\mathbf{X}_{21}, \mathbf{X}_{22}, \cdots, \mathbf{X}_{2n_2}$: random sample from $p-$variate population with mean vector $\boldsymbol{\mu}_2$ and covariance $\boldsymbol{\Sigma}_2$

$\mathbf{X}_{11}, \mathbf{X}_{12}, \cdots, \mathbf{X}_{1n_1}$ are independent of $\mathbf{X}_{21}, \mathbf{X}_{22}, \cdots, \mathbf{X}_{2n_2}$

Further assumptions when n_1 and n_2 small:

Both populations are multivariate normal

$\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$

Pooled Estimate of Population Covariance Matrix

$$\sum_{j=1}^{n_1} (\mathbf{x}_{j1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{j1} - \bar{\mathbf{x}}_1)' \approx (n_1 - 1)\boldsymbol{\Sigma}$$

$$\sum_{j=1}^{n_2} (\mathbf{x}_{j2} - \bar{\mathbf{x}}_2)(\mathbf{x}_{j2} - \bar{\mathbf{x}}_2)' \approx (n_2 - 1)\boldsymbol{\Sigma}$$

$$\mathbf{S}_{pooled} = \frac{\sum_{j=1}^{n_1} (\mathbf{x}_{j1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{j1} - \bar{\mathbf{x}}_1)' + \sum_{j=1}^{n_2} (\mathbf{x}_{j2} - \bar{\mathbf{x}}_2)(\mathbf{x}_{j2} - \bar{\mathbf{x}}_2)'}{n_1 + n_2 - 2}$$

$$= \frac{n_1 - 1}{n_1 + n_2 - 2}\mathbf{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\mathbf{S}_2$$

Result 6.2

$\mathbf{X}_{11}, \mathbf{X}_{12}, \cdots, \mathbf{X}_{1n_1} : N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$

$\mathbf{X}_{21}, \mathbf{X}_{22}, \cdots, \mathbf{X}_{2n_2} : N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$

$$\Rightarrow T^2 = \left[\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right]' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{pooled}\right]^{-1} \left[\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right]$$

is distributed as

$$\frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1+n_2-p-1}$$

Proof of Result 6.2

$$\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 = \frac{1}{n_1}\mathbf{X}_{11} + \cdots + \frac{1}{n_1}\mathbf{X}_{1n_1} - \frac{1}{n_2}\mathbf{X}_{21} + \cdots - \frac{1}{n_2}\mathbf{X}_{2n_2}$$

$$: N_p\left(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2, \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{\Sigma}\right)$$

$$(n_1 - 1)\mathbf{S}_1 : W_{n_1-1}(\boldsymbol{\Sigma}), \quad (n_2 - 1)\mathbf{S}_2 : W_{n_2-1}(\boldsymbol{\Sigma})$$

$$(n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2 : W_{n_1+n_2-2}(\boldsymbol{\Sigma})$$

$$T^2 = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1/2} \left[\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right]' \mathbf{S}_{pooled}^{-1} \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1/2} \left[\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right]$$

$$= N_p(0, \boldsymbol{\Sigma})' \left[\frac{W_{n_1+n_2-2}(\boldsymbol{\Sigma})}{n_1 + n_2 - 2}\right]^{-1} N_p(\mathbf{0}, \boldsymbol{\Sigma}) : \frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1+n_2-p-1}$$

Wishart Distribution

$$w_{n-1}(\mathbf{A} \mid \boldsymbol{\Sigma}) = \frac{|\mathbf{A}|^{(n-p-2)/2} e^{-\mathrm{tr}\left[\mathbf{A}\boldsymbol{\Sigma}^{-1}\right]/2}}{2^{p(n-1)/2} \pi^{p(p-1)/4} |\boldsymbol{\Sigma}|^{(n-1)/2} \prod_{i=1}^{p} \Gamma\left(\frac{1}{2}(n-i)\right)}$$

$\mathbf{A}$: positive definite

Properties :

$\mathbf{A}_1 : W_{m_1}(\mathbf{A}_1 \mid \boldsymbol{\Sigma}), \quad \mathbf{A}_2 : W_{m_2}(\mathbf{A}_2 \mid \boldsymbol{\Sigma}) \Rightarrow$

$\quad \mathbf{A}_1 + \mathbf{A}_2 : W_{m_1+m_2}(\mathbf{A}_1 + \mathbf{A}_2 \mid \boldsymbol{\Sigma})$

$\mathbf{A} : W_m(\mathbf{A} \mid \boldsymbol{\Sigma}) \Rightarrow \mathbf{CAC}' : W_m(\mathbf{CAC}' \mid \mathbf{C\Sigma\Sigma}')$

Test of Hypothesis

Reject $H_0 : \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \boldsymbol{\delta}_0$ in favor of $H_1 : \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 \neq \boldsymbol{\delta}_0$

if $\quad T^2 = (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2 - \boldsymbol{\delta}_0)' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{pooled}\right]^{-1} (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2 - \boldsymbol{\delta}_0)$

$$> \frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1} F_{p, n_1+n_2-p-1}(\alpha)$$

Note $E(\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2) = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$

$$\mathrm{Cov}(\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2)$$

$$= \mathrm{Cov}(\overline{\mathbf{X}}_1) - \mathrm{Cov}(\overline{\mathbf{X}}_1, \overline{\mathbf{X}}_2) - \mathrm{Cov}(\overline{\mathbf{X}}_2, \overline{\mathbf{X}}_1) + \mathrm{Cov}(\overline{\mathbf{X}}_2)$$

$$= \mathrm{Cov}(\overline{\mathbf{X}}_1) + \mathrm{Cov}(\overline{\mathbf{X}}_2) = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{\Sigma}$$

Example 6.3: Comparison of Soaps Manufactured in Two Ways

$n_1 = n_2 = 50$

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} 8.3 \\ 4.1 \end{bmatrix}, \quad \mathbf{S}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}, \quad \bar{\mathbf{x}}_2 = \begin{bmatrix} 10.2 \\ 3.9 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

$$\mathbf{S}_{pooled} = \frac{49}{98}\mathbf{S}_1 + \frac{49}{98}\mathbf{S}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, \quad \bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 = \begin{bmatrix} -1.9 \\ 0.2 \end{bmatrix}$$

Eigenvalues and eigenvectors of $\mathbf{S}_{pooled}$:

$\lambda_1 = 5.303, \quad \mathbf{e}_1 = [0.290 \quad 0.957]'$

$\lambda_2 = 1.697, \quad \mathbf{e}_1 = [0.957 \quad -0.290]'$

$$\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1} F_{p, n_1+n_2-p-1}(0.05) = 0.25$$

$$\sqrt{\lambda_1}\sqrt{0.25} = 1.15, \quad \sqrt{\lambda_2}\sqrt{0.25} = 0.65$$

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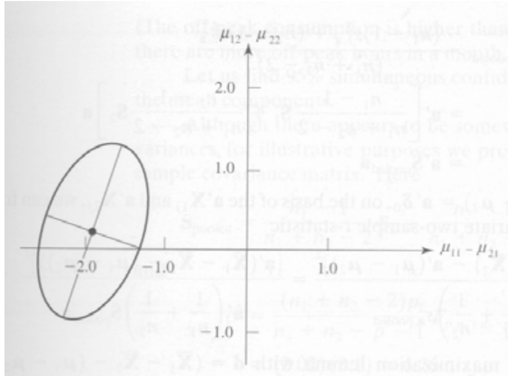
Example 6.3



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Result 6.3: Simultaneous Confidence Intervals

$$c^2 = \frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1} F_{p, n_1+n_2-p-1}(\alpha)$$

$$\mathbf{a}'(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2) \pm c\sqrt{\mathbf{a}'\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{pooled}\mathbf{a}}$$

will cover $\mathbf{a}'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$ for all $\mathbf{a}$

In particular, $\mu_{1i} - \mu_{2i}$ will be covered by

$$(\bar{X}_{1i} - \bar{X}_{2i}) \pm c\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)s_{ii,pooled}}$$

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Example 6.4: Electrical Usage of Homeowners with and without ACs

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} 204.4 \\ 556.6 \end{bmatrix}, \quad \mathbf{S}_1 = \begin{bmatrix} 13825.3 & 23823.4 \\ 23823.4 & 73107.4 \end{bmatrix}, \quad n_1 = 45$$

$$\bar{\mathbf{x}}_2 = \begin{bmatrix} 130.0 \\ 355.0 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 8632.0 & 19616.7 \\ 19616.7 & 55964.5 \end{bmatrix}, \quad n_1 = 55$$

$$\mathbf{S}_{pooled} = \frac{n_1 - 1}{n_1 + n_2 - 2}\mathbf{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\mathbf{S}_2 = \begin{bmatrix} 10963.7 & 21505.5 \\ 21505.5 & 63661.3 \end{bmatrix}$$

$$c^2 = \frac{98(2)}{97} F_{2,97}(0.05) = 6.26$$

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Example 6.4: Electrical Usage of Homeowners with and without ACs

95% simultaneous confidence intervals

$$\mu_{11}-\mu_{21}:(204.4-130.0)\pm\sqrt{6.26}\sqrt{\left(\frac{1}{45}+\frac{1}{55}\right)10963.7}$$

or $21.7\le\mu_{11}-\mu_{21}\le 127.1$

$$\mu_{12}-\mu_{22}:(556.6-355.0)\pm\sqrt{6.26}\sqrt{\left(\frac{1}{45}+\frac{1}{55}\right)63661.3}$$

or $74.7\le\mu_{12}-\mu_{22}\le 328.5$

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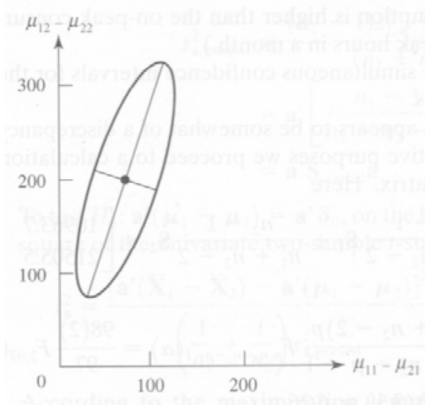
Example 6.4: 95% Confidence Ellipse



50

Bonferroni Simultaneous Confidence Intervals

$$\mu_{1i}-\mu_{2i}:(\bar{x}_1-\bar{x}_2)\pm t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)s_{ii,pooled}}$$

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Result 6.4

n_1-p and n_2-p are large

100% confidence ellipsoid for $\boldsymbol{\mu}_1-\boldsymbol{\mu}_2$:

$$[\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2-(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)]'\left[\frac{1}{n_1}\mathbf{S}_1+\frac{1}{n_2}\mathbf{S}_2\right]^{-1}[\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2-(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)]\le\chi_p^2(\alpha)$$

Simultaneous confidence intervals for $\mathbf{a}'(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)$:

$$\mathbf{a}'(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2)\pm\sqrt{\chi_p^2(\alpha)}\sqrt{\mathbf{a}'\left(\frac{1}{n_1}\mathbf{S}_1+\frac{1}{n_2}\mathbf{S}_2\right)\mathbf{a}}$$

