Multivariate Linear Regression Models

Shyh-Kang Jeng
Department of Electrical Engineering/
Graduate Institute of Communication/
Graduate Institute of Networking and Multimedia

1

Outline

- Introduction
- The Classical Linear Regression Model
- Least Square Estimation
- Inference about the Regression Model
- Inference from the Estimated Regression Function

2

Outline

- Model Checking and Other Aspects of Regression
- Multivariate Multiple Regression
- The Concept of Linear Regression
- Comparing the Two Formulations of the Regression Model
- Multiple Regression Models with Time Dependent Errors

3

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4

Questions

- What is regression analysis?
-

5

Regression Analysis

- A statistical methodology
  - For predicting value of one or more response (dependent) variables
  - Predict from a collection of predictor (independent) variable values

6

Example 7.1 Fitting a Straight Line

- Observed data

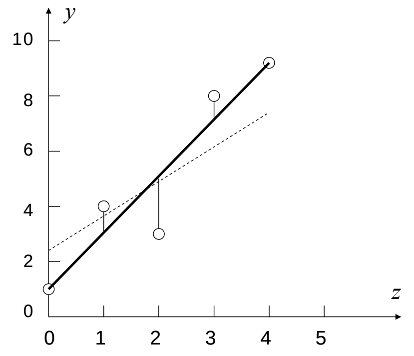
z_1	0	1	2	3	4
y	1	4	3	8	9

- Linear regression model

$$\text{Mean response} = E(Y) = \beta_0 + \beta_1 z_1$$

7

Example 7.1 Fitting a Straight Line



8

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9

Questions

- What is the classical regression model?
- How to treat a one-way ANOVA problem as the classical regression model?

10

Classical Linear Regression Model

$$Y = \beta_0 + \beta_1 z_1 + \cdots + \beta_r z_r + \varepsilon$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & z_{11} & z_{12} & \cdots & z_{1r} \\ 1 & z_{21} & z_{22} & \cdots & z_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_{n1} & z_{n2} & \cdots & z_{nr} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad z_{j0} = 1$$

11

Classical Linear Regression Model

$$E(\varepsilon_j) = 0$$

$$\mathrm{Var}(\varepsilon_j) = \sigma^2$$

$$\mathrm{Cov}(\varepsilon_j, \varepsilon_k) = 0, \quad j \neq k$$

$$\Rightarrow$$

$$E(\boldsymbol{\varepsilon}) = 0$$

$$\mathrm{Cov}(\boldsymbol{\varepsilon}) = E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \sigma^2 \mathbf{I}$$

12

Example 7.1

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_5 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & z_{11} \\ 1 & z_{21} \\ \vdots & \vdots \\ 1 & z_{51} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_5 \end{bmatrix}$$

$$\mathbf{y}' = \begin{bmatrix} 1 & 4 & 3 & 8 & 9 \end{bmatrix}$$

$$\mathbf{Z}' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

13

Examples 6.7 & 6.8

$$\begin{pmatrix} 9 & 6 & 9 \\ 0 & 2 & \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & \\ 4 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 4 \\ -3 & -3 & \\ -2 & -2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & \\ 1 & -1 & 0 \end{pmatrix}$$

$\mathrm{SS}_{obs} = 216, \ \mathrm{SS}_{mean} = 128$

$\mathrm{SS}_{tr} = 78, \ \text{d.f.} = 3 - 1 = 2$

$\mathrm{SS}_{res} = 10, \text{d.f.} = (3+2+3) - 3 = 5$

$$F = \frac{\mathrm{SS}_{tr} / (g-1)}{\mathrm{SS}_{res} / \left(\sum n_\ell - g\right)} = \frac{78/2}{10/5} = 19.5 > F_{2,5}(0.01) = 13.27$$

$H_0 : \tau_1 = \tau_2 = \tau_3 = 0$ is rejected at the 1% level

14

Example 7.2 One-Way ANOVA

$$X_{1j} = \mu + \tau_1 + e_{1j}, X_{2j} = \mu + \tau_2 + e_{2j}, X_{3j} = \mu + \tau_3 + e_{3j}$$

$$Y_j = \beta_0 + \beta_1 z_{j1} + \beta_2 z_{j2} + \beta_3 z_{j3} + \varepsilon_j$$

$$\beta_0 = \mu, \quad \beta_1 = \tau_1, \quad \beta_2 = \tau_2, \quad \beta_3 = \tau_3$$

$$z_j = \begin{cases} 1 & \text{if the observation is from population } j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{Y}' = \begin{bmatrix} 9 & 6 & 9 & 0 & 2 & 3 & 1 & 2 \end{bmatrix}$$

$$\mathbf{Z}' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

15

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16

Questions

- What is the method of least squares?
- What is the least square estimation about the assumed coefficients in the classical regression model? (Result 7.1)
- What is the coefficient of determination?
- How to explain the results of the least square estimation through geometry?

17

Questions

- What is the projection matrix?
- What are the expectation of the estimated coefficients and the residual? What are the covariance matrix and the variance of the residual? (Result 7.2)
- What is the Gauss least square theorem? (Result 7.3)

18

Method of Least Squares

Selects $\mathbf{b}$ so as to minimize

$$S(\mathbf{b}) = \sum_{j=1}^{n} \left(y_j - b_0 - b_1 z_{j1} - \cdots - b_r z_{jr}\right)^2$$

$$= (\mathbf{y} - \mathbf{Z}\mathbf{b})'(\mathbf{y} - \mathbf{Z}\mathbf{b})$$

$$\hat{\boldsymbol{\beta}} = \arg\min_{\mathbf{b}} S(\mathbf{b})$$

$$\text{residuals} = \hat{\varepsilon}_j = y_j - \hat{\beta}_0 - \hat{\beta}_1 z_{j1} - \cdots - \hat{\beta}_r z_{jr}$$

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}} = \mathbf{y} - \hat{\mathbf{y}}, \quad \text{fitted } \mathbf{y} = \hat{\mathbf{y}} = \mathbf{Z}\hat{\boldsymbol{\beta}}$$

19

Result 7.1

$$\mathbf{Z} \text{ has full rank } r+1 \le n, \quad \hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$$

$$\hat{\mathbf{y}} = \mathbf{Z}\hat{\boldsymbol{\beta}} = \mathbf{H}\mathbf{y}, \quad \mathbf{H} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$$

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{y} - \hat{\mathbf{y}} = (\mathbf{I} - \mathbf{H})\mathbf{y}$$

$$\mathbf{Z}'\hat{\boldsymbol{\varepsilon}} = 0, \quad \hat{\mathbf{y}}'\hat{\boldsymbol{\varepsilon}} = 0, \quad \mathbf{Z}'\mathbf{y} = \mathbf{Z}'\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \mathbf{y}'(\mathbf{y} - \hat{\mathbf{y}}) = \mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y} = \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{Z}\hat{\boldsymbol{\beta}}$$

20

Proof of Result 7.1

$$\hat{\boldsymbol\beta} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$$

$$\mathbf{y} - \mathbf{Z}\mathbf{b} = \mathbf{y} - \mathbf{Z}\hat{\boldsymbol\beta} + \mathbf{Z}\hat{\boldsymbol\beta} - \mathbf{Z}\mathbf{b} = \mathbf{y} - \mathbf{Z}\hat{\boldsymbol\beta} + \mathbf{Z}(\hat{\boldsymbol\beta} - \mathbf{b})$$

$$\begin{aligned} S(\mathbf{b}) &= (\mathbf{y} - \mathbf{Z}\mathbf{b})'(\mathbf{y} - \mathbf{Z}\mathbf{b}) \\ &= (\mathbf{y} - \mathbf{Z}\hat{\boldsymbol\beta})'(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol\beta}) + (\hat{\boldsymbol\beta} - \mathbf{b})'\mathbf{Z}'\mathbf{Z}(\hat{\boldsymbol\beta} - \mathbf{b}) \\ &\quad + 2(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol\beta})'\mathbf{Z}(\hat{\boldsymbol\beta} - \mathbf{b}) \end{aligned}$$

$$\begin{aligned} (\mathbf{y} - \mathbf{Z}\hat{\boldsymbol\beta})'\mathbf{Z} &= \mathbf{y}'(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')'\mathbf{Z} \\ &= \mathbf{y}'(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\mathbf{Z} = \mathbf{y}'(\mathbf{Z} - \mathbf{Z}) = 0 \end{aligned}$$

$$\hat{\boldsymbol\beta} = \arg\min_{\mathbf{b}} S(\mathbf{b})$$

Proof of Result 7.1

$$\begin{aligned} \mathbf{Z}'\hat{\boldsymbol\varepsilon} &= \mathbf{Z}'(\mathbf{y} - \hat{\mathbf{y}}) = \mathbf{Z}'(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol\beta}) \\ &= \mathbf{Z}'(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\mathbf{y} = 0 \end{aligned}$$

$$\hat{\mathbf{y}}'\hat{\boldsymbol\varepsilon} = \hat{\boldsymbol\beta}'\mathbf{Z}'\hat{\boldsymbol\varepsilon} = 0$$

$$\begin{aligned} \hat{\boldsymbol\varepsilon}'\hat{\boldsymbol\varepsilon} &= (\mathbf{y} - \hat{\mathbf{y}})'(\mathbf{y} - \hat{\mathbf{y}}) \\ &= \mathbf{y}'(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\mathbf{y} \\ &= \mathbf{y}'(\mathbf{I} - 2\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\mathbf{y} \\ &= \mathbf{y}'(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\mathbf{y} = \mathbf{y}'\mathbf{y} - \mathbf{y}'\hat{\mathbf{y}} \end{aligned}$$

