Multivariate Statistical Analysis Mid Term Solution

November 14, 2011

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1. Generalized variance $|\mathbf{S}| = 2$ and total sample variance = 1+5+2=8. (6%)

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2. $\cos\theta = \dfrac{\mathbf{d}_1'\mathbf{d}_2}{L_{\mathbf{d}_1} L_{\mathbf{d}_2}}$, $L_{\mathbf{d}_1} = \sqrt{(n-1)s_{11}}$, $L_{\mathbf{d}_2} = \sqrt{(n-1)s_{22}}$,

$\mathbf{d}_1'\mathbf{d}_2 = \sum_{j=1}^{n}(x_{j1} - \bar{x}_1)(x_{j2} - \bar{x}_2) = (n-1)s_{12}$, $\cos\theta = \dfrac{-4}{\sqrt{13}\sqrt{13}} = \dfrac{-4}{13}$. (6%)

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3. $\lambda_1 = 9$, $\mathbf{S}\mathbf{e}_1 = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 9/\sqrt{2} \\ 9/\sqrt{2} \\ 0 \end{bmatrix} = 9\mathbf{e}_1$,

$$\lambda_2 = 9, \mathbf{S}\mathbf{e}_2 = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix} \begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 9/3\sqrt{2} \\ -9/3\sqrt{2} \\ -36/3\sqrt{2} \end{bmatrix} = 9\mathbf{e}_2,$$

$$\lambda_3 = 18, \ \mathbf{S}\mathbf{e}_3 = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 36/3 \\ -36/3 \\ 18/3 \end{bmatrix} = 18\mathbf{e}_3. \quad (6\%)$$

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4. Eq.(4 − 7), p.153, Text book : major axes and directions $= c\sqrt{\lambda_i}\mathbf{e}_i$, where $\mathbf{e}_i$ is a normalized eigenvector.

$$\therefore \text{major axes and directions} = c3\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \ c3\begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}, \quad c3\sqrt{2}\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}. \quad (6\%)$$

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5. From Result 4.7 and Table 3 in Appendix of the Textbook, $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) = \chi_3^2(0.05) \approx 7.81$, $c = \sqrt{7.81}$,

$$\therefore \text{major axes and directions} = 3\sqrt{7.81}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \ 3\sqrt{7.81}\begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}, \quad 3\sqrt{2}\sqrt{7.81}\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

(6%)

Based on Eq. (5-6) or (5-7),

$$T^2 = n(\bar{\mathbf{x}}-\boldsymbol{\mu}_0)'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}_0) = 43[0.5 \quad -0.5 \quad -1.5]\frac{1}{162}\begin{bmatrix} 14 & 4 & -2 \\ 4 & 14 & 2 \\ -2 & 2 & 17 \end{bmatrix}\begin{bmatrix} 0.5 \\ -0.5 \\ -1.5 \end{bmatrix}$$

6. $$= \frac{43}{324}[0.5 \quad -0.5 \quad -1.5]\begin{bmatrix} 16 \\ -16 \\ -55 \end{bmatrix} = \frac{197\times 43}{648} \approx 13.073$$

$$\text{Critical value} = \frac{(n-1)p}{n-p}F_{p,n-p}(0.05) = \frac{42\times 3}{40}F_{3,40}(0.05) \approx \frac{126}{40}\times 2.84 \approx 8.946$$

$T^2 >$ critical value. $\therefore$ reject H_0

(6%)

From Eq. (5-18),

$$n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \le \frac{p(n-1)}{n-p}F_{p,n-p}(0.05) = 8.946$$

7. $$(\bar{\mathbf{x}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \le \frac{8.946}{43} \approx 0.208$$. (6%)

By Eq. (5-19), major axes and directions $= \sqrt{\lambda_i}\sqrt{\dfrac{p(n-1)}{n(n-p)}F_{p,n-p}(0.05)}\mathbf{e}_i$

namely, $3\sqrt{0.208}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, 3\sqrt{0.208}\begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}, 3\sqrt{2}\sqrt{0.208}\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

Use Eq. (5.24),

$$\bar{x}_1-\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{11}}{n}}\le\mu_1\le\bar{x}_1+\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{11}}{n}}$$

$$\bar{x}_2-\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{22}}{n}}\le\mu_2\le\bar{x}_2+\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{22}}{n}}$$

$$\bar{x}_3-\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{33}}{n}}\le\mu_3\le\bar{x}_3+\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{s_{33}}{n}}$$

$$\sqrt{\frac{p(n-1)}{n-p}F_{p,n-p}(0.05)}\approx\sqrt{8.946}\approx 2.991$$

$$0.5-\sqrt{8.946}\sqrt{\frac{13}{43}}\le\mu_1\le 0.5+\sqrt{8.946}\sqrt{\frac{13}{43}},\quad \mu_1\in(-1.14,2.14)$$

$$0.5-\sqrt{8.946}\sqrt{\frac{13}{43}}\le\mu_2\le 0.5+\sqrt{8.496}\sqrt{\frac{13}{43}},\quad \mu_2\in(-1.14,2.14)$$