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Proof of Result 6.4

$$E\left(\overline{\mathbf{X}}_1-\overline{\mathbf{X}}_2\right)=\boldsymbol{\mu}_1-\boldsymbol{\mu}_2$$

$$\text{Cov}\left(\overline{\mathbf{X}}_1-\overline{\mathbf{X}}_2\right)=\text{Cov}\left(\overline{\mathbf{X}}_1\right)+\text{Cov}\left(\overline{\mathbf{X}}_2\right)=\frac{1}{n_1}\boldsymbol{\Sigma}_1+\frac{1}{n_2}\boldsymbol{\Sigma}_2$$

$$\overline{\mathbf{X}}_1-\overline{\mathbf{X}}_2 : \text{nearly } N_p\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2, \frac{1}{n_1}\boldsymbol{\Sigma}_1+\frac{1}{n_2}\boldsymbol{\Sigma}_2\right)$$

$$\left[\overline{\mathbf{X}}_1-\overline{\mathbf{X}}_2-(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)\right]'\left(\frac{1}{n_1}\boldsymbol{\Sigma}_1+\frac{1}{n_2}\boldsymbol{\Sigma}_2\right)^{-1}$$
$$\left[\overline{\mathbf{X}}_1-\overline{\mathbf{X}}_2-(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)\right] : \chi_p^2$$

$$\boldsymbol{\Sigma}_1 \sim \mathbf{S}_1, \quad \boldsymbol{\Sigma}_2 \sim \mathbf{S}_2$$

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Remark

If $n_1 = n_2 = n$

$$\frac{n-1}{n+n-2}=\frac{1}{2}$$

$$\frac{1}{n_1}\mathbf{S}_1+\frac{1}{n_2}\mathbf{S}_2=\frac{1}{n}(\mathbf{S}_1+\mathbf{S}_2)$$

$$=\frac{(n-1)\mathbf{S}_1+(n-1)\mathbf{S}_2}{n+n-2}\left(\frac{1}{n}+\frac{1}{n}\right)=\mathbf{S}_{pooled}\left(\frac{1}{n}+\frac{1}{n}\right)$$

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Example 6.5

Example 6.4 Data

$$\frac{1}{n_1}\mathbf{S}_1+\frac{1}{n_2}\mathbf{S}_2=\begin{bmatrix}464.17 & 886.08\\ 886.08 & 2642.15\end{bmatrix}$$

$$\mu_{11}-\mu_{21} : 74.4 \pm \sqrt{5.99}\sqrt{464.17} \text{ or } (21.7, 127.1)$$

$$\mu_{12}-\mu_{22} : 201.6 \pm \sqrt{5.99}\sqrt{2642.15} \text{ or } (75.8, 327.4)$$

$$H_0 : \boldsymbol{\mu}_1-\boldsymbol{\mu}_2=0$$

$$T^2=[\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2]'\left[\frac{1}{n_1}\mathbf{S}_1+\frac{1}{n_2}\mathbf{S}_2\right]^{-1}[\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2]=15.66>\chi_2^2(0.05)=5.99$$

$$\text{Critical linear combination}: \left[\frac{1}{n_1}\mathbf{S}_1+\frac{1}{n_2}\mathbf{S}_2\right]^{-1}[\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2]=\begin{bmatrix}0.041\\ 0.063\end{bmatrix}$$

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Multivariate Behrens-Fisher Problem

- Test H_0: μ_1-μ_2=0
- Population covariance matrices are unequal
- Sample sizes are not large
- Populations are multivariate normal
- Both sizes are greater than the number of variables

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Approximation of T^2 Distribution

$$T^2 = (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2))' \left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1} (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2))$$

$$= \frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}$$

$$\nu = \frac{p + p^2}{\sum_{i=1}^{2} \frac{1}{n_i} \left\{ \mathrm{tr}\left[\left(\frac{1}{n_i}\mathbf{S}_i \left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1}\right)^2\right] + \left(\mathrm{tr}\left[\frac{1}{n_i}\mathbf{S}_i \left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1}\right]\right)^2 \right\}}$$

$$\min(n_1, n_2) \le \nu \le n_1 + n_2$$

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Confidence Region

$$(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2))' \left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2))$$

$$\le \frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}(\alpha)$$

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Example 6.6

- Example 6.4 data

$$\frac{1}{n_1}\mathbf{S}_1 \left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1} = \begin{bmatrix} 0.776 & -0.060 \\ -0.092 & 0.646 \end{bmatrix}$$

$$\frac{1}{n_2}\mathbf{S}_2 \left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1} = \begin{bmatrix} 0.224 & -0.060 \\ 0.092 & 0.354 \end{bmatrix}$$

$$\nu = 77.6$$

$$\frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}(0.05) = \frac{155.2}{76.6} \times 3.12 = 6.32$$

$$T^2 = 15.66 > 6.32, \quad H_0 : \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = 0 \text{ is rejected}$$

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Outline

- Introduction
- Comparison of Univariate Mean
- Paired Comparisons and a Repeated Measures Design
- Comparing Mean Vectors from Two Populations
- Comparison of Several Univariate Population Mean (One-Way ANOVA)

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Questions

- Why paired comparisons are not good ways to compare several population means?
- How to compute summed squares (between)?
- How to compute summed squares (within)?
- How to compute summed squares (total)?

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Questions

- How to calculate the degrees of freedom for summed squares (between)?
- How to calculate the degrees of freedom for summed squares (within)?
- How to calculate the degrees of freedom for summed squares (total)?

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Questions

- How to compute the F value for testing of the null hypothesis?
- How are the three kinds of summed squares related?
- How to explain the geometric meaning of the degrees of freedom for a treatment vector?
- What is an ANOVA table?

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Scenarios

- To test if the following statements are plausible
  - Music compressed by four MP3 compressors are with the same quality
  - Three new drugs are all as effective as a placebo
  - Four brands of beer are equally tasty
  - Lectures, group studying, and computer assisted instruction are equally effective for undergraduate students

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Comparing Four MP3 Compressors

- Test four brands, *A*, *B*, *C*, *D*
- 10 subjects each brand (40 in total) to provide a satisfaction rating on a 10-point scale
- Assume that the rating to each brand is a normal distribution, but all four distributions are with the same variance

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Hypotheses

- Null hypothesis

$$H_0 : \mu_A = \mu_B = \mu_C = \mu_D$$

- Alternative hypothesis

$$H_1 : \mathit{Not} \text{ all the } \mu\text{s are equal}$$

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Problem of Using a *t*-Test

- Must compare two brands at a time
- There are 6 possible comparisons
- Each has a 0.05 chance of being significant by chance
- Overall chance of significant result, even when no difference exist, approaches $1-(0.95)^6 \sim 0.26$

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Sample Data

Subject	A	B	C	D
1	4	5	7	2
2	4	5	8	1
3	5	6	7	2
4	5	6	9	3
5	6	7	6	3
6	3	6	3	4
7	4	4	2	5
8	4	5	2	4
9	3	6	2	4
10	4	3	3	3
Mean	4.2	5.3	4.9	3.1

Grand mean: 4.375

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*Adapted from: G. R. Norman and D. L. Streiner, *Biostatistics*, 3rd ed.

Thinking in Terms of Signals and Noises

- Signals
  - Overall difference among the means of the groups
  - Sum of all the squared differences between group means and the overall means
- Noises
  - Overall variability within the groups
  - Sum of all the squared differences between individual data and their group means

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Sum of Squares (Between)

$$SS(between) = n\sum\left(\bar{x}_\ell - \bar{x}\right)^2$$

$$SS(between) = 10[(4.2-4.375)^2 + (5.3-4.375)^2 + (4.9-4.375)^2 + (3.1-4.375)^2] = 27.875$$

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Sum of Squares (Within)

$$SS(within) = \sum_\ell \sum_j \left(x_{\ell j} - \bar{x}_\ell\right)^2$$

$$SS(within) = (4-4.2)^2 + (4-4.2)^2 + \cdots + (4-4.2)^2 + (5-5.3)^2 + (5-5.3)^2 + \cdots + (3-5.3)^2 + (7-4.9)^2 + (8-4.9)^2 + \cdots + (3-4.9)^2 + (2-3.1)^2 + (1-3.1)^2 + \cdots + (3-3.1)^2 \text{ [40 terms]} = 101.50$$

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Sum of Squares (Total)

$$SS(total) = \sum_\ell \sum_j (x_{\ell j} - \bar{x})^2$$

$$x_{\ell j} - \bar{x} = (x_{\ell j} - \bar{x}_\ell) + (\bar{x}_\ell - \bar{x})$$

$$\left(x_{\ell j} - \bar{x}\right)^2 = (x_{\ell j} - \bar{x}_\ell)^2 + 2(x_{\ell j} - \bar{x}_\ell)(\bar{x}_\ell - \bar{x}) + (\bar{x}_\ell - \bar{x})^2$$

$$\sum_j (x_{\ell j} - \bar{x}_\ell) = 0$$

$$\sum_j \left(x_{\ell j} - \bar{x}\right)^2 = \sum_j (x_{\ell j} - \bar{x}_\ell)^2 + n(\bar{x}_\ell - \bar{x})^2$$

$$SS(total) = SS(within) + SS(between)$$

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Sum of Squares (Total)

$$SS(total) = (4-4.375)^2 + (4-4.375)^2 + \cdots + (4-4.375)^2 + (5-4.375)^2 + (5-4.375)^2 + \cdots + (3-4.375)^2 + (7-4.375)^2 + (8-4.375)^2 + \cdots + (3-4.375)^2 + (2-4.375)^2 + (1-4.375)^2 + \ldots + (3-4.375)^2$$

[40 terms]

$$= 129.375 = 101.50 + 27.875$$

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χ^2 Distribution

$X_1 : N(\mu_1, \sigma_1^2), \quad X_2 : N(\mu_2, \sigma_2^2), \quad \cdots,$

$X_\nu : N(\mu_\nu, \sigma_\nu^2); \quad Z_i = \dfrac{X_i - \mu_i}{\sigma_i} : N(0,1)$

$\chi^2 = \sum_{i=1}^{\nu} \left(\dfrac{x_i - \mu_i}{\sigma_i} \right)^2, \quad \nu$: degrees of freedom (d.f.)

$$f_\nu(\chi^2) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} \left(\chi^2\right)^{\nu/2-1} e^{-\chi^2/2}, & \chi^2 > 0 \\ 0, & \chi^2 \le 0 \end{cases}$$

(Gamma distribution with $\alpha = \nu/2$)

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Distribution of Sum of Squares

$X : N(\mu, \sigma^2)$

$S^2 = \dfrac{1}{n-1} \sum_{j=1}^{n} \left(X_j - \bar{X} \right)^2$

$(n-1)\dfrac{S^2}{\sigma^2} : \chi^2$ distribution with $n-1$ degrees of freedom

[proved by moment generating function, see P. G. Hoel, *Introduction to Mathematical Statistics,* 5th ed., John Wiley & Sons, 1984, p. 281]

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Distribution of Sum of Squares

$SS(within) = \sum_{\ell=1}^{g} \sum_{j=1}^{n} \left(x_{\ell j} - \bar{x} \right)^2$

$\dfrac{SS(within)}{\sigma^2} : \chi^2$ distribution with

degree of freedom $df(within) = gn - 1$

$SS(between) = \sum_{\ell=1}^{g} \left(\bar{x}_\ell - \bar{x} \right)^2$

$\bar{x} = \dfrac{1}{gn} \sum_{\ell=1}^{g} \sum_{j=1}^{n} x_{\ell j} = \dfrac{1}{g} \sum_{\ell=1}^{g} \left(\dfrac{1}{n} \sum_{j=1}^{n} x_{\ell j} \right) = \dfrac{1}{g} \sum_{\ell=1}^{g} \bar{x}_\ell$