Example 7.1 Fitting a Straight Line

- Observed data

z_1	0	1	2	3	4
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- Linear regression model

$$\text{Mean response} = E(Y) = \beta_0 + \beta_1 z_1$$

Example 7.3

$$\mathbf{Z}'\mathbf{Z} = \begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix}, \quad (\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.1 \end{bmatrix}$$

$$\mathbf{Z}'\mathbf{y} = \begin{bmatrix} 25 \\ 70 \end{bmatrix}, \quad \hat{\boldsymbol\beta} = \begin{bmatrix} \hat\beta_0 \\ \hat\beta_1 \end{bmatrix} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{y} = 1 + 2z, \quad \hat{\mathbf{y}} = \mathbf{Z}\hat{\boldsymbol\beta} = [1 \quad 3 \quad 5 \quad 7 \quad 9]'$$

$$\hat{\boldsymbol\varepsilon} = \mathbf{y} - \hat{\mathbf{y}} = [0 \quad 1 \quad -2 \quad 1 \quad 0]'$$

$$\text{residual sum of equations } \hat{\boldsymbol\varepsilon}'\hat{\boldsymbol\varepsilon} = 6$$

Coefficient of Determination

$\hat{\mathbf{y}}'\hat{\boldsymbol{\varepsilon}} = 0$

$\mathbf{y}'\mathbf{y} = (\hat{\mathbf{y}} + \mathbf{y} - \hat{\mathbf{y}})'(\hat{\mathbf{y}} + \mathbf{y} - \hat{\mathbf{y}}) = (\hat{\mathbf{y}} + \hat{\boldsymbol{\varepsilon}})'(\hat{\mathbf{y}} + \hat{\boldsymbol{\varepsilon}}) = \hat{\mathbf{y}}'\hat{\mathbf{y}} + \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}$

$\mathbf{Z}'\hat{\boldsymbol{\varepsilon}} = 0 \Rightarrow 0 = \mathbf{1}'\hat{\boldsymbol{\varepsilon}} = \sum_{j=1}^{n} \hat{\varepsilon}_j = \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} \hat{y}_j \Rightarrow \bar{y} = \bar{\hat{y}}$

$\mathbf{y}'\mathbf{y} - n\bar{y}^2 = \hat{\mathbf{y}}'\hat{\mathbf{y}} - n\left(\bar{\hat{y}}\right)^2 + \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}$

$\sum_{j=1}^{n} \left(y_j - \bar{y}\right)^2 = \sum_{j=1}^{n} \left(\hat{y}_j - \bar{y}\right)^2 + \sum_{j=1}^{n} \hat{\varepsilon}_j^2$

$$R^2 = 1 - \frac{\sum_{j=1}^{n} \hat{\varepsilon}_j^2}{\sum_{j=1}^{n} \left(y_j - \bar{y}\right)^2} = \frac{\sum_{j=1}^{n} \left(\hat{y}_j - \bar{y}\right)^2}{\sum_{j=1}^{n} \left(y_j - \bar{y}\right)^2}$$

25

Geometry of Least Squares

$$E(\mathbf{Y}) = \mathbf{Z}\boldsymbol{\beta} = \beta_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} z_{11} \\ z_{21} \\ \vdots \\ z_{n1} \end{bmatrix} + \cdots + \beta_r \begin{bmatrix} z_{1r} \\ z_{2r} \\ \vdots \\ z_{nr} \end{bmatrix}$$

$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$

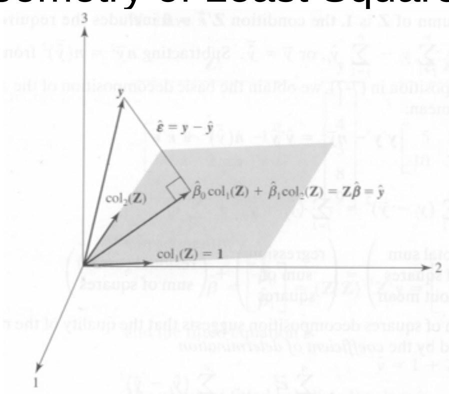
$\mathbf{y} - \mathbf{Zb} =$ (observed vector) - (vector in model plane)

$S(\mathbf{b}) = (\mathbf{y} - \mathbf{Zb})'(\mathbf{y} - \mathbf{Zb})$

$\hat{\boldsymbol{\beta}} = \arg\min_{\mathbf{b}} S(\mathbf{b})$, $\hat{\mathbf{y}} = \mathbf{Z}\hat{\boldsymbol{\beta}}$ on model plane, $\hat{\boldsymbol{\varepsilon}} \perp$ model plane

26

Geometry of Least Squares



27

Projection Matrix

$\mathbf{Z}'\mathbf{Z} = \lambda_1 \mathbf{e}_1 \mathbf{e}_1^{'} + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^{'} + \cdots + \lambda_{r+1} \mathbf{e}_r \mathbf{e}_r^{'}$

$(\mathbf{Z}'\mathbf{Z})^{-1} = \frac{1}{\lambda_1} \mathbf{e}_1 \mathbf{e}_1^{'} + \frac{1}{\lambda_2} \mathbf{e}_2 \mathbf{e}_2^{'} + \cdots + \frac{1}{\lambda_{r+1}} \mathbf{e}_r \mathbf{e}_r^{'}$

$\mathbf{q}_i = \lambda_i^{-1/2} \mathbf{Z}\mathbf{e}_i \Rightarrow \mathbf{q}_i^{'}\mathbf{q}_k = \lambda_i^{-1/2} \lambda_k^{-1/2} \mathbf{e}_i^{'} \mathbf{Z}'\mathbf{Z}\mathbf{e}_k = \lambda_i^{-1/2} \lambda_k^{-1/2} \mathbf{e}_i^{'} \lambda_k \mathbf{e}_k = \delta_{ik}$

$\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' = \sum_{i=1}^{r+1} \lambda_i^{-1} \mathbf{Z}\mathbf{e}_i \mathbf{e}_i^{'} \mathbf{Z}' = \sum_{i=1}^{r+1} \mathbf{q}_i \mathbf{q}_i^{'}$

projection of $\mathbf{y}$ on the model plane constructed by $\{\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{r+1}\}$ is

$$\sum_{i=1}^{r+1} \mathbf{q}_i \left(\mathbf{q}_i^{'} \mathbf{y}\right) = \left(\sum_{i=1}^{r+1} \mathbf{q}_i \mathbf{q}_i^{'}\right) \mathbf{y} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y} = \mathbf{Z}\hat{\boldsymbol{\beta}}$$

28

Result 7.2

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$$
$$\Rightarrow E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}, \quad \mathrm{Cov}(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1}$$
$$\hat{\boldsymbol{\varepsilon}} = \mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$$
$$\Rightarrow E(\hat{\boldsymbol{\varepsilon}}) = 0, \quad \mathrm{Cov}(\hat{\boldsymbol{\varepsilon}}) = \sigma^2\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right] = \sigma^2[\mathbf{I} - \mathbf{H}]$$
$$E(\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}) = (n - r - 1)\sigma^2$$
$$s^2 = \frac{\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}}{n-(r+1)} = \frac{\mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y}}{n-r-1}, \quad E(s^2) = \sigma^2$$

$\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\varepsilon}}$ are uncorrelated

29

Proof of Result 7.2*

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$
$$\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) = \boldsymbol{\beta} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\varepsilon}$$
$$\hat{\boldsymbol{\varepsilon}} = \left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)\mathbf{Y} = \left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)(\mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon})$$
$$= \left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)\boldsymbol{\varepsilon}$$
$$E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'E(\boldsymbol{\varepsilon}) = \boldsymbol{\beta}$$
$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathrm{Cov}(\boldsymbol{\varepsilon})\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1} = \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1}$$
$$E(\hat{\boldsymbol{\varepsilon}}) = \left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)E(\boldsymbol{\varepsilon}) = 0$$
$$\mathrm{Cov}(\hat{\boldsymbol{\varepsilon}}) = \left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)\mathrm{Cov}(\boldsymbol{\varepsilon})\left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)$$
$$= \sigma^2\left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)$$

30

Proof of Result 7.2*

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\varepsilon}}) = E\left[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\hat{\boldsymbol{\varepsilon}}'\right] = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]$$
$$= \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right] = 0$$
$$\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\varepsilon}}' = \boldsymbol{\varepsilon}'\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\boldsymbol{\varepsilon}$$
$$= \boldsymbol{\varepsilon}'\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\boldsymbol{\varepsilon} = \mathrm{tr}\left[\boldsymbol{\varepsilon}'\left(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)\boldsymbol{\varepsilon}\right]$$
$$= \mathrm{tr}\left(\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\right)$$
$$E(\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\varepsilon}}') = \mathrm{tr}\left(\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\right)$$
$$= \sigma^2\,\mathrm{tr}\left(\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\right) = \sigma^2\,\mathrm{tr}(\mathbf{I}) - \sigma^2 tr\left((\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Z}\right)$$
$$= \sigma^2(n - r - 1)$$