8. $$-0.5-\sqrt{8.946}\sqrt{\frac{10}{43}}\le\mu_3\le -0.5+\sqrt{8.946}\sqrt{\frac{10}{43}},\quad \mu_3\in(-1.94,0.94)$$. (6%)

From Eq. (5-29),

$$\bar{x}_1-t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{11}}{n}}\le\mu_1\le\bar{x}_1+t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{11}}{n}}$$

$$\bar{x}_2-t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{22}}{n}}\le\mu_2\le\bar{x}_2+t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{22}}{n}}$$

$$\bar{x}_3-t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{33}}{n}}\le\mu_3\le\bar{x}_3+t_{n-1}(\frac{0.05}{2p})\sqrt{\frac{s_{33}}{n}}$$

$$p=3,\quad 0.05/(2p)\approx 0.00833,\quad t_{40}(0.00833)\approx 2.499,\quad t_{60}(0.00833)\approx 2.463$$

$$t_{42}(0.00833)\approx 2.499+((2.463-2.499)/(60-40))*(42-40)=2.495$$

$$0.5-2.495\sqrt{\frac{13}{43}}\le\mu_1\le 0.5+2.495\sqrt{\frac{13}{43}},\quad \mu_1\in(-0.87,1.87)$$

$$0.5-2.495\sqrt{\frac{13}{43}}\le\mu_2\le 0.5+2.495\sqrt{\frac{13}{43}},\quad \mu_2\in(-0.87,1.87)$$

9. $$-0.5-2.495\sqrt{\frac{10}{43}}\le\mu_3\le -0.5+2.495\sqrt{\frac{10}{43}},\quad \mu_3\in(-1.70,0.70)$$ (6%)

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10. From the equation in pp. 249, LCL $=0$, UCL $=\frac{(n-1)p}{n-p}F_{p,n-p}(0.05)$ (6%)

$$\therefore \text{UCL}=\frac{42\times 3}{40}F_{3,40}(0.05)=\frac{126}{40}\times 2.84\approx 8.946$$

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11. Result 4.2: Given that a random vector $\mathbf{X}$ follows $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the probability distribution of $\mathbf{a}'\mathbf{X}$ will be $N(\mathbf{a}'\boldsymbol{\mu}, \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a})$. Since with $\boldsymbol{\mu}' = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix}$, $\mathbf{a}' = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$, $\mathbf{a}'\boldsymbol{\mu} = 3$, $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} = 58$. Thus, $\mathbf{a}'\mathbf{X}$ is $N(3,58)$. (6%)

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12. (a) X_1 and X_2 are dependent, because $\sigma_{12} \neq 0$ and Result 4.5 (3%)

    (b) (X_1, X_2) and X_3 are independent, because $(\sigma_{13}, \sigma_{23}) = (0,0)$ and Result 4.5 (3%)

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13. The log-likelihood function is $l(\theta) = \sum_{k=1}^{n} \ln P(x_k) = n \ln \theta - \theta \sum_{k=1}^{n} x_k$

$$\frac{dl(\theta)}{d\theta} = \frac{n}{\theta} - \sum_{k=1}^{n} x_k = 0, \quad \hat{\theta} = \frac{n}{\sum_{k=1}^{n} x_k} = \frac{1}{\bar{x}}. \qquad (6\%)$$

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14. $\Lambda = \dfrac{L(\theta_0)}{L(\hat{\theta})} = \dfrac{\theta_0^n e^{-\theta_0 (x_1 + \cdots + x_n)}}{\hat{\theta}^n e^{-n}} = \dfrac{\theta_0^n e^{-n\theta_0/\hat{\theta}}}{\hat{\theta}^n e^{-n}} = \left(\dfrac{\theta_0}{\hat{\theta}} \right)^n e^{-n\left(\theta_0/\hat{\theta} - 1\right)}$. (6%)