$\dfrac{SS(between)}{\sigma^2} : \chi^2$ distribution with

degree of freedom $df(between) = g - 1$

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F-Distribution

χ_1^2, χ_2^2 : independent, with d.f. f_1 and f_2, respectively

$$F = \frac{\chi_1^2 / f_1}{\chi_2^2 / f_2}, F > 0$$

$$f(F) = \frac{\Gamma\left(\frac{f_1+f_2}{2}\right)}{\Gamma\left(\frac{f_1}{2}\right)\Gamma\left(\frac{f_2}{2}\right)}\left(\frac{f_1}{f_2}\right)^{\frac{f_1}{2}} \frac{F^{\frac{f_1}{2}-1}}{\left(1+\frac{f_1 F}{f_2}\right)^{(f_1+f_2)/2}}$$

: F_{f_1, f_2}

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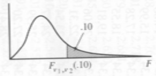
F-Distribution



ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.05	62.26	62.53	62.79
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.17	5.17	5.16	5.15
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.81	2.80	2.78	2.76
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.57	2.56	2.54	2.51
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.27	2.25	2.23	2.21
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.17	2.16	2.13	2.11
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03

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Distribution of *F*

$$F = \frac{SS(between) / df(between)}{SS(within) / df(within)} :$$

F distribution of degree of freedoms

$df(between)$ and $df(within)$

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Expected Values of Sum of Squares

if no difference between groups

$$E[SS(between) / df(between)] = \sigma_{err}^2$$

if no difference within groups

$$E[SS(between) / df(between)] = n\sigma_{bet}^2$$

if both differences can happen

$$E[SS(between) / df(between)] = n\sigma_{bet}^2 + \sigma_{err}^2$$

Thus, if H_0 is invalid

$$\frac{E[SS(between) / df(between)]}{E[SS(within) / df(within)]} = \frac{n\sigma_{bet}^2 + \sigma_{err}^2}{\sigma_{err}^2} > 1$$

$F > 1$

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Degrees of Freedom

$$df(between) = g - 1 = 4 - 1 = 3$$
$$df(within) = g(n-1) = 4(10-1) = 36$$
$$df(total) = gn - 1 = gn - g + g - 1$$
$$= df(within) + df(between)$$
$$= 40 - 1 = 39 = 36 + 3$$

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ANOVA Summary

Source	Sum of Squares	*df*	Mean square	F
Between	27.875	3	9.292	3.296
Within	101.500	36	2.819	
Total	129.375	39		

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Hypothesis Testing

$$F = 3.296 > F_{3,36}(0.05) = 2.86$$
reject $H_0 : \mu_A = \mu_B = \mu_C = \mu_D$
at 0.05 significance level

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Univariate ANOVA

$X_{\ell 1}, X_{\ell 2}, \cdots, X_{\ell n_\ell}$: random sample from $N(\mu_\ell, \sigma^2)$

$\ell = 1, 2, \cdots, g$

Null hypothesis $H_0 : \mu_1 = \mu_2 = \cdots = \mu_g$

Reparameterization

$\mu_\ell = \mu + \tau_\ell$

$H_0 : \tau_1 = \tau_2 = \cdots = \tau_g = 0$

$X_{\ell j} = \mu + \tau_\ell + e_{\ell j}, \quad e_{\ell j} : N(0, \sigma^2), \quad \sum_{\ell=1}^{g} n_\ell \tau_\ell = 0$

$x_{\ell j} = \bar{x} + (\bar{x}_\ell - \bar{x}) + (x_{\ell j} - \bar{x}_\ell)$

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Univariate ANOVA

$$(x_{\ell j}-\bar{x})^2 = (\bar{x}_\ell-\bar{x})^2 + (x_{\ell j}-\bar{x}_\ell)^2 + 2(\bar{x}_\ell-\bar{x})(x_{\ell j}-\bar{x}_\ell)$$

$$\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x}_\ell)=0$$

$$\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x})^2 = n_\ell(\bar{x}_\ell-\bar{x})^2+\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x}_\ell)^2$$

$$\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x})^2 = \sum_{\ell=1}^{g} n_\ell(\bar{x}_\ell-\bar{x})^2 + \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x}_\ell)^2$$

$$(\mathrm{SS}_{cor}) = (\mathrm{SS}_{tr}) + (\mathrm{SS}_{res})$$

$$\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}x_{\ell j}^2 = (n_1+n_2+\cdots+n_\ell)\bar{x}^2 + \sum_{\ell=1}^{g} n_\ell(\bar{x}_\ell-\bar{x})^2 + \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x}_\ell)^2$$

$$(\mathrm{SS}_{obs}) = (\mathrm{SS}_{mean}) + (\mathrm{SS}_{tr}) + (\mathrm{SS}_{res})$$

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Concept of Degrees of Freedom

$$\mathbf{y}' = [x_{11},\cdots,x_{1n_1},x_{21},\cdots,x_{2n_2},\cdots,x_{gn_g}] : \text{d.f. } n = n_1+n_2+\cdots+n_g$$

Treatment vector

$$(\bar{x}_1-\bar{x})\begin{bmatrix}1\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix} + (\bar{x}_2-\bar{x})\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} + \cdots + (\bar{x}_g-\bar{x})\begin{bmatrix}0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\\ 1\\ \vdots\\ 1\end{bmatrix}$$

$$= (\bar{x}_1-\bar{x})\mathbf{u}_1 + (\bar{x}_2-\bar{x})\mathbf{u}_2 + \cdots + (\bar{x}_g-\bar{x})\mathbf{u}_g$$

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Concept of Degrees of Freedom

$\mathbf{1} = [1,\cdots,1] = \mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_g$

Treatment vector and $\mathbf{1}$ are all on the hyperplane spanned by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_g$: d.f. g

$\mathbf{1}$ is perpendicular to the treatment vector

$\therefore$ mean vector $\bar{x}\mathbf{1}$: d.f. $g-1$

Residual vector

$\mathbf{e} = \mathbf{y} - \bar{x}\mathbf{1} - [(\bar{x}_1-\bar{x})\mathbf{u}_1 + (\bar{x}_2-\bar{x})\mathbf{u}_2 + \cdots + (\bar{x}_g-\bar{x})\mathbf{u}_g]$

perpendicular to the hyperplane spanned by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_g$

$\therefore$ d.f. of $\mathbf{e}$: $n-g$

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Univariate ANOVA

ANOVA TABLE FOR COMPARING UNIVARIATE POPULATION MEANS

Source of variation	Sum of squares (SS)	Degrees of freedom (d.f.)
Treatments	$\mathrm{SS}_{tr} = \sum_{\ell=1}^{g} n_\ell(\bar{x}_\ell-\bar{x})^2$	$g-1$
Residual (Error)	$\mathrm{SS}_{res} = \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x}_\ell)^2$	$\sum_{\ell=1}^{g} n_\ell - g$
Total (corrected for the mean)	$\mathrm{SS}_{cor} = \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(x_{\ell j}-\bar{x})^2$	$\sum_{\ell=1}^{g} n_\ell - 1$

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Univariate ANOVA

Reject $H_0 : \tau_1 = \tau_2 = \cdots = \tau_g = 0$ at level α if

$$F = \frac{\mathrm{SS}_{tr} / (g-1)}{\mathrm{SS}_{res} / \left(\sum_{\ell=1}^{g} n_\ell - g \right)} > F_{g-1, \sum n_\ell - g}(\alpha)$$

$$\frac{1}{1 + \mathrm{SS}_{tr} / \mathrm{SS}_{res}} = \frac{\mathrm{SS}_{res}}{\mathrm{SS}_{res} + \mathrm{SS}_{tr}}$$

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Examples 6.7 & 6.8

$$\begin{pmatrix} 9 & 6 & 9 \\ 0 & 2 & \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & \\ 4 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 4 \\ -3 & -3 & \\ -2 & -2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & \\ 1 & -1 & 0 \end{pmatrix}$$

$\mathrm{SS}_{obs} = 216, \ \mathrm{SS}_{mean} = 128$

$\mathrm{SS}_{tr} = 78, \ \text{d.f.} = 3 - 1 = 2$

$\mathrm{SS}_{res} = 10, \text{d.f.} = (3 + 2 + 3) - 3 = 5$

$$F = \frac{\mathrm{SS}_{tr} / (g-1)}{\mathrm{SS}_{res} / \left(\sum n_\ell - g \right)} = \frac{78/2}{10/5} = 19.5 > F_{2,5}(0.01) = 13.27$$

$H_0 : \tau_1 = \tau_2 = \tau_3 = 0$ is rejected at the 1% level

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Outline

- Comparing Several Multivariate Population Means (One-Way Manova)
- Simultaneous Confidence Intervals for Treatment Effects
- Testing for Equality of Covariance Matrices
- Two-Way ANOVA
- Two-Way Multivariate Analysis of Variance

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Questions

- What is the one-way MANOVA table?
- How to compute Wilk's lambda for MANOVA?
- How to test the equality of several mean vectors from the Wilk's lambda?
- How to test the equality of several mean vectors for large sample size?

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Questions

- What are other statistics used in statistical software package for one-way MANOVA?

Scenario: Example 6.10, Nursing Home Data

- Nursing homes can be classified by the owners: private (271), non-profit (138), government (107)
- Costs: nursing labor, dietary labor, plant operation and maintenance labor, housekeeping and laundry labor
- To investigate the effects of ownership on costs

One-Way MANOVA

Population 1: $\mathbf{X}_{11}, \mathbf{X}_{12}, \cdots, \mathbf{X}_{1n_1}$

Population 2: $\mathbf{X}_{21}, \mathbf{X}_{22}, \cdots, \mathbf{X}_{2n_2}$

$\vdots \qquad \vdots$

Population g: $\mathbf{X}_{g1}, \mathbf{X}_{g2}, \cdots, \mathbf{X}_{gn_g}$

MANOVA (Multivariate ANalysis Of VAriance) is used to investigate whether the population mean vectors are the same, and, if not, which mean components differ significantly

Assumptions about the Data

$\mathbf{X}_{\ell 1}, \mathbf{X}_{\ell_2}, \cdots, \mathbf{X}_{\ell n_\ell}$: random sample from a population with mean $\boldsymbol{\mu}_\ell$, $\ell = 1, 2, \cdots, g$

Random sample from different populations are independent

All populations have a common covariance matrix $\boldsymbol{\Sigma}$

Each population is multivariate normal

MANOVA

$$\mathbf{X}_{\ell j} = \boldsymbol{\mu} + \boldsymbol{\tau}_\ell + \mathbf{e}_{\ell j};\ j = 1, 2, \cdots, n_\ell;\ \ell = 1, 2, \cdots, g$$

$\mathbf{e}_{\ell j} : N_p(\mathbf{0}, \boldsymbol{\Sigma})$, $\boldsymbol{\mu}$: overall mean (level)

$\boldsymbol{\tau}_\ell$: ℓth treatment effect, $\sum_{\ell=1}^{g} n_\ell \boldsymbol{\tau}_\ell = \mathbf{0}$

$$\mathbf{x}_{\ell j} = \bar{\mathbf{x}} + (\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}}) + (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell) = \hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\tau}}_\ell + \hat{\mathbf{e}}_{\ell j}$$

$$\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}})(\mathbf{x}_{\ell j} - \bar{\mathbf{x}})' = \sum_{\ell=1}^{g} n_\ell (\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})'$$

$$+ \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)' = \mathbf{B} + \mathbf{W}$$

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MANOVA

MANOVA TABLE FOR COMPARING POPULATION MEAN VECTORS

Source of variation	Matrix of sum of squares and cross products (SSP)	Degrees of freedom (d.f.)
Treatment	$\mathbf{B} = \sum_{\ell=1}^{g} n_\ell (\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})'$	$g - 1$
Residual (Error)	$\mathbf{W} = \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)'$	$\sum_{\ell=1}^{g} n_\ell - g$
Total (corrected for the mean)	$\mathbf{B} + \mathbf{W} = \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}})(\mathbf{x}_{\ell j} - \bar{\mathbf{x}})'$	$\sum_{\ell=1}^{g} n_\ell - 1$