31

Result 7.3
Gauss Least Square Theorem

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = 0, \quad \mathrm{Cov}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$$

$\mathbf{c}'\hat{\boldsymbol{\beta}} = c_0\hat{\beta}_0 + c_1\hat{\beta}_1 + \cdots + c_r\hat{\beta}_r$ as an estimator of $\mathbf{c}'\boldsymbol{\beta}$ has the smallest possible variance among all estimator of the form

$$\mathbf{a}'\mathbf{Y} = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$$

that are unbiased for $\mathbf{c}'\boldsymbol{\beta}$

32

Proof of Result 7.3

For $\mathbf{a}'\mathbf{Y}$ as an unbiased estimator of $\mathbf{c}'\boldsymbol{\beta}$,

$$E(\mathbf{a}'\mathbf{Y}) = E(\mathbf{a}'\mathbf{Z}\boldsymbol{\beta} + \mathbf{a}'\boldsymbol{\varepsilon}) = \mathbf{a}'\mathbf{Z}\boldsymbol{\beta} = \mathbf{c}'\boldsymbol{\beta} \Rightarrow \mathbf{a}'\mathbf{Z} = \mathbf{c}'$$

$$E(\mathbf{c}'\hat{\boldsymbol{\beta}}) = \mathbf{c}'\boldsymbol{\beta}$$

$$\mathbf{c}'\hat{\boldsymbol{\beta}} = \mathbf{c}'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} = \mathbf{a}^{*\prime}\mathbf{Y}, \quad \mathbf{a}^{*\prime}\mathbf{Z} = \mathbf{c}'$$

$$\mathrm{Var}(\mathbf{a}'\mathbf{Y}) = \mathrm{Var}(\mathbf{a}'\mathbf{Z}\boldsymbol{\beta} + \mathbf{a}'\boldsymbol{\varepsilon}) = \mathrm{Var}(\mathbf{a}'\boldsymbol{\varepsilon}) = \mathbf{a}'\mathbf{I}\sigma^2\mathbf{a}$$

$$= \sigma^2(\mathbf{a} - \mathbf{a}^* + \mathbf{a}^*)'(\mathbf{a} - \mathbf{a}^* + \mathbf{a}^*)$$

$$= \sigma^2\left[(\mathbf{a} - \mathbf{a}^*)'(\mathbf{a} - \mathbf{a}^*) + \mathbf{a}^{*\prime}\mathbf{a}^*\right]$$

is minimum when $\mathbf{a}'\mathbf{Y} = \mathbf{a}^{*\prime}\mathbf{Y} = \mathbf{c}'\hat{\boldsymbol{\beta}}$ (BLUE)

33

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34

Questions

- What are the maximum likelihood estimator to the coefficients and the assumed variance? (Result 7.4)
- What are the confidence region and the simultaneous confidence intervals for the assumed coefficients? (Result 7.5)
- How to know that the number of the coefficients has been enough? (Result 7.6)

35

Questions

- How to modify Result 7.6 when the rank of the $\mathbf{Z}$ matrix is not full?
- How to generalize Result 7.6 to the case that the coefficient vector is multiplied by a matrix?

36

Result 7.4

$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} : N_n(\mathbf{0}, \sigma^2\mathbf{I})$

maximul likelihood estimator of $\boldsymbol{\beta}$ is the same as the least squares estimator $\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$

$\hat{\boldsymbol{\beta}} : N_{r+1}\left(\boldsymbol{\beta}, \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1}\right)$

independent of $\hat{\boldsymbol{\varepsilon}} = \mathbf{Y} - \mathbf{Z}\boldsymbol{\beta}$

$\hat{\sigma}^2$: maximum likelihood estimator of σ^2

$n\hat{\sigma}^2 = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} : \sigma^2\chi^2_{n-r-1}$

37

Proof of Result 7.4*

$$L(\boldsymbol{\beta}, \sigma^2) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\varepsilon_j^2/2\sigma^2} = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}/2\sigma^2}$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-(\mathbf{y}-\mathbf{Z}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{Z}\boldsymbol{\beta})/2\sigma^2}$$

For fixed σ^2,

$$\arg\max_{\boldsymbol{\beta}} L(\boldsymbol{\beta}, \sigma^2) = \arg\min_{\boldsymbol{\beta}} (\mathbf{y} - \mathbf{Z}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{Z}\boldsymbol{\beta})$$

$$= \hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}, \text{ independent of } \sigma$$

38

Proof of Result 7.4*

$$\hat{\sigma}^2 = \arg\max_{\sigma^2} L(\hat{\boldsymbol{\beta}}, \sigma^2)$$

$$= \frac{(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})}{n}$$

$$= \frac{\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}}{n}$$

39

Proof of Result 4.11*

Exponent of $L(\boldsymbol{\mu}, \boldsymbol{\Sigma})$:

$$-\frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'\right)\right] - \frac{1}{2}n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu})$$

$$\Rightarrow \hat{\boldsymbol{\mu}} = \bar{\mathbf{x}}$$

$$L(\hat{\boldsymbol{\mu}}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{np/2}|\boldsymbol{\Sigma}|^{n/2}} e^{-\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'\right)\right]}$$

$$\Rightarrow \hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' = \frac{n-1}{n}\mathbf{S}$$

40

Proof of Result 7.4*

$$\begin{bmatrix} \hat{\boldsymbol{\beta}} \\ -- \\ \hat{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta} \\ -- \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\ -------- \\ \mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \end{bmatrix} \boldsymbol{\varepsilon} = \boldsymbol{\alpha} + \mathbf{A}\boldsymbol{\varepsilon}$$

$$\mathrm{Cov}\left(\begin{bmatrix} \hat{\boldsymbol{\beta}} \\ -- \\ \hat{\boldsymbol{\varepsilon}} \end{bmatrix}\right) = \mathbf{A}\,\mathrm{Cov}(\boldsymbol{\varepsilon})\mathbf{A}'$$

$$= \sigma^2 \begin{bmatrix} (\mathbf{Z}'\mathbf{Z})^{-1} & | & \mathbf{0} \\ ---- & + & -------- \\ \mathbf{0}' & | & \mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \end{bmatrix}$$

41

Proof of Result 7.4*

$$\left(\mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)\mathbf{e} = \lambda\mathbf{e}$$

$$\left(\mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)^2 = \mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$$

$$\lambda^2\mathbf{e} = \lambda\mathbf{e} \Rightarrow \lambda = 0, 1$$

$$\mathrm{tr}\left(\mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right) = n-r-1$$

$$\mathrm{tr}\left(\mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right) = \lambda_1+\lambda_2+\cdots+\lambda_n$$

$$\lambda_1=\lambda_2=\cdots=\lambda_{n-r-1}=1, \quad \lambda_{n-r}=\lambda_{n-r+1}=\cdots=\lambda_n=0$$

$$\mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' = \mathbf{e}_1\mathbf{e}_1' + \mathbf{e}_2\mathbf{e}_2' + \cdots + \mathbf{e}_{n-r-1}\mathbf{e}_{n-r-1}'$$

42

Proof of Result 7.4*

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{n-r-1} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \vdots \\ \mathbf{e}_{n-r-1}' \end{bmatrix} \boldsymbol{\varepsilon} : N_{n-r-1}(\mathbf{0}, \mathrm{Cov}(\mathbf{V}))$$

$$\mathrm{Cov}(V_i, V_k) = \mathbf{e}_i'\,\mathrm{Cov}(\boldsymbol{\varepsilon})\mathbf{e}_k' = \sigma^2\delta_{ik}$$

V_i : independent $N(0, \sigma^2)$

$$n\hat{\sigma}^2 = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}'\left(\mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right)\boldsymbol{\varepsilon} = \sum_{i=1}^{n-r-1} \boldsymbol{\varepsilon}'\mathbf{e}_i\mathbf{e}_i'\boldsymbol{\varepsilon}$$

$$= V_1^2 + V_2^2 + \cdots + V_{n-r-1}^2 : \sigma^2\chi^2_{n-r-1}$$

43

χ^2 Distribution*

$$X_1 : N(\mu_1, \sigma_1^2), \quad X_2 : N(\mu_2, \sigma_2^2), \quad \cdots,$$

$$X_\nu : N(\mu_\nu, \sigma_\nu^2); \quad Z_i = \frac{X_i - \mu_i}{\sigma_i} : N(0,1)$$

$$\chi^2 = \sum_{i=1}^{\nu}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2, \quad \nu : \text{degrees of freedom (d.f.)}$$

$$f_n(\chi^2) = \begin{cases} \dfrac{1}{2^{n/2}\Gamma(n/2)}\left(\chi^2\right)^{n/2-1} e^{-\chi^2/2}, & \chi^2 > 0 \\ 0, & \chi^2 \le 0 \end{cases}$$

(Gamma distribution with $\alpha = n/2-1, \beta = 2$)

44

Result 7.5

$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} : N_n(0, \sigma^2\mathbf{I})$

100(1-α)% confidence region for $\boldsymbol{\beta}$

$(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})'\mathbf{Z}'\mathbf{Z}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) \le (r+1)s^2 F_{r+1,n-r-1}$