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15. Initial estimate $\tilde{\mu}_1 = \dfrac{2+4}{2} = 3$, $\tilde{\mu}_2 = \dfrac{6+4+8}{3} = 6$,

$$\tilde{\sigma}_{11} = \frac{(2-3)^2 + (4-3)^2 + (3-3)^2}{3} = \frac{2}{3}, \quad \tilde{\sigma}_{22} = \frac{(6-6)^2 + (4-6)^2 + (8-6)^2}{3} = \frac{8}{3},$$

$$\tilde{\sigma}_{12} = \frac{(2-3)(6-6) + (4-3)(4-6) + (3-3)(8-6)}{3} = -\frac{2}{3}.$$

Partition $\hat{\boldsymbol{\mu}} = \begin{bmatrix} \mu_1 \\ -- \\ \mu_2 \end{bmatrix}$, $\tilde{\boldsymbol{\Sigma}} = \begin{bmatrix} \tilde{\sigma}_{11} & | & \tilde{\sigma}_{12} \\ -- & + & -- \\ \tilde{\sigma}_{12} & | & \tilde{\sigma}_{22} \end{bmatrix}$, and predict

$$\tilde{x}_{31} = \tilde{\mu}_1 + \tilde{\sigma}_{12} \tilde{\sigma}_{22}^{-1} (x_{32} - \tilde{\mu}_2) = 3 + (-\frac{2}{3}) \frac{3}{8} (8-6) = 3 - \frac{1}{2} = \frac{5}{2}$$

$$\tilde{x^2_{31}} = \tilde{\sigma}_{11} - \tilde{\sigma}_{12} \tilde{\sigma}_{22}^{-1} \bar{\sigma}_{12} + \tilde{x}_{31}^2 = \frac{2}{3} - \left(-\frac{2}{3} \right) \frac{3}{8} \left(-\frac{2}{3} \right) + \left(\frac{5}{2} \right)^2 = \frac{27}{4}$$

$$x_{31}\tilde{x}_{32} = \tilde{x}_{31}x_{32} = \frac{5}{2}\times 8 = 20$$

$$\tilde{\mathbf{T}}_1 = \begin{bmatrix} x_{11}+x_{21}+\tilde{x}_{31} \\ x_{12}+x_{22}+x_{32} \end{bmatrix} = \begin{bmatrix} 6+\frac{5}{2} \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{17}{2} \\ 18 \end{bmatrix}$$

$$\tilde{\mathbf{T}}_2 = \begin{bmatrix} x_{11}^2+x_{21}^2+\tilde{x}_{31}^2 & x_{11}x_{12}+x_{21}x_{22}+x_{31}\tilde{x}_{32} \\ x_{11}x_{12}+x_{21}x_{22}+x_{31}\tilde{x}_{32} & x_{12}^2+x_{22}^2+x_{32}^2 \end{bmatrix} = \begin{bmatrix} \frac{107}{4} & 48 \\ 48 & 116 \end{bmatrix}$$

$$\tilde{\boldsymbol{\mu}} = \frac{\tilde{\mathbf{T}}_1}{n} = \begin{bmatrix} \frac{17}{6} \\ 6 \end{bmatrix},$$

$$\tilde{\boldsymbol{\Sigma}} = \frac{\tilde{\mathbf{T}}_2}{n} - \tilde{\boldsymbol{\mu}}\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \frac{107}{12} & 16 \\ 16 & \frac{116}{3} \end{bmatrix} - \begin{bmatrix} \frac{17}{6} \\ 6 \end{bmatrix}\begin{bmatrix} \frac{17}{6} & 6 \end{bmatrix} = \begin{bmatrix} \frac{8}{9} & -1 \\ -1 & \frac{8}{3} \end{bmatrix}$$

. (8%)

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16. $$\Pr\left\{ \chi_{n-1}^2(\frac{\alpha}{2}) < \frac{(n-1)s^2}{\sigma^2} < \chi_{n-1}^2(1-\frac{\alpha}{2}) \right\} = 1-\alpha$$

$$\Pr\left\{ \frac{1}{\chi_{n-1}^2(1-\frac{\alpha}{2})} < \frac{\sigma^2}{(n-1)s^2} < \frac{1}{\chi_{n-1}^2(\frac{\alpha}{2})} \right\} = 1-\alpha$$

$$\Pr\left\{ \frac{(n-1)s^2}{\chi_{n-1}^2(1-\frac{\alpha}{2})} < \sigma^2 < \frac{(n-1)s^2}{\chi_{n-1}^2(\frac{\alpha}{2})} \right\} = 1-\alpha$$

$\therefore$ confidence interval for the population variance $\sigma^2 = \left(\frac{(n-1)s^2}{\chi_{n-1}^2(1-\frac{\alpha}{2})}, \frac{(n-1)s^2}{\chi_{n-1}^2(\frac{\alpha}{2})} \right)$

(8%)