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MANOVA

$$\mathbf{W} = \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)'$$

$$= (n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2 + \cdots + (n_g - 1)\mathbf{S}_g$$

Reject $H_0 : \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \cdots = \boldsymbol{\tau}_g = 0$ if Wilk's lambda

$$\Lambda^* = \frac{|W|}{|B+W|} = \frac{\left|\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)'\right|}{\left|\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}})(\mathbf{x}_{\ell j} - \bar{\mathbf{x}})'\right|}$$

is too small

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Distribution of Wilk's Lambda

No. of variables	No. of groups	Sampling distribution for multivariate normal data
$p = 1$	$g \ge 2$	$\left(\frac{\Sigma n_\ell - g}{g - 1}\right)\left(\frac{1 - \Lambda^*}{\Lambda^*}\right) \sim F_{g-1, \Sigma n_\ell - g}$
$p = 2$	$g \ge 2$	$\left(\frac{\Sigma n_\ell - g - 1}{g - 1}\right)\left(\frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2(g-1), 2(\Sigma n_\ell - g - 1)}$
$p \ge 1$	$g = 2$	$\left(\frac{\Sigma n_\ell - p - 1}{p}\right)\left(\frac{1 - \Lambda^*}{\Lambda^*}\right) \sim F_{p, \Sigma n_\ell - p - 1}$
$p \ge 1$	$g = 3$	$\left(\frac{\Sigma n_\ell - p - 2}{p}\right)\left(\frac{1 - \sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2p, 2(\Sigma n_\ell - p - 2)}$

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Test of Hypothesis for Large Size

If H_0 is true and $\sum n_\ell = n$ is large,

$$-\left(n-1-\frac{p+g}{2}\right)\ln\Lambda^* : \chi^2_{p(g-1)}$$

Reject H_0 at significance level α if

$$-\left(n-1-\frac{p+g}{2}\right)\ln\left(\frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|}\right) > \chi^2_{p(g-1)}(\alpha)$$

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Popular MANOVA Statistics Used in Statistical Packages

Wilk's lambda $\Lambda^* = \dfrac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|}$

Lawley-Hotelling trace $= tr\left[\mathbf{B}\mathbf{W}^{-1}\right]$

Pillai trace $= tr\left[\mathbf{B}(\mathbf{B}+\mathbf{W})^{-1}\right]$

Roy's largest root =

maximum eigenvalue of $\mathbf{W}(\mathbf{B}+\mathbf{W})^{-1}$

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Example 6.9

$$\begin{pmatrix} \begin{bmatrix}9\\3\end{bmatrix} & \begin{bmatrix}6\\2\end{bmatrix} & \begin{bmatrix}9\\7\end{bmatrix} \\ \begin{bmatrix}0\\4\end{bmatrix} & \begin{bmatrix}2\\0\end{bmatrix} & \\ \begin{bmatrix}3\\8\end{bmatrix} & \begin{bmatrix}1\\9\end{bmatrix} & \begin{bmatrix}2\\7\end{bmatrix} \end{pmatrix} \bar{\mathbf{x}}_1 = \begin{bmatrix}8\\4\end{bmatrix}, \bar{\mathbf{x}}_2 = \begin{bmatrix}1\\2\end{bmatrix}, \bar{\mathbf{x}}_3 = \begin{bmatrix}4\\5\end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix}4\\5\end{bmatrix}$$

$$\begin{pmatrix} 9 & 6 & 9 \\ 0 & 2 & \\ 3 & 1 & 2 \end{pmatrix}, \mathrm{SS}_{obs} = \mathrm{SS}_{mean} + \mathrm{SS}_{tr} + \mathrm{SS}_{res} = 128 + 78 + 10 = 216$$

$$\begin{pmatrix} 3 & 2 & 7 \\ 4 & 0 & \\ 8 & 9 & 7 \end{pmatrix}, \mathrm{SS}_{obs} = \mathrm{SS}_{mean} + \mathrm{SS}_{tr} + \mathrm{SS}_{res} = 200 + 48 + 24 = 272$$

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Example 6.8

$$\begin{pmatrix} 9 & 6 & 9 \\ 0 & 2 & \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & \\ 4 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 4 \\ -3 & -3 & \\ -2 & -2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & \\ 1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 7 \\ 4 & 0 & \\ 8 & 9 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & \\ 5 & 5 & 5 \end{pmatrix} + \begin{pmatrix} -1 & -1 & -1 \\ -3 & -3 & \\ 3 & 3 & 3 \end{pmatrix} + \begin{pmatrix} -1 & -2 & 3 \\ 2 & -2 & \\ 0 & 1 & -1 \end{pmatrix}$$

Cross products

Mean : $8 \times 4 \times 5 = 160$

Treatment : $3 \times 4 \times (-1) + 2 \times (-3) \times (-3) + 3 \times (-2) \times 3 = -12$

Residual : $1 \times (-1) + (-2) \times (-2) + 1 \times 3 + \cdots + 0 \times (-1) = 1$

Total : $9 \times 3 + 6 \times 2 + 9 \times 7 + \cdots + 2 \times 7 = 149$

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Example 6.9

Source of variation	Matrix of sum of squares and cross products	Degrees of freedom
Treatment	$\begin{bmatrix} 78 & -12 \\ -12 & 48 \end{bmatrix}$	$3 - 1 = 2$
Residual	$\begin{bmatrix} 10 & 1 \\ 1 & 24 \end{bmatrix}$	$3 + 2 + 3 - 3 = 5$
Total (corrected)	$\begin{bmatrix} 88 & -11 \\ -11 & 72 \end{bmatrix}$	7

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Example 6.9

$$\Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|} = \frac{\begin{vmatrix} 10 & 1 \\ 1 & 24 \end{vmatrix}}{\begin{vmatrix} 88 & -11 \\ -11 & 72 \end{vmatrix}} = 0.0385$$

$$\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right)\frac{\sum n_\ell - g - 1}{g-1} = \left(\frac{1-\sqrt{0.0385}}{\sqrt{0.0385}}\right)\frac{8-3-1}{3-1}$$

$$= 8.19 > F_{2(g-1),2\sum n_\ell - g - 1}(0.01) = F_{4,8}(0.01) = 7.01$$

Reject H_0

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Example 6.10: Nursing Home Data

- Nursing homes can be classified by the owners: private (271), non-profit (138), government (107)
- Costs: nursing labor, dietary labor, plant operation and maintenance labor, housekeeping and laundry labor
- To investigate the effects of ownership on costs

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Example 6.10

Group	Number of observations	Sample mean vectors
$\ell = 1$ (private)	$n_1 = 271$	
$\ell = 2$ (nonprofit)	$n_2 = 138$	$\bar{\mathbf{x}}_1 = \begin{bmatrix} 2.066 \\ .480 \\ .082 \\ .360 \end{bmatrix};\ \bar{\mathbf{x}}_2 = \begin{bmatrix} 2.167 \\ .596 \\ .124 \\ .418 \end{bmatrix};\ \bar{\mathbf{x}}_3 = \begin{bmatrix} 2.273 \\ .521 \\ .125 \\ .383 \end{bmatrix}$
$\ell = 3$ (government)	$n_3 = 107$	
	$\sum_{\ell=1}^{3} n_\ell = 516$	

Sample covariance matrices

$$\mathbf{S}_1 = \begin{bmatrix} .291 & & & \\ -.001 & .011 & & \\ .002 & .000 & .001 & \\ .010 & .003 & .000 & .010 \end{bmatrix};\quad \mathbf{S}_2 = \begin{bmatrix} .561 & & & \\ .011 & .025 & & \\ .001 & .004 & .005 & \\ .037 & .007 & .002 & .019 \end{bmatrix};$$

$$\mathbf{S}_3 = \begin{bmatrix} .261 & & & \\ .030 & .017 & & \\ .003 & -.000 & .004 & \\ .018 & .006 & .001 & .013 \end{bmatrix}$$

Source: Data courtesy of State of Wisconsin Department of Health and Social Services.

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Example 6.10

$$\mathbf{W} = (n_1-1)\mathbf{S}_1 + (n_2-1)\mathbf{S}_2 + (n_3-1)\mathbf{S}_3$$

$$= \begin{bmatrix} 182.962 & & & \\ 4.408 & 8.200 & & \\ 1.695 & 0.633 & 1.484 & \\ 9.581 & 2.428 & 0.394 & 6.538 \end{bmatrix}$$

$$\bar{\mathbf{x}} = \frac{n_1\bar{\mathbf{x}}_1 + n_2\bar{\mathbf{x}}_2 + n_3\bar{\mathbf{x}}_3}{n_1+n_2+n_3} = [2.136 \quad 0.519 \quad 0.102 \quad 0.380]'$$

$$\mathbf{B} = \sum_{\ell=1}^{g} n_\ell (\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})' = \begin{bmatrix} 3.475 & & & \\ 1.111 & 1.225 & & \\ 0.821 & 0.453 & 0.235 & \\ 0.584 & 0.610 & 0.230 & 0.304 \end{bmatrix}$$

Example 6.10

$$\Lambda^* = \frac{|W|}{|B+W|} = 0.7714$$

$$\left(\frac{\sum n_\ell - p - 2}{p}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) = 17.67$$

$$F_{2\times4,2\times510}(0.01) \approx \chi_8^2(0.01)/8 = 2.51$$

or, approximate analysis

$$-(n-1-(p+g)/2)\ln\left(\frac{|W|}{|B+W|}\right) = 132.76$$

$$\chi^2_{p(g-1)}(0.01) = \chi_8^2(0.01) = 20.09$$

Reject H_0 by both analyses

Outline

- Comparing Several Multivariate Population Means (One-Way Manova)
- Simultaneous Confidence Intervals for Treatment Effects
- Testing for Equality of Covariance Matrices
- Two-Way ANOVA
- Two-Way Multivariate Analysis of Variance

Questions

- What are the Bonferroni Intervals for Treatment Effects?