Simultaneous 100(1-α)% confidence intervals for β_i

$\hat{\beta}_i \pm \sqrt{\widehat{\mathrm{Var}}(\hat{\beta}_i)}\sqrt{(r+1)F_{r+1,n-r-1}(\alpha)}$

$\widehat{\mathrm{Var}}(\hat{\beta}_i)$: diagonal element of $s^2(\mathbf{Z}'\mathbf{Z})^{-1}$ corresponding to $\hat{\beta}_i$

45

Proof of Result 7.5

$\mathbf{V} = (\mathbf{Z}'\mathbf{Z})^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}), \quad E(\mathbf{V}) = 0$

$\mathrm{Cov}(\mathbf{V}) = (\mathbf{Z}'\mathbf{Z})^{1/2}\,\mathrm{Cov}(\hat{\boldsymbol{\beta}})(\mathbf{Z}'\mathbf{Z})^{1/2} = \sigma^2\mathbf{I}$

$\mathbf{V} : N_{r+1}(0, \sigma^2\mathbf{I}), \quad \mathbf{V}'\mathbf{V} : \sigma^2\chi^2_{r+1}$

$(n-r-1)s^2 : \sigma^2\chi^2_{n-r-1}$

$$\frac{\mathbf{V}'\mathbf{V}/(r+1)}{(n-r-1)s^2/(n-r-1)} : F_{r+1,n-r-1}$$

Confidence region

$(\boldsymbol{\beta}\text{-}\hat{\boldsymbol{\beta}})'\mathbf{Z}'\mathbf{Z}(\boldsymbol{\beta}\text{-}\hat{\boldsymbol{\beta}}) = \mathbf{V}'\mathbf{V} \le (r+1)s^2 F_{r+1,n-r-1}(\alpha)$

46

Example 7.4 (Real Estate Data)

- 20 homes in a Milwaukee, Wisconsin, neighborhood
- Regression model

$Y_j = \beta_0 + \beta_1 z_{j1} + \beta_2 z_{j2} + \varepsilon$

Y : selling price (thousands of dollars)

z_1 : total dwelling size (hundereds of squared feet)

z_2 : assessed value (thousands of dollars)

47

Example 7.4

$$(\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} 5.1523 & & \\ 0.2544 & 0.0512 & \\ -0.1463 & -0.0172 & 0.0067 \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y} = \begin{bmatrix} 30.967 \\ 2.634 \\ 0.045 \end{bmatrix}, \quad s = 3.473$$

$s\sqrt{5.1523} = 7.88, \quad s\sqrt{0.0512} = 0.785, \quad s\sqrt{0.0067} = 0.285$

$\hat{y} = \underset{(7.88)}{30.967} + \underset{(0.785)}{2.634} z_1 + \underset{(0.285)}{0.045} z_2, \quad R^2 = 0.834$

$\hat{\beta}_2 \pm t_{17}(0.025)\sqrt{\widehat{\mathrm{Var}}(\hat{\beta}_2)} = 0.045 \pm 2.110 \times 0.285$

95% confidence interval for β_2 : $(-0.556, 0.647)$

48

Result 7.6

Likelihood ratio test rejects $H_0 : \boldsymbol{\beta}_{(2)} = 0$ if

$$\frac{\left(\mathrm{SS}_{res}(\mathbf{Z}_1) - \mathrm{SS}_{res}(\mathbf{Z})\right)/(r-q)}{s^2} > F_{r-q,n-r-1}(\alpha)$$

$$\boldsymbol{\beta}_{(2)}' = \begin{bmatrix}\beta_{q+1} & \beta_{q+2} & \cdots & \beta_r\end{bmatrix}, \quad \boldsymbol{\varepsilon} : N_n(\mathbf{0}, \sigma^2\mathbf{I})$$

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \begin{bmatrix}\mathbf{Z}_1 & \mathbf{Z}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\beta}_{(1)} \\ \boldsymbol{\beta}_{(2)}\end{bmatrix} + \boldsymbol{\varepsilon} = \mathbf{Z}_1\boldsymbol{\beta}_{(1)} + \mathbf{Z}_2\boldsymbol{\beta}_{(2)} + \boldsymbol{\varepsilon}$$

$$\mathrm{SS}_{res}(\mathbf{Z}_1) - \mathrm{SS}_{res}(\mathbf{Z})$$

$$= (\mathbf{y} - \mathbf{Z}_1\hat{\boldsymbol{\beta}}_{(1)})'(\mathbf{y} - \mathbf{Z}_1\hat{\boldsymbol{\beta}}_{(1)}) - (\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})$$

$$\hat{\boldsymbol{\beta}}_{(1)} = \left(\mathbf{Z}_1'\mathbf{Z}_1\right)^{-1}\mathbf{Z}_1'\mathbf{y}$$

49

Effect of Rank

- In situations where $\mathbf{Z}$ is not of full rank, $\mathrm{rank}(\mathbf{Z})$ replaces $r+1$ and $\mathrm{rank}(\mathbf{Z}_1)$ replaces $q+1$ in Result 7.6

50

Proof of Result 7.6

$$\max_{\boldsymbol{\beta},\sigma^2} L(\boldsymbol{\beta}, \sigma^2) = \max_{\boldsymbol{\beta},\sigma^2} \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-(\mathbf{y}-\mathbf{Z}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{Z}\boldsymbol{\beta})/2\sigma^2}$$

$$= \frac{1}{(2\pi)^{n/2}\hat{\sigma}^n} e^{-n/2}$$

which occurs at $\hat{\boldsymbol{\beta}} = \left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{Z}'\mathbf{y}$ and $\hat{\sigma}^2 = (\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}})$

Under $H_0 : \boldsymbol{\beta}_{(2)} = 0, \quad \mathbf{Y} = \mathbf{Z}_1\boldsymbol{\beta}_{(1)} + \boldsymbol{\varepsilon}$ and

$$\max_{\boldsymbol{\beta}_{(1)},\sigma^2} L(\boldsymbol{\beta}_{(1)}, \sigma^2) = \frac{1}{(2\pi)^{n/2}\hat{\sigma}_1^n} e^{-n/2},$$ where the maximum

occurs at $\hat{\boldsymbol{\beta}}_{(1)} = \left(\mathbf{Z}_1'\mathbf{Z}_1\right)^{-1}\mathbf{Z}_1'\mathbf{y}$ and $\hat{\sigma}_1^2 = (\mathbf{y} - \mathbf{Z}_1\hat{\boldsymbol{\beta}}_{(1)})'(\mathbf{y} - \mathbf{Z}_1\hat{\boldsymbol{\beta}}_{(1)})$

51

Proof of Result 7.6

Reject $H_0 : \boldsymbol{\beta}_{(2)} = 0$ for small values of

$$\frac{\max_{\boldsymbol{\beta}_{(1)},\sigma^2} L(\boldsymbol{\beta}_{(1)}, \sigma^2)}{\max_{\boldsymbol{\beta},\sigma^2} L(\boldsymbol{\beta}, \sigma^2)} = \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}^2}\right)^{-n/2} = \left(1 + \frac{\hat{\sigma}_1^2 - \hat{\sigma}^2}{\hat{\sigma}^2}\right)^{-n/2}$$

is equivalent to reject H_0 for large $\left(\hat{\sigma}_1^2 - \hat{\sigma}^2\right)/\hat{\sigma}^2$ or

$$\frac{n\left(\hat{\sigma}_1^2 - \hat{\sigma}^2\right)/(r-q)}{n\hat{\sigma}^2/(n-r-1)} = \frac{\left(\mathrm{SS}_{res}(\mathbf{Z}_1) - \mathrm{SS}_{res}(\mathbf{Z})\right)/(r-q)}{s^2} = F$$

$$n\hat{\sigma}^2 : \sigma^2\chi^2_{n-r-1}, \quad n\hat{\sigma}_1^2 : \sigma^2\chi^2_{n-q-1}, \quad F : F_{r-q,n-r-1}$$

52

Wishart Distribution

$$w_{n-1}(\mathbf{A}\mid\boldsymbol{\Sigma})=\frac{|\mathbf{A}|^{(n-p-2)/2}e^{-\mathrm{tr}\left[\mathbf{A}\boldsymbol{\Sigma}^{-1}\right]/2}}{2^{p(n-1)/2}\pi^{p(p-1)/4}|\boldsymbol{\Sigma}|^{(n-1)/2}\prod_{i=1}^{p}\Gamma\left(\frac{1}{2}(n-i)\right)}$$

$\mathbf{A}$: positive definite

Properties :

$\mathbf{A}_1 : W_{m_1}(\mathbf{A}_1\mid\boldsymbol{\Sigma}),\quad \mathbf{A}_2 : W_{m_2}(\mathbf{A}_2\mid\boldsymbol{\Sigma}) \Rightarrow$

$\quad \mathbf{A}_1+\mathbf{A}_2 : W_{m_1+m_2}(\mathbf{A}_1+\mathbf{A}_2\mid\boldsymbol{\Sigma})$

$\mathbf{A} : W_m(\mathbf{A}\mid\boldsymbol{\Sigma}) \Rightarrow \mathbf{CAC}' : W_m(\mathbf{CAC}'\mid\mathbf{C}\boldsymbol{\Sigma}\boldsymbol{\Sigma}')$

53

Generalization of Result 7.6

$\mathbf{C} : (r-q)\times(r+1)$ matrix

Reject $H_0 : \mathbf{C}\boldsymbol{\beta}=\mathbf{0}$ at level α if

$$\frac{\left(\mathbf{C}\hat{\boldsymbol{\beta}}\right)(\mathbf{C}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{C}')^{-1}\left(\mathbf{C}\hat{\boldsymbol{\beta}}\right)}{s^2} > (r-q)F_{r-q,n-r-1}$$

since $C\hat{\boldsymbol{\beta}} : N_{r-q}(\mathbf{C}\boldsymbol{\beta},\sigma^2\mathbf{C}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{C}')$ and

$\left(\mathbf{C}\hat{\boldsymbol{\beta}}\right)(\mathbf{C}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{C}')^{-1}\left(\mathbf{C}\hat{\boldsymbol{\beta}}\right) : \sigma^2\chi^2_{n-r}$

54

Example 7.5 (Service Ratings Data)

Location	Gender	Service (Y)
1	0	15.2
1	0	21.2
1	0	27.3
1	0	21.2
1	0	21.2
1	1	36.4
1	1	92.4
2	0	27.3
2	0	15.2
2	0	9.1
2	0	18.2
2	0	50.0
2	1	44.0
2	1	63.6
3	0	15.2
3	0	30.3
3	1	36.4
3	1	40.9

55

Example 7.5: Design Matrix

$$\mathbf{Z}=\left[\begin{array}{c|ccc|cc|cccccc}
\text{constant} & & \text{location} & & \text{gender} & & & & \text{interaction} & & & \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{array}{l}
\\
\left.\begin{array}{l}\\ \\ \\ \\ \\ \end{array}\right\}\text{5 responses}\\
\left.\begin{array}{l}\\ \\ \end{array}\right\}\text{2 responses}\\
\left.\begin{array}{l}\\ \\ \\ \\ \\ \end{array}\right\}\text{5 responses}\\
\left.\begin{array}{l}\\ \\ \end{array}\right\}\text{2 responses}\\
\left.\begin{array}{l}\\ \\ \end{array}\right\}\text{2 responses}\\
\left.\begin{array}{l}\\ \\ \end{array}\right\}\text{2 responses}
\end{array}$$

56

Example 7.5

$\boldsymbol{\beta}' = [\beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \tau_1 \quad \tau_2 \quad \gamma_{11} \quad \gamma_{12} \quad \gamma_{21} \quad \gamma_{22} \quad \gamma_{31} \quad \gamma_{32}]$

$\text{rank}(\mathbf{Z}) = 6, \quad \text{SS}_{res}(\mathbf{Z}) = 2977.4, \quad n - \text{rank}(\mathbf{Z}) = 12$

$\mathbf{Z}_1$: first six columns of $\mathbf{Z}$

$\text{SS}_{res}(\mathbf{Z}_1) = 3419.1, \quad n - \text{rank}(\mathbf{Z}_1) = 18 - 4 = 14$

$H_0 : \gamma_{11} = \gamma_{12} = \gamma_{21} = \gamma_{22} = \gamma_{31} = \gamma_{32} = 0$

$$F = \frac{(\text{SS}_{res}(\mathbf{Z}_1) - \text{SS}_{res}(\mathbf{Z}))/(6-4)}{s^2} = \frac{(\text{SS}_{res}(\mathbf{Z}_1) - \text{SS}_{res}(\mathbf{Z}))/2}{\text{SS}_{res}(\mathbf{Z})/12}$$

$= 0.89$: insignificant for any appropriate level α

We can further verify that there is no location effect, but that the gender is significant

57

Outline

- Introduction
- The Classical Linear Regression Model
- Least Square Estimation
- Inference about the Regression Model
- <u>Inference from the Estimated Regression Function</u>

58

Questions

- What is the unbiased estimator of $E(\boldsymbol{Y}_0|\mathbf{z}_0)$ with minimum variance and its corresponding confidence intervals? (Result 7.7)
- What are the unbiased predictor and its prediction intervals of $\boldsymbol{Y}_0$? (Result 7.8)

59

Result 7.7

$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} : N_n(0, \sigma^2\mathbf{I})$

$\mathbf{z}_0' = [1 \quad z_{01} \quad \cdots \quad z_{0r}], \quad Y_0$: response at $\mathbf{z}_0$

$E(Y_0 \mid \mathbf{z}_0) = \beta_0 + \beta_1 z_{01} + \cdots + \beta_r z_{0r} = \mathbf{z}_0'\boldsymbol{\beta}$

$\mathbf{z}_0'\hat{\boldsymbol{\beta}}$ is the unbiased estimator of $E(Y_0 \mid \mathbf{z}_0)$ with minimum variance.

$\text{Var}(\mathbf{z}_0'\hat{\boldsymbol{\beta}}) = \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\sigma^2$

$100(1-\alpha)\%$ congidence interval of $E(Y_0 \mid \mathbf{z}_0)$ is

$$\mathbf{z}_0'\hat{\boldsymbol{\beta}} \pm t_{n-r-1}(\alpha/2)\sqrt{(\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0)s^2}$$

60

Proof of Result 7.7

$\mathbf{z}_0'\boldsymbol{\beta}$ is a linear combination of β_i's $\Rightarrow$

$\mathbf{z}_0'\hat{\boldsymbol{\beta}}$ is the unbiased estimator of $\mathbf{z}_0'\boldsymbol{\beta}$ with the minimum variance by Result 7.3

$\text{Var}\left(\mathbf{z}_0\hat{\boldsymbol{\beta}}\right) = \mathbf{z}_0' \,\text{Cov}(\hat{\boldsymbol{\beta}})\mathbf{z}_0 = \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\sigma^2$

$\hat{\boldsymbol{\beta}} : N_{r+1}(\boldsymbol{\beta}, \sigma^2(\mathbf{Z}'\mathbf{Z})^{-1})$

which is independent of $s^2/\sigma^2 : \chi^2_{n-r-1}/(n-r-1)$

$\mathbf{z}_0'\hat{\boldsymbol{\beta}} : N(\mathbf{z}_0'\boldsymbol{\beta}, \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\sigma^2)$

$$\frac{\left(\mathbf{z}_0'\hat{\boldsymbol{\beta}} - \mathbf{z}_0'\boldsymbol{\beta}\right)/\sqrt{\sigma^2\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}}{\sqrt{s^2/\sigma^2}} = \frac{\left(\mathbf{z}_0'\hat{\boldsymbol{\beta}} - \mathbf{z}_0'\boldsymbol{\beta}\right)}{\sqrt{s^2\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}} : t_{n-r-1}$$

61

Result 7.8

$Y_0 = \mathbf{z}_0'\boldsymbol{\beta} + \varepsilon_0$

$\varepsilon_0 : N(0, \sigma^2)$ independent of $\boldsymbol{\varepsilon}, \hat{\boldsymbol{\beta}}, s^2$

unbiased predictor of $Y_0 : \mathbf{z}_0'\hat{\boldsymbol{\beta}}$

$\text{Var}(Y_0 - \mathbf{z}_0'\hat{\boldsymbol{\beta}}) = \sigma^2\left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)$

$100(1-\alpha)\%$ prediction interval for Y_0 :

$$\mathbf{z}_0'\hat{\boldsymbol{\beta}} \pm t_{n-r-1}(\alpha/2)\sqrt{s^2\left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)}$$

62

Proof of Result 7.8

Forecast error $Y_0 - \mathbf{z}_0'\hat{\boldsymbol{\beta}} = \mathbf{z}_0'\boldsymbol{\beta} + \varepsilon_0 - \mathbf{z}_0'\hat{\boldsymbol{\beta}} = \varepsilon_0 + \mathbf{z}_0'(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$

$E(Y_0 - \mathbf{z}_0'\hat{\boldsymbol{\beta}}) = E(\varepsilon_0) + E(\mathbf{z}_0'(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})) = 0$

$\text{Var}(Y_0 - \mathbf{z}_0'\hat{\boldsymbol{\beta}}) = \text{Var}(\varepsilon_0) + \text{Var}(\mathbf{z}_0'(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))$

$= \sigma^2\left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)$

$(Y_0 - \mathbf{z}_0'\hat{\boldsymbol{\beta}}) : N(0, \sigma^2\left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right))$

$(n-r-1)s^2 : \sigma^2\chi^2_{n-r-1}$

$$\frac{(Y_0 - \mathbf{z}_0'\hat{\boldsymbol{\beta}})}{\sqrt{s^2\left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)}} : t_{n-r-1}$$

63

Example 7.6 (Computer Data)

z_1 (Orders)	z_2 (Add–delete items)	Y (CPU time)
123.5	2.108	141.5
146.1	9.213	168.9
133.9	1.905	154.8
128.5	.815	146.5
151.5	1.061	172.8
136.2	8.603	160.1
92.0	1.125	108.5

Source: Data taken from H. P. Artis, *Forecasting Computer Requirements: A Forecaster's Dilemma* (Piscataway, NJ: Bell Laboratories, 1979).