Bonferroni Intervals for Treatment Effects

$$\hat{\tau}_{ki} = \bar{x}_{ki} - \bar{x}_i, \quad \hat{\tau}_{ki} - \hat{\tau}_{\ell i} = \bar{x}_{ki} - \bar{x}_{\ell i}$$

$$\text{Var}(\hat{\tau}_{ki} - \hat{\tau}_{\ell i}) = \text{Var}(\bar{x}_{ki} - \bar{x}_{\ell i}) = \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)\sigma_{ii}$$

$$\mathbf{W} = (n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2 + \cdots + (n_g - 1)\mathbf{S}_g$$

$$= (n - g)\mathbf{S}_{pooled} \approx (n - g)\mathbf{\Sigma}$$

$$\text{Var}(\hat{\tau}_{ki} - \hat{\tau}_{\ell i}) \approx \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)\frac{w_{ii}}{(n-g)}$$

$$m = pg(g-1)/2$$

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Result 6.5: Bonferroni Intervals for Treatment Effects

With confidence at least $(1-\alpha)$

$\tau_{ki} - \tau_{\ell i}$ belongs to

$$\bar{x}_{ki} - \bar{x}_{\ell i} \pm t_{n-g}\left(\frac{\alpha}{pg(g-1)}\right)\sqrt{\frac{w_{ii}}{n-g}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$$

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Example 6.11: Example 6.10 Data

$$\hat{\boldsymbol{\tau}}_1 = \bar{\mathbf{x}}_1 - \bar{\mathbf{x}} = [-0.070 \quad -0.039 \quad -0.020 \quad -0.020]'$$

$$\hat{\boldsymbol{\tau}}_3 = \bar{\mathbf{x}}_3 - \bar{\mathbf{x}} = [0.137 \quad 0.002 \quad 0.023 \quad 0.003]'$$

$$\hat{\tau}_{13} - \hat{\tau}_{33} = -0.20 - 0.023 = -0.043, \; n = 516$$

$$\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_3}\right)\frac{w_{33}}{n-g}} = \sqrt{\left(\frac{1}{271} + \frac{1}{107}\right)\frac{1.484}{516-3}} = 0.00614$$

$$t_{513}(0.05/4\times 3\times 2) = 2.87$$

95% simultaneous confidence interval for $\tau_{13} - \tau_{33}$

$-0.043 \pm 2.87 \times 0.00614$ or $(-0.061, -0.025)$

95% simultaneous confidence intervals for

$\tau_{13} - \tau_{23}$ and $\tau_{23} - \tau_{33}$: $(-0.058, -0.026)$, $(-0.021, 0.019)$

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Outline

- Comparing Several Multivariate Population Means (One-Way Manova)
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- Two-Way Multivariate Analysis of Variance

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Questions

- How to test if the covariance matrices of several populations are equal? (Box's *M*-Test)

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Test for Equality of Covariance Matrices

- With g populations, null hypothesis

  H_0: $\Sigma_1 = \Sigma_2 = \ldots = \Sigma_g = \Sigma$
- Assume multivariate normal populations
- Likelihood ratio statistic for testing H_0

$$\Lambda = \prod_{\ell} \left(\frac{|\mathbf{S}_\ell|}{|\mathbf{S}_{pooled}|} \right)^{(n_\ell - 1)/2}$$

$$\mathbf{S}_{pooled} = \frac{1}{\sum_{\ell} (n_\ell - 1)} \left\{ (n_1 - 1)\mathbf{S}_1 + \cdots + (n_g - 1)\mathbf{S}_g \right\}$$

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Box's *M*-Test

$$M = -2\ln\Lambda = \left[\sum_{\ell} (n_\ell - 1) \right] \ln|\mathbf{S}_{pooled}| - \sum_{\ell} \left[(n_\ell - 1) \ln|\mathbf{S}_\ell| \right]$$

$$u = \left[\sum_{\ell} \frac{1}{(n_\ell - 1)} - \frac{1}{\sum_{\ell} (n_\ell - 1)} \right] \left[\frac{2p^2 + 3p - 1}{6(p+1)(g-1)} \right]$$

$C = (1-u)M$: approximate χ^2_ν

$$\nu = \frac{1}{2} p(p+1)(g-1)$$

Reject H_0 if $C > \chi^2_\nu(\alpha)$

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Example 6.12

- Example 6.10 - nursing home data

$g = 3, \quad p = 4, \quad n_1 = 271, \quad n_2 = 138, \quad n_3 = 107$

$\ln|\mathbf{S}_1| = -17.397, \quad \ln|\mathbf{S}_2| = -13.926$

$\ln|\mathbf{S}_3| = -15.741, \quad \ln|\mathbf{S}_{pooled}| = -15.564$

$$u = \left[\frac{1}{270} + \frac{1}{137} + \frac{1}{106} - \frac{1}{270+137+106} \right] \left[\frac{2(4)^2 + 3(4) - 1}{6(4+1)(3-1)} \right] = 0.0133$$

$$M = [270+137+106](-15.564) - [270(-17.397) + 137(-13.926) + 106(-15.741)] = 289.3$$

$C = (1 - 0.0133) \times 289.3 = 285.5$

$\nu = 4(4+1)(3-1)/2 = 20$

H_0 is rejected at any reasonable lelel of significance from χ^2_ν table for comparison with C

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Example 6.13: Plastic Film Data

x_1 = tear resistance, x_2 = gloss, and x_3 = opacity

		Factor 2: Amount of additive					
		Low (1.0%)			High (1.5%)		
		x_1	x_2	x_3	x_1	x_2	x_3
Factor 1: Change in rate of extrusion	Low (−10)%	[6.5	9.5	4.4]	[6.9	9.1	5.7]
		[6.2	9.9	6.4]	[7.2	10.0	2.0]
		[5.8	9.6	3.0]	[6.9	9.9	3.9]
		[6.5	9.6	4.1]	[6.1	9.5	1.9]
		[6.5	9.2	0.8]	[6.3	9.4	5.7]
		x_1	x_2	x_3	x_1	x_2	x_3
	High (10%)	[6.7	9.1	2.8]	[7.1	9.2	8.4]
		[6.6	9.3	4.1]	[7.0	8.8	5.2]
		[7.2	8.3	3.8]	[7.2	9.7	6.9]
		[7.1	8.4	1.6]	[7.5	10.1	2.7]
		[6.8	8.5	3.4]	[7.6	9.2	1.9]

Outline

- Comparing Several Multivariate Population Means (One-Way Manova)
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- Two-Way Multivariate Analysis of Variance

Questions

- How to determine if a factor and its interaction with the other factor is significant if two factors are involved in an experiment?
- What are the four types of interactions of two factors?
- What is the two-way ANOVA table?

Scenarios

- To observe if effects of factors in the following scenarios are significant
  - Ratings of music compressed by MP3 compressors: brands vs. ages of the subjects
  - Performance of Teaching: methods (Lectures, group studying, and computer assisted instruction) vs. genders of undergraduate students

Teaching Methods vs. Gender: Knowing only Overall Mean

Gender	CAI	Lecture	Group Studying	Mean
Boys	50	50	50	50
Girls	50	50	50	50
Mean	50	50	50	50

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Teaching Methods vs. Gender:
Knowing Overall Mean and Row Effects

Gender	CAI	Lecture	Group Studying	Mean
Boys	40	40	40	40
Girls	60	60	60	60
Mean	50	50	50	50

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Teaching Methods vs. Gender:
Knowing Overall Mean, Row Effects, and Column Effects

Gender	CAI	Lecture	Group Studying	Mean
Boys	50	40	30	40
Girls	70	60	50	60
Mean	60	50	40	50

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Teaching Methods vs. Gender: Including Interaction Terms

Gender	CAI	Lecture	Group Studying	Mean
Boys	65	40	15	40
Girls	55	60	65	60
Mean	60	50	40	50

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Comparing Four MP3 Compressors

- Test four brands, *A*, *B*, *C*, *D*
- 10 subjects, 5 young and 5 senior, each brand (40 in total) to provide a satisfaction rating on a 10-point scale
- Assume that the rating to each brand is a normal distribution, but all four distributions are with the same variance

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Sample Data

		A	B	C	D	Mean
Young Subjects	1~4	4	5	7	2	5.05
	5~8	4	5	8	1	
	9~12	5	6	7	2	
	13~16	5	6	9	3	
	17~20	6	7	6	3	
	Mean	4.8	5.8	7.4	2.2	

*Adapted from: G. R. Norman and D. L. Streiner, *Biostatistics*, 3rd ed.

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Sample Data

		A	B	C	D	Mean
Senior Subjects	21~24	3	6	3	4	3.70
	25~28	4	4	2	5	
	29~32	4	5	2	4	
	33~36	3	6	2	4	
	37~40	4	3	3	3	
	Mean	3.6	4.8	2.4	4.0	

	A	B	C	D	Mean
Brand Mean	4.2	5.3	4.9	3.1	4.375

*Adapted from: G. R. Norman and D. L. Streiner, *Biostatistics*, 3rd ed.

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Sum of Squares (Young/Senior)

$$SS(young\,/\,senior) = bn\sum_{\ell=1}^{g}\left(\bar{x}_{\ell\bullet} - \bar{x}\right)^2$$

$$SS(young\,/\,senior) = 20[(5.05-4.375)^2 + (3.70-4.375)^2]$$

$$= 18.225$$

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Sum of Squares (Brands)

$$SS(brands) = gn\sum_{k=1}^{b}(\bar{x}_{\bullet k} - \bar{x})^2$$

$$SS(brands) = 10[(4.2-4.375)^2 + (5.3-4.375)^2 + (4.9-4.375)^2 + (3.1-4.375)^2] = 27.875$$

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Sum of Squares (Within)

$$SS(within) = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{\gamma=1}^{n}\left(x_{\ell k\gamma} - \bar{x}_{\ell k}\right)$$

$$SS(within) = (4-4.8)^2 + (4-4.8)^2 + \cdots + (6-4.8)^2 + (5-5.8)^2 + (5-5.8)^2 + \cdots + (7-5.9)^2 + \cdots + (4-4.0)^2 + (5-4.0)^2 + \cdots + (3-4.0)^2$$

[40 terms]

$$= 24.80$$

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Sum of Squares (Total)

$$x_{\ell kr} = \bar{x} + (\bar{x}_{\ell\bullet} - \bar{x}) + (\bar{x}_{\bullet k} - \bar{x}) + (\bar{x}_{\ell k} - \bar{x}_{\ell\bullet} - \bar{x}_{\bullet k} + \bar{x}) + (x_{\ell kr} - \bar{x}_{\ell k})$$

$$\sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{\ell kr} - \bar{x})^2 = \sum_{\ell=1}^{g} bn(\bar{x}_{\ell\bullet} - \bar{x})^2 + \sum_{k=1}^{b} gn(\bar{x}_{\bullet k} - \bar{x})^2 + \sum_{\ell=1}^{g}\sum_{k=1}^{b} n(\bar{x}_{\ell k} - \bar{x}_{\ell\bullet} - \bar{x}_{\bullet k} + \bar{x})^2 + \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{\ell kr} - \bar{x}_{\ell k})^2$$

$$\mathrm{SS}(total) = \mathrm{SS}(young/senior) + \mathrm{SS}(brand) + \mathrm{SS}(interactions) + \mathrm{SS}(within)$$

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Sum of Squares (Interactions)

$$SS(interactions) = n\sum_{\ell=1}^{g}\sum_{k=1}^{b}(\bar{x}_{\ell k} - \bar{x}_{\ell\bullet} - \bar{x}_{\bullet k} + \bar{x})^2$$

$$SS(interactions) = 5[(4.8-4.875)^2 + (3.6-3.525)^2 + \cdots + (4.0-2.425)^2]$$

[8 terms]

$$= 58.475$$

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Sum of Squares (Total)

$$SS(total) = (4-4.375)^2 + (4-4.375)^2 + \cdots + (4-4.375)^2 +$$
$$(5-4.375)^2 + (5-4.375)^2 + \cdots + (3-4.375)^2 +$$
$$(7-4.375)^2 + (8-4.375)^2 + \cdots + (3-4.375)^2 +$$
$$(2-4.375)^2 + (1-4.375)^2 + \ldots + (3-4.375)^2$$
$$[\text{40 terms}]$$
$$= 129.375 = 18.225 + 58.475 + 24.80 + 27.875$$

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Expected Values of Sum of Squares

$$E[SS(brand)/df(brand)] \text{ contains } \sigma^2_{brand}, \sigma^2_{interactions}, \sigma^2_{err}$$
$$E[SS(within)/df(within)] = \sigma^2_{err}$$
Thus, if brand effect is significant
$$\frac{E[SS(brand)/df(brand)]}{E[SS(within)/df(within)]} > 1$$
$$F_{brand} > 1$$

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Degrees of Freedom

$$df(young/senior) = g-1 = 2-1 = 1$$
$$df(brand) = b-1 = 4-1 = 3$$
$$df(within) = bg(n-1) = 8(5-1) = 32$$
$$df(interactions) = (b-1)(g-1) = (4-1)(2-1) = 3$$
$$df(total) = bgn-1 = bg(n-1) + (b-1)(g-1) + b-1 + g-1$$
$$= df(within) + df(interactions) +$$
$$df(brand) + df(young/senior)$$
$$= 40-1 = 39 = 32+3+3+1$$