64

Example 7.6

$\mathbf{z}_0' = [1 \quad 130 \quad 7.5], \quad \hat{y} = 8.42 + 1.08 z_1 + 0.42 z_2$

$$(\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} 8.17969 & & \\ -0.06411 & 0.00052 & \\ 0.08831 & -0.00107 & 0.01440 \end{bmatrix}$$

$s = 1.204, \quad \mathbf{z}_0'\hat{\boldsymbol{\beta}} = 8.42 + 1.08*130 + 0.42*7.5 = 151.97$

$s\sqrt{\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0} = 0.71, \quad t_4(0.025) = 2.776$

95% confidence interval for the mean CPU time

$\mathbf{z}_0'\hat{\boldsymbol{\beta}} \pm t_4(0.025) s \sqrt{\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}$ or $(150.00, 153.94)$

95% prediction interval at $\mathbf{z}_0$

$\mathbf{z}_0'\hat{\boldsymbol{\beta}} \pm t_4(0.025) s \sqrt{1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}$ or $(148.08, 155.86)$

65

Outline

- Model Checking and Other Aspects of Regression
- Multivariate Multiple Regression
- The Concept of Linear Regression
- Comparing the Two Formulations of the Regression Model
- Multiple Regression Models with Time Dependent Errors

66

Questions

- How to know the adequacy of the linear regression model?
- How to test independence of time?
- What is the leverage?
- What is the Mallow's C_p Statistic? How to use it?
- What is the stepwise regression?
- How to treat colinearity?

67

Questions

- What is the bias caused by a mis-specified model?

68

Adequacy of the Model

$$\hat{\varepsilon}_1 = y_1 - \hat{\beta}_0 - \hat{\beta}_1 z_{11} - \cdots - \hat{\beta}_r z_{1r}$$

$$\hat{\varepsilon}_2 = y_2 - \hat{\beta}_0 - \hat{\beta}_1 z_{21} - \cdots - \hat{\beta}_r z_{2r}$$

$$\vdots \qquad\qquad \vdots$$

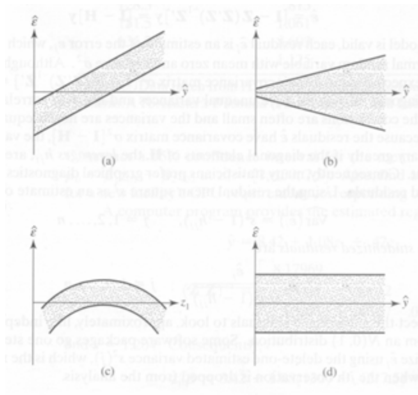
$$\hat{\varepsilon}_n = y_n - \hat{\beta}_0 - \hat{\beta}_1 z_{n1} - \cdots - \hat{\beta}_r z_{nr}$$

$$\hat{\boldsymbol{\varepsilon}} = \left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\mathbf{y} = \left[\mathbf{I} - \mathbf{H}\right]\mathbf{y}$$

$\hat{\varepsilon}_j$ is an estimate of $\varepsilon_j : N(0, \sigma^2)$

69

Residual Plots



70

Q-*Q* Plots and Histograms

- Used to detect the presence of unusual observations or severe departures from normality that may require special attention in the analysis
- If n is large, minor departures from normality will not greatly affect inferences about β

71

Test of Independence of Time

Test constructed from the first autocorrelation

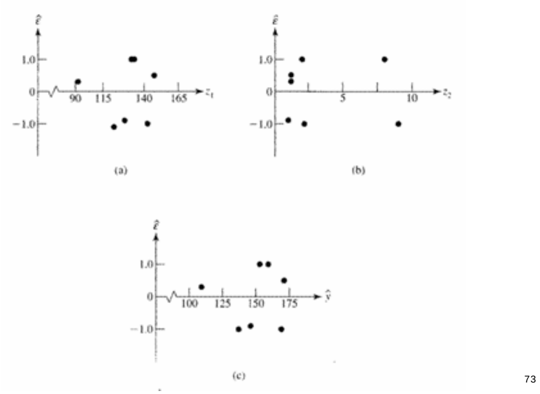
$$r_1 = \frac{\sum_{j=2}^{n} \hat{\varepsilon}_j \hat{\varepsilon}_{j-1}}{\sum_{j=1}^{n} \hat{\varepsilon}_j^2}$$

Durbin - Watson Test

$$\frac{\sum_{j=2}^{n} \left(\hat{\varepsilon}_j - \hat{\varepsilon}_{j-1}\right)^2}{\sum_{j=1}^{n} \hat{\varepsilon}_j^2} \approx 2(1 - r_1)$$

72

Example 7.7: Residual Plot



73

Leverage

- "Outliers" in either the response or explanatory variables may have a considerable effect on the analysis and determine the fit
- Leverage for simple linear regression with one explanatory variable z

$$h_{jj} = \frac{1}{n} + \frac{(z_j - \bar{z})^2}{\sum_{j=1}^{n} (z_j - \bar{z})^2}, \quad \text{average} = \frac{r+1}{n}$$

74

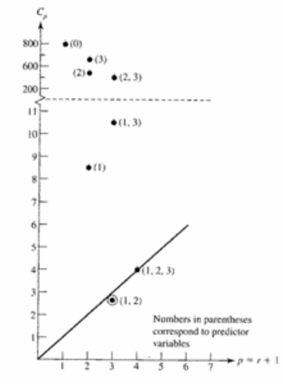
Mallow's C_p Statistic

- Select variables from all possible combinations

$$C_p = \left(\frac{\left(\begin{array}{l} \text{residual sum of squares for subset models} \\ \text{with } p \text{ parameters, including an intercept} \end{array} \right)}{(\text{residual variance for full model})} \right) - (n - 2p)$$

75

Usage of Mallow's C_p Statistic



76

Stepwise Regression

- 1. The predictor variable that explains the largest significant proportion of the variation in Y is the first variable to enter
- 2. The next to enter is the one that makes the highest contribution to the regression sum of squares. Use Result 7.6 to determine the significance (*F*-test)

77

Stepwise Regression

- 3. Once a new variable is included, the individual contributions to the regression sum of squares of the other variables already in the equation are checked using F-tests. If the *F*-statistic is small, the variable is deleted
- 4. Steps 2 and 3 are repeated until all possible additions are non-significant and all possible deletions are significant

78

Treatment of Colinearity

- If $\mathbf{Z}$ is not of full rank, $\mathbf{Z'Z}$ does not have an inverse → Colinear
- Not likely to have exact colinearity
- Possible to have a linear combination of columns of $\mathbf{Z}$ that are nearly 0
- Can be overcome somewhat by
  - Delete one of a pair of predictor variables that are strongly correlated
  - Relate the response Y to the principal components of the predictor variables

79

Bias Caused by a Misspecified Model

$$\mathbf{Z} = [\mathbf{Z}_1 \quad \mathbf{Z}_2]$$

$$\mathbf{Y} = [\mathbf{Z}_1 \quad \mathbf{Z}_2]\begin{bmatrix}\boldsymbol{\beta}_{(1)} \\ \boldsymbol{\beta}_{(2)}\end{bmatrix} + \boldsymbol{\varepsilon} = \mathbf{Z}_1\boldsymbol{\beta}_{(1)} + \mathbf{Z}_2\boldsymbol{\beta}_{(2)} + \boldsymbol{\varepsilon}$$

$$\hat{\boldsymbol{\beta}}_{(1)} = (\mathbf{Z}_1'\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\mathbf{Y}$$

$$E(\hat{\boldsymbol{\beta}}_{(1)}) = (\mathbf{Z}_1'\mathbf{Z}_1)^{-1}\mathbf{Z}_1'E(\mathbf{Y}) = (\mathbf{Z}_1'\mathbf{Z}_1)^{-1}\mathbf{Z}_1'(\mathbf{Z}_1\boldsymbol{\beta}_{(1)} + \mathbf{Z}_2\boldsymbol{\beta}_{(2)})$$

$$= \boldsymbol{\beta}_{(1)} + (\mathbf{Z}_1'\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\mathbf{Z}_2\boldsymbol{\beta}_{(2)}$$

biased estimator of $\boldsymbol{\beta}_{(1)}$

80

Outline

- Model Checking and Other Aspects of Regression
- Multivariate Multiple Regression
- The Concept of Linear Regression
- Comparing the Two Formulations of the Regression Model
- Multiple Regression Models with Time Dependent Errors

81

Questions

- How to do multivariate multiple regression?
- What are the expectation of the estimated matrix of coefficients and the covariance matrix of the residuals? (Result 7.9)
- What is the forecast error?

82

Questions

- What is the maximum likelihood estimator of the matrix of coefficients? (Result 7.10)
- How to know that number of variables is enough in the multivariate multiple regression? (Result 7.11)
- How to do Predictions from Regressions?

83

Example 7.8

- Observed data

z_1	0	1	2	3	4
y_1	1	4	3	8	9
y_2	-1	-1	2	3	2

- Regression model

$$Y_1 = \beta_{01} + \beta_{11} z_1 + \varepsilon_1$$

$$Y_2 = \beta_{02} + \beta_{12} z_1 + \varepsilon_2$$

84

Multivariate Multiple Regression

$$Y_1 = \beta_{01} + \beta_{11} z_1 + \cdots + \beta_{r1} z_r + \varepsilon_1$$
$$Y_2 = \beta_{02} + \beta_{12} z_1 + \cdots + \beta_{r2} z_r + \varepsilon_2$$
$$\vdots \qquad \vdots$$
$$Y_m = \beta_{0m} + \beta_{1m} z_1 + \cdots + \beta_{rm} z_r + \varepsilon_m$$
$$\boldsymbol{\varepsilon}' = [\varepsilon_1 \quad \varepsilon_2 \quad \cdots \quad \varepsilon_m], \quad E(\boldsymbol{\varepsilon}) = 0, \quad \mathrm{Var}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}$$
$$\mathbf{Y}_j' = [Y_{j1} \quad Y_{j2} \quad \cdots \quad Y_{jm}], \quad \boldsymbol{\varepsilon}_j' = [\varepsilon_{j1} \quad \varepsilon_{j2} \quad \cdots \quad \varepsilon_{jm}]$$
$$\mathbf{Z} = \begin{bmatrix} z_{10} & z_{11} & \cdots & z_{1r} \\ z_{20} & z_{21} & \cdots & z_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n0} & z_{n1} & \cdots & z_{nr} \end{bmatrix}$$

85

Multivariate Multiple Regression

$$\mathbf{Y} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nm} \end{bmatrix} = [\mathbf{Y}_{(1)} \quad \mathbf{Y}_{(2)} \quad \cdots \quad \mathbf{Y}_{(m)}]$$
$$\boldsymbol{\beta} = \begin{bmatrix} \beta_{01} & \beta_{02} & \cdots & \beta_{0m} \\ \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{r1} & \beta_{r2} & \cdots & \beta_{rm} \end{bmatrix} = [\boldsymbol{\beta}_{(1)} \quad \boldsymbol{\beta}_{(2)} \quad \cdots \quad \boldsymbol{\beta}_{(m)}]$$

86

Multivariate Multiple Regression

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1m} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n1} & \varepsilon_{n2} & \cdots & \varepsilon_{nm} \end{bmatrix} = [\boldsymbol{\varepsilon}_{(1)} \quad \boldsymbol{\varepsilon}_{(2)} \quad \cdots \quad \boldsymbol{\varepsilon}_{(m)}] = \begin{bmatrix} \boldsymbol{\varepsilon}_1' \\ \boldsymbol{\varepsilon}_2' \\ \vdots \\ \boldsymbol{\varepsilon}_n' \end{bmatrix}$$
$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$
$$E(\boldsymbol{\varepsilon}_{(i)}) = 0, \quad \mathrm{Cov}(\boldsymbol{\varepsilon}_{(i)}, \boldsymbol{\varepsilon}_{(k)}) = \sigma_{ik}\mathbf{I}, \quad i, k = 1, 2, \cdots, m$$
$$\boldsymbol{\Sigma} = \{\sigma_{ik}\}$$

87

Multivariate Multiple Regression

$$\mathbf{Y}_{(i)} = \mathbf{Z}\boldsymbol{\beta}_{(i)} + \boldsymbol{\varepsilon}_{(i)}, \quad \mathrm{Cov}(\boldsymbol{\varepsilon}_{(i)}) = \sigma_{ii}\mathbf{I}, \quad i = 1, 2, \cdots, m$$
$$\hat{\boldsymbol{\beta}}_{(i)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}_{(i)}$$
$$\hat{\boldsymbol{\beta}} = [\hat{\boldsymbol{\beta}}_{(1)} \quad \hat{\boldsymbol{\beta}}_{(2)} \quad \cdots \quad \hat{\boldsymbol{\beta}}_{(m)}] = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'[\mathbf{Y}_{(1)} \quad \mathbf{Y}_{(2)} \quad \cdots \quad \mathbf{Y}_{(m)}]$$
$$\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}, \quad \mathbf{B} = [\mathbf{b}_{(1)} \quad \mathbf{b}_{(2)} \quad \cdots \quad \mathbf{b}_{(m)}]$$
$$(\mathbf{Y} - \mathbf{Z}\mathbf{B})'(\mathbf{Y} - \mathbf{Z}\mathbf{B}) =$$
$$\begin{bmatrix} (\mathbf{Y}_{(1)} - \mathbf{Z}\mathbf{b}_{(1)})'(\mathbf{Y}_{(1)} - \mathbf{Z}\mathbf{b}_{(1)}) & \cdots & (\mathbf{Y}_{(1)} - \mathbf{Z}\mathbf{b}_{(1)})'(\mathbf{Y}_{(m)} - \mathbf{Z}\mathbf{b}_{(m)}) \\ \vdots & & \vdots \\ (\mathbf{Y}_{(m)} - \mathbf{Z}\mathbf{b}_{(m)})'(\mathbf{Y}_{(1)} - \mathbf{Z}\mathbf{b}_{(1)}) & \cdots & (\mathbf{Y}_{(m)} - \mathbf{Z}\mathbf{b}_{(m)})'(\mathbf{Y}_{(m)} - \mathbf{Z}\mathbf{b}_{(m)}) \end{bmatrix}$$

88

Multivariate Multiple Regression

$\mathbf{b}_{(i)} = \hat{\boldsymbol\beta}_{(i)}$ minimizes $\left(\mathbf{Y}_{(i)} - \mathbf{Z}\mathbf{b}_{(i)}\right)'\left(\mathbf{Y}_{(i)} - \mathbf{Z}\mathbf{b}_{(i)}\right)$

$\therefore \operatorname{tr}\left[(\mathbf{Y}-\mathbf{Z}\mathbf{B})'(\mathbf{Y}-\mathbf{Z}\mathbf{B})\right]$ is minimized by $\mathbf{B} = \hat{\boldsymbol\beta}$

Generalized variance $\left|(\mathbf{Y}-\mathbf{Z}\mathbf{B})'(\mathbf{Y}-\mathbf{Z}\mathbf{B})\right|$ is also minimized by $\mathbf{B} = \hat{\boldsymbol\beta}$

89

Multivariate Multiple Regression

Predicted values: $\hat{\mathbf{Y}} = \mathbf{Z}\hat{\boldsymbol\beta} = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$

Residuals: $\hat{\boldsymbol\varepsilon} = \mathbf{Y} - \hat{\mathbf{Y}} = \left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\mathbf{Y}$

$$\mathbf{Z}'\hat{\boldsymbol\varepsilon} = \mathbf{Z}'\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\mathbf{Y} = \mathbf{0}$$

$$\hat{\mathbf{Y}}'\hat{\boldsymbol\varepsilon} = \hat{\boldsymbol\beta}'\mathbf{Z}'\left[\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\right]\mathbf{Y} = \mathbf{0}$$

$$\mathbf{Y}'\mathbf{Y} = \left(\hat{\mathbf{Y}} + \hat{\boldsymbol\varepsilon}\right)'\left(\hat{\mathbf{Y}} + \hat{\boldsymbol\varepsilon}\right) = \hat{\mathbf{Y}}'\hat{\mathbf{Y}} + \hat{\boldsymbol\varepsilon}'\hat{\boldsymbol\varepsilon}$$

$$\hat{\boldsymbol\varepsilon}'\hat{\boldsymbol\varepsilon} = \mathbf{Y}'\mathbf{Y} - \hat{\mathbf{Y}}'\hat{\mathbf{Y}} = \mathbf{Y}'\mathbf{Y} - \hat{\boldsymbol\beta}'\mathbf{Z}'\mathbf{Z}\hat{\boldsymbol\beta}$$

90

Example 7.8

$$\mathbf{Z}' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}, \quad (\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.1 \end{bmatrix}$$

$$\hat{\boldsymbol\beta}_{(1)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_{(1)} = [1 \quad 2]'$$

$$\hat{\boldsymbol\beta}_{(2)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_{(2)} = [-1 \quad 1]'$$

$$\hat{\boldsymbol\beta} = \left[\hat{\boldsymbol\beta}_{(1)} \quad \hat{\boldsymbol\beta}_{(2)}\right] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left[\mathbf{y}_{(1)} \quad \mathbf{y}_{(2)}\right]$$

$$\hat{y}_1 = 1 + 2z_1, \quad \hat{y}_2 = -1 + z_2$$

91

Example 7.8

$$\hat{\mathbf{Y}} = \mathbf{Z}\hat{\boldsymbol\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 5 & 1 \\ 7 & 2 \\ 9 & 3 \end{bmatrix}$$

$$\hat{\boldsymbol\varepsilon} = \mathbf{Y} - \hat{\mathbf{Y}} = \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 \end{bmatrix}', \quad \hat{\boldsymbol\varepsilon}'\hat{\mathbf{Y}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{Y}'\mathbf{Y} = \begin{bmatrix} 171 & 43 \\ 43 & 19 \end{bmatrix}, \quad \hat{\mathbf{Y}}'\hat{\mathbf{Y}} = \begin{bmatrix} 165 & 45 \\ 45 & 15 \end{bmatrix}, \quad \hat{\boldsymbol\varepsilon}'\hat{\boldsymbol\varepsilon} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$$

$\mathbf{Y}'\mathbf{Y} = \hat{\mathbf{Y}}'\hat{\mathbf{Y}} + \hat{\boldsymbol\varepsilon}'\hat{\boldsymbol\varepsilon}$ is verified

92

Result 7.9

$E(\hat{\boldsymbol{\beta}}_{(i)}) = \boldsymbol{\beta}_{(i)}$ or $E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$