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Two-way ANOVA Summary

Source	Sum of Squares	*df*	Mean square	F
Brand	27.875	3	9.29	11.99
Young/ Senior	18.225	1	18.23	23.52
Brand X Y/S	58.475	3	19.49	25.15
Within	24.80	32	0.78	
Total	129.375	39		

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Hypothesis Testing

$$F_{brand} = 11.99 > F_{3,32}(0.05) \approx 2.92$$

$$F_{Y/S} = 23.52 > F_{1,32}(0.05) \approx 4.17$$

$$F_{interactions} = 25.15 > F_{3,32}(0.05) \approx 2.92$$

All factors and interactions are significant at 0.05 significance level

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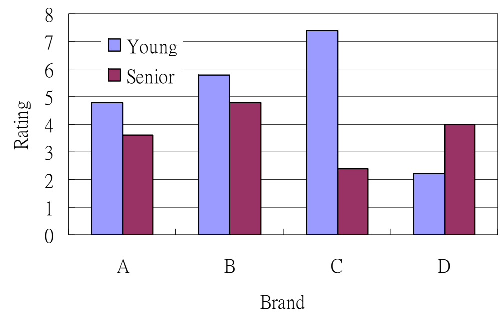
Histogram of Means



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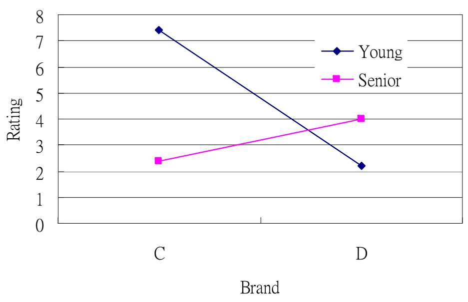
Effect of Interaction



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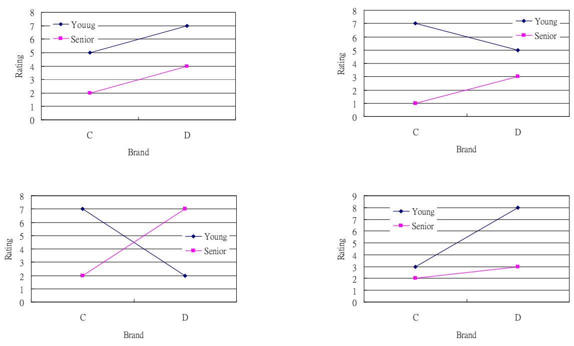
Possible Types of Interactions



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de Groot's Experiment (1965)

- Observed the ability of chess masters and novices to recall piece positions
- Experts
  - Recalled about 90% of the pieces in a typical mid-game
- Novices
  - Recalled about 20%
- Many factors might have been introduced
- Randomized piece positions
  - Everybody recalled about 20%
  - No effect of expertise

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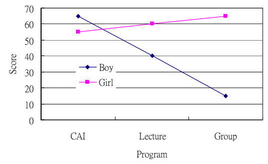
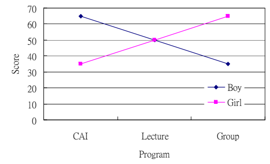
Interpretation by Adjusted Data





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Two-Way ANOVA

$$X_{\ell kr} = \mu + \tau_\ell + \beta_k + \gamma_{\ell k} + e_{\ell kr}$$

$$\ell = 1, 2, \cdots, g; \quad k = 1, 2, \cdots, b; \quad r = 1, 2, \cdots, n$$

$$\sum_{\ell=1}^{g} \tau_\ell = \sum_{k=1}^{b} \beta_k = \sum_{\ell=1}^{g} \gamma_{\ell k} = \sum_{k=1}^{b} \gamma_{\ell k} = 0, \quad e_{\ell kr} : N(0, \sigma^2)$$

$$E(X_{\ell kr}) = \mu + \tau_\ell + \beta_k + \gamma_{\ell k}$$

$$x_{\ell kr} = \bar{x} + (\bar{x}_{\ell \bullet} - \bar{x}) + (\bar{x}_{\bullet k} - \bar{x}) + (\bar{x}_{\ell k} - \bar{x}_{\ell \bullet} - \bar{x}_{\bullet k} + \bar{x}) + (x_{\ell kr} - \bar{x}_{\ell k})$$

$$\sum_{\ell=1}^{g} \sum_{k=1}^{b} \sum_{r=1}^{n} (x_{\ell kr} - \bar{x})^2 = \sum_{\ell=1}^{g} bn(\bar{x}_{\ell \bullet} - \bar{x})^2 + \sum_{k=1}^{b} gn(\bar{x}_{\bullet k} - \bar{x})^2$$

$$+ \sum_{\ell=1}^{g} \sum_{k=1}^{b} n(\bar{x}_{\ell k} - \bar{x}_{\ell \bullet} - \bar{x}_{\bullet k} + \bar{x})^2 + \sum_{\ell=1}^{g} \sum_{k=1}^{b} \sum_{r=1}^{n} (x_{\ell kr} - \bar{x}_{\ell k})^2$$

$$SS_{cor} = SS_{fac1} + SS_{fac2} + SS_{int} + SS_{res}$$

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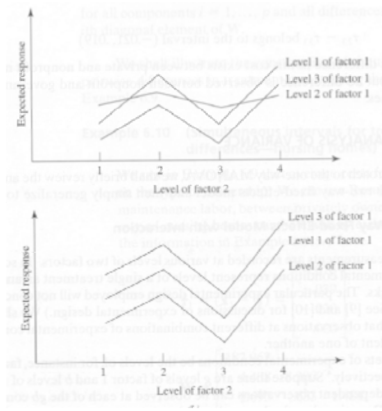
Effect of Interactions



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Two-Way ANOVA

ANOVA TABLE FOR COMPARING EFFECTS OF TWO FACTORS AND THEIR INTERACTION

Source of variation	Sum of squares (SS)	Degrees of freedom (d.f.)
Factor 1	$SS_{fac1} = \sum_{\ell=1}^{g} bn(\bar{x}_{\ell\cdot} - \bar{x})^2$	$g - 1$
Factor 2	$SS_{fac2} = \sum_{k=1}^{b} gn(\bar{x}_{\cdot k} - \bar{x})^2$	$b - 1$
Interaction	$SS_{int} = \sum_{\ell=1}^{g}\sum_{k=1}^{b} n(\bar{x}_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})^2$	$(g-1)(b-1)$
Residual (Error)	$SS_{res} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n} (x_{\ell kr} - \bar{x}_{\ell k})^2$	$gb(n-1)$
Total (corrected)	$SS_{cor} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n} (x_{\ell kr} - \bar{x})^2$	$gbn - 1$

Two-Way ANOVA

F – ratio tests

$$\frac{SS_{fac1}/(g-1)}{SS_{res}/\left(gb(n-1)\right)} : \text{for effects of factor 1}$$

$$\frac{SS_{fac2}/(b-1)}{SS_{res}/\left(gb(n-1)\right)} : \text{for effects of factor 2}$$

$$\frac{SS_{int}/(g-1)(b-1)}{SS_{res}/\left(gb(n-1)\right)} : \text{for effects of}$$

factor 1 - factor 2 intercation

Outline

- Comparing Several Multivariate Population Means (One-Way Manova)
- Simultaneous Confidence Intervals for Treatment Effects
- Testing for Equality of Covariance Matrices
- Two-Way ANOVA
- Two-Way Multivariate Analysis of Variance

Questions

- What is the two-way MANOVA table?
- How to determine if the interaction effect exists?
- How to test the effect of each factor by the two-way MANOVA?
- How to determine the Bonferroni confidence intervals if the interaction effect is negligible?

Two-Way MANOVA

$$\mathbf{X}_{\ell kr} = \boldsymbol{\mu} + \boldsymbol{\tau}_\ell + \boldsymbol{\beta}_k + \boldsymbol{\gamma}_{\ell k} + \mathbf{e}_{\ell kr}$$

$$\ell = 1, 2, \cdots, g; \quad k = 1, 2, \cdots, b; \quad r = 1, 2, \cdots, n$$

$$\sum_{\ell=1}^{g} \boldsymbol{\tau}_\ell = \sum_{k=1}^{b} \boldsymbol{\beta}_k = \sum_{\ell=1}^{g} \boldsymbol{\gamma}_{\ell k} = \sum_{k=1}^{b} \boldsymbol{\gamma}_{\ell k} = 0, \quad \mathbf{e}_{\ell kr} : N_p(\mathbf{0}, \boldsymbol{\Sigma})$$

$$\mathbf{x}_{\ell kr} = \bar{\mathbf{x}} + (\bar{\mathbf{x}}_{\ell\bullet} - \bar{\mathbf{x}}) + (\bar{\mathbf{x}}_{\bullet k} - \bar{\mathbf{x}}) + (\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\bullet} - \bar{\mathbf{x}}_{\bullet k} + \bar{\mathbf{x}}) + (\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})$$

$$\sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n} (\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})' =$$

$$\sum_{\ell=1}^{g} bn(\bar{\mathbf{x}}_{\ell\bullet} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell\bullet} - \bar{\mathbf{x}})' + \sum_{k=1}^{b} gn(\bar{\mathbf{x}}_{\bullet k} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\bullet k} - \bar{\mathbf{x}})'$$

$$+ \sum_{\ell=1}^{g}\sum_{k=1}^{b} n(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\bullet} - \bar{\mathbf{x}}_{\bullet k} + \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\bullet} - \bar{\mathbf{x}}_{\bullet k} + \bar{\mathbf{x}})'$$

$$+ \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n} (\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})'$$

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Two-Way MANOVA

MANOVA TABLE FOR COMPARING FACTORS AND THEIR INTERACTION

Source of variation	Matrix of sum of squares and cross products (SSP)	Degrees of freedom (d.f.)
Factor 1	$\mathrm{SSP}_{fac1} = \sum_{\ell=1}^{g} bn(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})'$	$g - 1$
Factor 2	$\mathrm{SSP}_{fac2} = \sum_{k=1}^{b} gn(\bar{\mathbf{x}}_{\cdot k} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\cdot k} - \bar{\mathbf{x}})'$	$b - 1$
Interaction	$\mathrm{SSP}_{int} = \sum_{\ell=1}^{g}\sum_{k=1}^{b} n(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}}_{\cdot k} + \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}}_{\cdot k} + \bar{\mathbf{x}})'$	$(g - 1)(b - 1)$
Residual (Error)	$\mathrm{SSP}_{res} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n} (\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})'$	$gb(n - 1)$
Total (corrected)	$\mathrm{SSP}_{cor} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n} (\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})'$	$gbn - 1$

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Two-Way MANOVA

Test for interaction

For large samples, reject $H_0 : \boldsymbol{\gamma}_{11} = \boldsymbol{\gamma}_{12} = \cdots = \boldsymbol{\gamma}_{gb} = 0$ if

$$-\left[gb(n-1) - \frac{p+1-(g-1)(b-1)}{2}\right]\ln\Lambda^* > \chi^2_{(g-1)(b-1)}(\alpha)$$

$$\text{Wilk's lambda } \Lambda^* = \frac{|\mathrm{SSP}_{res}|}{|\mathrm{SSP}_{int} + \mathrm{SSP}_{res}|}$$

If interaction effects exist, the factor effects do not have a clear interpretation

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Two-Way MANOVA

Test for factor 1 effect

For large samples, reject $H_0 : \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \cdots = \boldsymbol{\tau}_g = 0$ if

$$-\left[gb(n-1) - \frac{p+1-(g-1)(b-1)}{2}\right]\ln\Lambda^* > \chi^2_{(g-1)p}(\alpha)$$

$$\text{Wilk's lambda } \Lambda^* = \frac{|\mathrm{SSP}_{res}|}{|\mathrm{SSP}_{fac1} + \mathrm{SSP}_{res}|}$$

Test for factor 2 effect

For large samples, reject $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_b = 0$ if

$$-\left[gb(n-1) - \frac{p+1-(g-1)(b-1)}{2}\right]\ln\Lambda^* > \chi^2_{(b-1)p}(\alpha)$$

$$\text{Wilk's lambda } \Lambda^* = \frac{|\mathrm{SSP}_{res}|}{|\mathrm{SSP}_{fac2} + \mathrm{SSP}_{res}|}$$

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Bonferroni Confidence Intervals

With negligible interactions,
the simultaneus confidence intervals are

$$(\bar{x}_{\ell \bullet i} - \bar{x}_{m \bullet i}) \pm t_p\left(\frac{\alpha}{pg(g-1)}\right)\sqrt{\frac{E_{ii}}{\nu}\frac{2}{bn}} \quad \text{for } \tau_{\ell i} - \tau_{mi}$$

and

$$(\bar{x}_{\bullet ki} - \bar{x}_{\bullet qi}) \pm t_p\left(\frac{\alpha}{pb(b-1)}\right)\sqrt{\frac{E_{ii}}{\nu}\frac{2}{gn}} \quad \text{for } \beta_{ki} - \beta_{qi}$$

$$\nu = gb(n-1), \quad \mathbf{E} = \text{SSP}_{res}$$

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Example 6.13: MANOVA Table

Source of variation	SSP	d.f.
Factor 1: change in rate of extrusion	$\begin{bmatrix} 1.7405 & -1.5045 & .8555 \\ & 1.3005 & -.7395 \\ & & .4205 \end{bmatrix}$	1
Factor 2: amount of additive	$\begin{bmatrix} .7605 & .6825 & 1.9305 \\ & .6125 & 1.7325 \\ & & 4.9005 \end{bmatrix}$	1
Interaction	$\begin{bmatrix} .0005 & .0165 & .0445 \\ & .5445 & 1.4685 \\ & & 3.9605 \end{bmatrix}$	1
Residual	$\begin{bmatrix} 1.7640 & .0200 & -3.0700 \\ & 2.6280 & -.5520 \\ & & 64.9240 \end{bmatrix}$	16
Total (corrected)	$\begin{bmatrix} 4.2655 & -.7855 & -.2395 \\ & 5.0855 & 1.9095 \\ & & 74.2055 \end{bmatrix}$	19

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Example 6.13: Interaction

$$\Lambda^* = \frac{|\text{SSP}_{res}|}{|\text{SSP}_{int} + \text{SSP}_{res}|} = 0.7771$$

$$(g-1)(b-1) = 1$$

$$F = \left(\frac{1-\Lambda^*}{\Lambda^*}\right)\frac{(gb(n-1)-p+1)/2}{(|(g-1)(b-1)-p|+1)/2} : F_{\nu_1,\nu_2}$$

$$\nu_1 = |(g-1)(b-1)-p|+1 = 3$$

$$\nu_2 = gb(n-1)-p+1 = 14$$

$$F = 1.34 < F_{3,14}(0.05) = 3.34$$

$H_0 : \boldsymbol{\gamma}_{11} = \boldsymbol{\gamma}_{12} = \boldsymbol{\gamma}_{21} = \boldsymbol{\gamma}_{22} = 0$ (no interaction) is not rejected

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Example 6.13: Effects of Factors 1 & 2

$$\Lambda_1^* = \frac{|\text{SSP}_{res}|}{|\text{SSP}_{fac1} + \text{SSP}_{res}|} = 0.3819$$

$$\Lambda_2^* = \frac{|\text{SSP}_{res}|}{|\text{SSP}_{fac2} + \text{SSP}_{res}|} = 0.5230$$

$$F_1 = \left(\frac{1-\Lambda_1^*}{\Lambda_1^*}\right)\frac{\nu_2/2}{\nu_1/2} = 7.55, \quad \nu_1 = |(g-1)-p|+1 = 3$$

$$F_2 = \left(\frac{1-\Lambda_2^*}{\Lambda_2^*}\right)\frac{\nu_2/2}{\nu_1/2} = 4.26, \quad \nu_1 = |(b-1)-p|+1 = 3$$

$$\nu_2 = gb(n-1)-p+1 = 14$$

$F_1 > F_{3,14}(0.05) = 3.34$, reject $H_0 : \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = 0$

$F_2 > F_{3,14}(0.05) = 3.34$, reject $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = 0$

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Outline

- Profile Analysis
- ANOVA for Repeated Measures
- Repeated Measures Designs and Growth Curves
- Perspectives and Strategy for Analyzing Multivariate Models

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Questions

- What is the profile analysis?
- How to carry out the profile analysis?

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Profile Analysis

- A battery of p treatments (tests, questions, etc.) are administered to two or more group of subjects
- The question of equality of mean vectors is divided into several specific possibilities
  - Are the profiles parallel?
  - Are the profiles coincident?
  - Are the profiles level?

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Example 6.14: Love and Marriage Data

Extremely negative	Very negative	Moderately negative	Slightly negative	Slightly positive	Moderately positive	Very positive	Extremely positive
1	2	3	4	5	6	7	8

1. All things considered, how would you describe *your contributions* to the marriage?
2. All things considered, how would you describe *your outcomes* from the marriage?

Subjects were also asked to respond to the following questions, using the 5-point scale shown.

3. What is the level of *passionate* love that you feel for your partner?
4. What is the level of *companionate* love that you feel for your partner?

None at all	Very little	Some	A great deal	Tremendous amount
1	2	3	4	5

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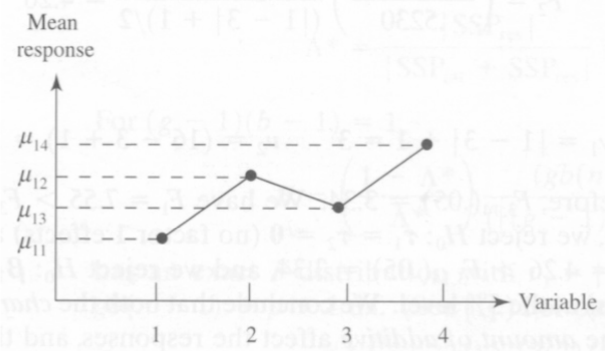
Population Profile



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Profile Analysis

Assume two populations

Are the profiles parallel?

$$H_{01}: \mu_{1i} - \mu_{1i-1} = \mu_{2i} - \mu_{2i-1},\ i = 2, 3, \cdots, p$$

Are the profiles coincident?

$$H_{02}: \mu_{1i} = \mu_{2i},\ i = 1, 2, \cdots, p$$

Are the profiles level?

$$H_{03}: \mu_{11} = \mu_{12} = \cdots = \mu_{1p} = \mu_{21} = \mu_{22} = \cdots = \mu_{2p}$$

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Test for Parallel Profiles

$$\underset{(p-1)\times p}{\mathbf{C}} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$\mathbf{CX}_{1j} : N_{p-1}(\mathbf{C\mu}_1, \mathbf{C\Sigma\Sigma}'),\ \mathbf{CX}_{2j} : N_{p-1}(C\boldsymbol{\mu}_2, \mathbf{C\Sigma\Sigma}')$$

Reject $H_{01}: \mathbf{C\mu}_1 = \mathbf{C\mu}_2$ at level α if

$$T^2 = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'\mathbf{C}'\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{CS}_{pooled}\mathbf{C}'\right]^{-1}\mathbf{C}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) > c^2$$

$$c^2 = \frac{(n_1 + n_2 - 2)(p-1)}{n_1 + n_2 - p} F_{p-1, n_1+n_2-p}(\alpha)$$

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Test for Coincident Profiles

Given parallel profiles

Reject $H_{02}: \mathbf{1}'\boldsymbol{\mu}_1 = \mathbf{1}'\boldsymbol{\mu}_2$ at level α if

$$T^2 = \mathbf{1}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{1}'\mathbf{S}_{pooled}\mathbf{1}\right]^{-1}\mathbf{1}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$$

$$= \left(\frac{\mathbf{1}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)}{\sqrt{\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{1}'\mathbf{S}_{pooled}\mathbf{1}\right]}}\right)^2 > t^2_{n_1+n_2-2}\left(\frac{\alpha}{2}\right) = F_{1, n_1+n_2-2}(\alpha)$$

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Test for Level Profiles

Given coincident profiles

Reject $H_{03}: \mathbf{C}\boldsymbol{\mu} = \mathbf{0}$ at level α if

$$(n_1 + n_2)\bar{\mathbf{x}}'\mathbf{C}'\left[\mathbf{CSC}'\right]^{-1}\mathbf{C}\bar{\mathbf{x}} > c^2$$

$$c^2 = \frac{(n_1 + n_2 - 1)(p-1)}{n_1 + n_2 - p - 1} F_{p-1, n_1+n_2-p+1}(\alpha)$$

$$\bar{\mathbf{x}} = \frac{n_1}{n_1 + n_2}\bar{\mathbf{x}}_1 + \frac{n_2}{n_1 + n_2}\bar{\mathbf{x}}_2$$

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Example 6.14

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} 6.833 \\ 7.033 \\ 3.967 \\ 4.700 \end{bmatrix}, \bar{\mathbf{x}}_2 = \begin{bmatrix} 6.633 \\ 7.000 \\ 4.000 \\ 4.533 \end{bmatrix}$$

$$\mathbf{S}_{pooled} = \begin{bmatrix} 0.606 & 0.262 & 0.066 & 0.161 \\ 0.262 & 0.637 & 0.173 & 0.143 \\ 0.066 & 0.173 & 0.810 & 0.029 \\ 0.161 & 0.143 & 0.029 & 0.306 \end{bmatrix}$$

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Example 6.14: Test for Parallel Profiles

$$\mathbf{CS}_{pooled}\mathbf{C}' = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \mathbf{S}_{pooled} \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0.200 \\ 0.033 \\ -0.033 \\ 0.167 \end{bmatrix} = \begin{bmatrix} -0.167 \\ -0.066 \\ 0.200 \end{bmatrix}$$

$$T^2 = 1.005 < \frac{(30+30-2)(4-1)}{30+30-4} F_{3,56}(0.05) = 8.7$$

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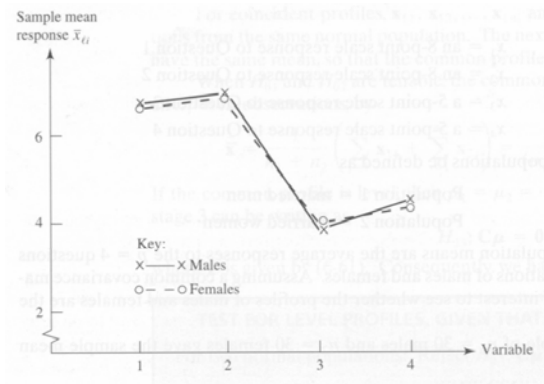
Example 6.14: Sample Profiles



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Example 6.14: Test for Coincident Profiles

$$\mathbf{1}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = 0.367$$

$$\mathbf{1}'\mathbf{S}_{pooled}\mathbf{1} = 4.207$$

$$T^2 = \left(\frac{0.367}{\sqrt{\left(\frac{1}{30} + \frac{1}{30}\right)4.207}} \right)^2 = 0.501 < F_{1,58}(0.05) = 4.0$$

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Outline

- Profile Analysis
- ANOVA for Repeated Measures
- Repeated Measures Designs and Growth Curves
- Perspectives and Strategy for Analyzing Multivariate Models

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Questions

- What are repeated measures?
- How to view the data for repeated measures in a two-way ANOVA view?
- How to test the null hypothesis in repeated measures?

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Repeated-Measures ANOVA

- Drugs A, B, C are tested to see if they are equally effective for pain relief
- Subjects are to take all of the drugs, in turn, suitably blinded and after a suitable washout period
- Subjects rate the degree of pain belief on a 1 to 6 scale (1: no relief, 6 complete relief)

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Avoiding Order Effects

- Randomize the order of treatment
  - 1/3 get drug A first, 1/3 get drug B first, 1/3 get drug C first
- People in a long, natural healing course may grow tolerant of the irritant and learn to tune them out
  - The last medication may work the best
  - Order effects

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Sample Data

Subject	A	B	C	Average
1	5	3	2	3.33
2	5	4	3	4.00
3	5	6	5	5.33
4	6	4	2	4.00
5	6	6	6	6.00
6	4	2	1	2.33
7	4	4	3	3.67
8	4	5	5	4.67
9	4	2	2	2.67
10	5	3	1	3.00
Means	4.80	3.90	3.00	3.90

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*Adapted from: G. R. Norman and D. L. Streiner, *Biostatistics*, 3rd ed.

Two-Way ANOVA View

- Individual subjects as one factor
- Pain reliever as a second factor
- Cells are defined by
  - Subjects: 10 levels
  - Drug: 3 levels
- One observation per cell
- Special case of two-way ANOVA
  - $n = 1, g = 10, b = 3$

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Sum of Squares (Drug)

$$SS(drug) = g\sum_{k=1}^{b}(\bar{x}_{\bullet k} - \bar{x})^2$$

$$SS(drug) = 10[(4.8-3.9)^2 + (3.9-3.9)^2 + (3.0-3.9)^2]$$

$$= 16.2$$

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Sum of Squares (Subjects)

$$SS(subjects) = b\sum_{\ell=1}^{g}(\bar{x}_{\ell\bullet} - \bar{x})^2$$

$$SS(subjects) = 3[(3.33-3.90)^2 + (4.00-3.90)^2 + \cdots + (3.00-3.90)^2] = 36.7$$

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Sum of Squares (Interaction)

$$SS(interaction) = \sum_{\ell=1}^{g}\sum_{k=1}^{b}(\bar{x}_{\ell k} - \bar{x}_{\ell\bullet} - \bar{x}_{\bullet k} + \bar{x})^2$$

$$SS(interaction) = [(5-4.23)^2 + (3-3.33)^2 + \cdots + (1-2.10)^2]$$

[30 terms]

$$= 15.8$$

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Sum of Squares (Within)

$$SS(within) = \sum_{\ell=1}^{g}\sum_{k=1}^{b}(x_{\ell k\gamma} - \bar{x}_{\ell k}) = 0$$

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Degrees of Freedom

$$df(subject) = g - 1 = 10 - 1 = 9$$

$$df(drug) = b - 1 = 3 - 1 = 2$$

$$df(within) = bg(n-1) = 0$$

$$df(interaction) = (b-1)(g-1) = (3-1)(10-1) = 18$$

$$df(total) = bg - 1 = (b-1)(g-1) + b - 1 + g - 1$$

$$= df(interaction) + df(drug) + df(subject)$$

$$= 30 - 1 = 29 = 18 + 2 + 9$$

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Signal vs. Noise

- To determine if there is any significant difference in relief from different pain relievers
  - Main effect of Drug
- $SS(within) = 0$
- Choose $SS(interaction)$ as error term
  - Reflects the extent to which different subjects respond differently to the different drug types

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ANOVA Table

Source	Sum of Squares	*df*	Mean square	F
Drug	16.2	2	8.100	9.225
Subject	36.7	9	4.078	
Drug X Subject	15.8	18	0.878	
Totals	68.7	29		

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Hypothesis Testing

$$F_{Drug} = 9.225 > F_{2,18}(0.05) \approx 3.55$$

Drug effect is significant (i.e., difference exists) at 0.05 significance level

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ANOVA Table for Same Data as a One-Way ANOVA Test

Source	Sum of Squares	*df*	Mean square	F
Drug	16.2	2	8.100	4.107
Error	52.5	27	1.944	
Totals	68.7	29		

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Outline

- Profile Analysis
- ANOVA for Repeated Measured
- Repeated Measures Designs and Growth Curves
- Perspectives and Strategy for Analyzing Multivariate Models

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Questions

- How to compare growth curves?

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Example 6.15: Ulna Data, Control Group

Subject	Initial	1 year	2 year	3 year
1	87.3	86.9	86.7	75.5
2	59.0	60.2	60.0	53.6
3	76.7	76.5	75.7	69.5
4	70.6	76.1	72.1	65.3
5	54.9	55.1	57.2	49.0
6	78.2	75.3	69.1	67.6
7	73.7	70.8	71.8	74.6
8	61.8	68.7	68.2	57.4
9	85.3	84.4	79.2	67.0
10	82.3	86.9	79.4	77.4
11	68.6	65.4	72.3	60.8
12	67.8	69.2	66.3	57.9
13	66.2	67.0	67.0	56.2
14	81.0	82.3	86.8	73.9
15	72.3	74.6	75.3	66.1
Mean	72.38	73.29	72.47	64.79

Source: Data courtesy of Everett Smith.

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Example 6.15: Ulna Data, Treatment Group

Subject	Initial	1 year	2 year	3 year
1	83.8	85.5	86.2	81.2
2	65.3	66.9	67.0	60.6
3	81.2	79.5	84.5	75.2
4	75.4	76.7	74.3	66.7
5	55.3	58.3	59.1	54.2
6	70.3	72.3	70.6	68.6
7	76.5	79.9	80.4	71.6
8	66.0	70.9	70.3	64.1
9	76.7	79.0	76.9	70.3
10	77.2	74.0	77.8	67.9
11	67.3	70.7	68.9	65.9
12	50.3	51.4	53.6	48.0
13	57.7	57.0	57.5	51.5
14	74.3	77.7	72.6	68.0
15	74.0	74.7	74.5	65.7
16	57.3	56.0	64.7	53.0
Mean	69.29	70.66	71.18	64.53

Source: Data courtesy of Everett Smith.

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Comparison of Growth Curves

$\mathbf{X}_{\ell j}$: vector of p measurements on subject j in group ℓ

$j = 1, 2, \cdots, n_\ell; \quad \ell = 1, 2, \cdots, g$

$\mathbf{X}_{\ell j}$: Multivariate normal with covariance $\mathbf{\Sigma}$

Putthoff - Roy model

$$E(\mathbf{X}_{\ell j}) = \begin{bmatrix} \beta_{\ell 0} + \beta_{\ell 1} t_1 + \cdots + \beta_{\ell q} t_1^q \\ \beta_{\ell 0} + \beta_{\ell 1} t_2 + \cdots + \beta_{\ell q} t_2^q \\ \vdots \\ \beta_{\ell 0} + \beta_{\ell 1} t_p + \cdots + \beta_{\ell q} t_p^q \end{bmatrix} = \begin{bmatrix} 1 & t_1 & \cdots & t_1^q \\ 1 & t_2 & \cdots & t_2^q \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_p & \cdots & t_p^q \end{bmatrix} \begin{bmatrix} \beta_{\ell 0} \\ \beta_{\ell 1} \\ \vdots \\ \beta_{\ell q} \end{bmatrix}$$

$$= \mathbf{B}\boldsymbol{\beta}_\ell$$

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Comparison of Growth Curves

Maximum likelihood estimators of $\boldsymbol{\beta}_\ell$:

$$\hat{\boldsymbol{\beta}}_\ell = \left(\mathbf{B}'\mathbf{S}_{pooled}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{S}_{pooled}^{-1}\overline{\mathbf{X}}_\ell$$

$$\mathbf{S}_{pooled} = \frac{1}{N-g}\left((n_1 - 1)\mathbf{S}_1 + \cdots + (n_g - 1)\mathbf{S}_g\right) = \frac{\mathbf{W}}{N-g}$$

$$N = \sum_{\ell=1}^{g} n_\ell, \quad \hat{\text{Cov}}(\hat{\boldsymbol{\beta}}_\ell) = \frac{k}{n_\ell}\left(\mathbf{B}'\mathbf{S}_{pooled}^{-1}\mathbf{B}\right)^{-1}$$

$$k = (N-g)(N-g-1)/(N-g-p+q)(N-g-p+q+1)$$

$$\mathbf{W}_q = \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}\left(\mathbf{X}_{\ell j} - \mathbf{B}\hat{\boldsymbol{\beta}}_\ell\right)\left(\mathbf{X}_{\ell j} - \mathbf{B}\hat{\boldsymbol{\beta}}_\ell\right)', \quad \Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{W}_q|}$$

Reject the null hypothesis that the polynomial is adequate if

$$-\left(N - (p - q + g)/2\right)\ln\Lambda^* > \chi^2_{(p-q-1)g}(\alpha)$$

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Example 6.15

Use quadratic growth model

$$\begin{bmatrix}\hat{\beta}_1 & \hat{\beta}_2\end{bmatrix} = \begin{bmatrix} 73.0701\,(2.58) & 70.1387\,(2.50) \\ 3.6444\,(0.83) & 4.0900\,(0.80) \\ -2.0274\,(0.28) & -1.8534\,(0.27) \end{bmatrix}$$

Control Group: $73.07 + 3.64t - 2.03t^2$

Treatment Group: $70.14 + 4.09t - 1.85t^2$

$\Lambda^* = 0.7627$

$$-\left(N - (p - q + g)/2\right)\ln\Lambda^* = 7.86 < \chi^2_{(4-2-1)2}(0.01) = 9.21$$

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Outline

- Profile Analysis
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- Perspectives and Strategy for Analyzing Multivariate Models

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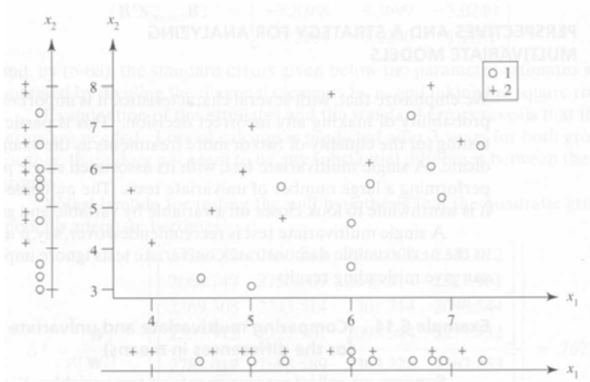
Questions

- What are the strategies in multivariate analysis?
- Why is the experimental design important?

Example 6.16: Comparing Multivariate and Univariate Tests



Example 6.16: Comparing Multivariate and Univariate Tests

$$\text{Univariate test on } x_1: F = 2.46 < F_{1,18}(0.10) = 3.01$$

$$\text{Univariate test on } x_2: F = 2.68 < F_{1,18}(0.10) = 3.01$$

$$\text{Accept } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$$

Hotelling's test:

$$T^2 = 17.29 > c^2 = \frac{18 \times 2}{17} F_{2,17}(0.01) = 12.94$$

$$\text{Reject } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$$

Strategy for Multivariate Comparison of Treatments

- Try to identify outliers
  - Perform calculations with and without the outliers
- Perform a multivariate test of hypothesis
- Calculate the Bonferroni simultaneous confidence intervals
  - For all pairs of groups or treatments, and all characteristics

Importance of Experimental Design

- Differences could appear in only one of the many characteristics or a few treatment combinations
- Differences may become lost among all the inactive ones
- Best preventative is a good experimental design
  - Do not include too many other variables that are not expected to show differences

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