$\text{Cov}(\hat{\boldsymbol{\beta}}_{(i)}, \hat{\boldsymbol{\beta}}_{(k)}) = \sigma_{ik}(\mathbf{Z}'\mathbf{Z})^{-1}$

$\hat{\boldsymbol{\varepsilon}} = [\hat{\boldsymbol{\varepsilon}}_{(1)} \quad \hat{\boldsymbol{\varepsilon}}_{(2)} \quad \cdots \quad \hat{\boldsymbol{\varepsilon}}_{(m)}] = \mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}$

$E(\hat{\boldsymbol{\varepsilon}}_{(i)}) = \mathbf{0}, \quad E(\hat{\boldsymbol{\varepsilon}}'_{(i)}\hat{\boldsymbol{\varepsilon}}_{(k)}) = (n-r-1)\sigma_{ik}$

$E(\hat{\boldsymbol{\varepsilon}}) = 0, \quad E\left(\dfrac{\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}}{n-r-1}\right) = \boldsymbol{\Sigma}$

$\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\beta}}$ are uncorrelated

93

Proof of Result 7.9

$\mathbf{Y}_{(i)} = \mathbf{Z}\boldsymbol{\beta}_{(i)} + \boldsymbol{\varepsilon}_{(i)}, \quad E(\boldsymbol{\varepsilon}_{(i)}) = 0, \quad E(\boldsymbol{\varepsilon}_{(i)}\boldsymbol{\varepsilon}'_{(i)}) = \sigma_{ii}\mathbf{I}$

$\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}_{(i)} - \boldsymbol{\beta}_{(i)} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\varepsilon}_{(i)}$

$\hat{\boldsymbol{\varepsilon}}_{(i)} = \mathbf{Y}_{(i)} - \hat{\mathbf{Y}}_{(i)} = [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\mathbf{Y}_{(i)}$

$= [\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}']\boldsymbol{\varepsilon}_{(i)}$

$E(\hat{\boldsymbol{\beta}}_{(i)}) = \boldsymbol{\beta}_{(i)}, \quad E(\hat{\boldsymbol{\varepsilon}}_{(i)}) = \mathbf{0}$

$\text{Cov}(\hat{\boldsymbol{\beta}}_{(i)}, \hat{\boldsymbol{\beta}}_{(k)}) = E(\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)})(\hat{\boldsymbol{\beta}}_{(k)} - \boldsymbol{\beta}_{(k)})'$

$= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'E(\boldsymbol{\varepsilon}_{(i)}\boldsymbol{\varepsilon}'_{(k)})\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1} = \sigma_{ik}(\mathbf{Z}'\mathbf{Z})^{-1}$

94

Proof of Result 7.9

$\mathbf{U}$: random vector, $\quad \mathbf{A}$: fixed matrix

$E(\mathbf{U}'\mathbf{A}\mathbf{U}) = E[\text{tr}(\mathbf{U}'\mathbf{A}\mathbf{U})] = \text{tr}[\mathbf{A}E(\mathbf{U}\mathbf{U}')]$

$E(\hat{\boldsymbol{\varepsilon}}'_{(i)}\hat{\boldsymbol{\varepsilon}}_{(k)}) = E(\boldsymbol{\varepsilon}'_{(i)}(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\boldsymbol{\varepsilon}_{(k)})$

$= \text{tr}[(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\sigma_{ik}\mathbf{I}]$

$= \sigma_{ik}\,\text{tr}[(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')] = \sigma_{ik}(n-r-1)$

$E\left(\dfrac{\hat{\boldsymbol{\varepsilon}}'_{(i)}\hat{\boldsymbol{\varepsilon}}_{(k)}}{n-r-1}\right) = \boldsymbol{\Sigma}$

95

Proof of Result 7.9

$\text{Cov}(\hat{\boldsymbol{\beta}}_{(i)}, \hat{\boldsymbol{\varepsilon}}_{(k)})$

$= E((\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\varepsilon}_{(i)}\boldsymbol{\varepsilon}'_{(k)}(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'))$

$= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'E(\boldsymbol{\varepsilon}_{(i)}\boldsymbol{\varepsilon}'_{(k)})(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')$

$= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\sigma_{ik}\mathbf{I}(\mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')$

$= \sigma_{ik}((\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' - (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}') = \mathbf{0}$

96

Forecast Error

$$\mathbf{z}_0' = [1 \quad z_{01} \quad \cdots \quad z_{0r}], \quad Y_{0i} = \mathbf{z}_0'\boldsymbol{\beta}_{(i)} + \varepsilon_{0i}$$

$$E\left(\mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(i)}\right) = \mathbf{z}_0' E\left(\hat{\boldsymbol{\beta}}_{(i)}\right) = \mathbf{z}_0'\boldsymbol{\beta}_{(i)}, \quad E(\mathbf{z}_0'\hat{\boldsymbol{\beta}}) = \mathbf{z}_0'\boldsymbol{\beta}$$

$$E\left[\mathbf{z}_0'\left(\boldsymbol{\beta}_{(i)} - \hat{\boldsymbol{\beta}}_{(i)}\right)\left(\boldsymbol{\beta}_{(k)} - \hat{\boldsymbol{\beta}}_{(k)}\right)'\mathbf{z}_0\right] = \mathbf{z}_0' E\left(\left(\boldsymbol{\beta}_{(i)} - \hat{\boldsymbol{\beta}}_{(i)}\right)\left(\boldsymbol{\beta}_{(k)} - \hat{\boldsymbol{\beta}}_{(k)}\right)'\right)\mathbf{z}_0$$

$$= \sigma_{ik}\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0$$

$\boldsymbol{\varepsilon}_0' = [\varepsilon_{01} \quad \varepsilon_{02} \quad \cdots \quad \varepsilon_{0m}]$ independent of $\boldsymbol{\varepsilon}$

$$E(\varepsilon_{0i}) = 0, \quad E(\varepsilon_{0i}\varepsilon_{0k}) = \sigma_{ik}$$

$$Y_{0i} - \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(i)} = Y_{0i} - \mathbf{z}_0'\boldsymbol{\beta}_{(i)} + \mathbf{z}_0'\boldsymbol{\beta}_{(i)} - \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(i)} = \varepsilon_{0i} - \mathbf{z}_0'\left(\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)}\right)$$

$$E(Y_{0i} - \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(i)}) = 0$$

97

Forecast Error

$$E\left(Y_{0i} - \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(i)}\right)\left(Y_{0k} - \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(k)}\right)$$

$$= E\left(\varepsilon_{0i} - \mathbf{z}_0'\left(\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)}\right)\right)\left(\varepsilon_{0k} - \mathbf{z}_0'\left(\hat{\boldsymbol{\beta}}_{(k)} - \boldsymbol{\beta}_{(k)}\right)\right)$$

$$= E\left(\varepsilon_{0i}\varepsilon_{0k}\right) + \mathbf{z}_0' E\left(\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)}\right)\left(\hat{\boldsymbol{\beta}}_{(k)} - \boldsymbol{\beta}_{(k)}\right)'\mathbf{z}_0$$

$$- \mathbf{z}_0' E\left((\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)})\varepsilon_{0k}\right) - E\left(\varepsilon_{0i}\left(\hat{\boldsymbol{\beta}}_{(k)} - \boldsymbol{\beta}_{(k)}\right)'\right)\mathbf{z}_0$$

$$= \sigma_{ik}\left(1 + \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\right)$$

$$\hat{\boldsymbol{\beta}}_{(i)} = \boldsymbol{\beta}_{(i)} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\varepsilon}_{(i)}, \quad E\left((\hat{\boldsymbol{\beta}}_{(i)} - \boldsymbol{\beta}_{(i)})\varepsilon_{0k}\right) = 0$$

98

Result 7.10

$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon}$: normal distribution

$\hat{\boldsymbol{\beta}}$ is the maximum likelihood estimator of $\boldsymbol{\beta}$

$\hat{\boldsymbol{\beta}}$: normal distribution with $E\left(\hat{\boldsymbol{\beta}}\right) = \boldsymbol{\beta}$

$\quad \mathrm{Cov}\left(\hat{\boldsymbol{\beta}}_{(i)}, \hat{\boldsymbol{\beta}}_{(k)}\right) = \sigma_{ik}(\mathbf{Z}'\mathbf{Z})^{-1}$

$\hat{\boldsymbol{\beta}}$ is independent of the maximum likelihood estimator of $\boldsymbol{\Sigma}$

given by $\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \frac{1}{n}\left(\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}\right)'\left(\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}\right)$

and $n\hat{\boldsymbol{\Sigma}}$ is distributed as $W_{m,n-r-1}(\boldsymbol{\Sigma})$

99

Result 7.11

$$\boldsymbol{\beta} = \begin{bmatrix} \underset{((q+1)\times m)}{\boldsymbol{\beta}_{(1)}} \\ --- \\ \underset{((r-q)\times m)}{\boldsymbol{\beta}_{(2)}} \end{bmatrix}, \quad H_0 : \boldsymbol{\beta}_{(2)} = \mathbf{0}$$

$n\hat{\boldsymbol{\Sigma}} = \left(\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}\right)'\left(\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}\right) : W_{m,n-r-1}(\boldsymbol{\Sigma})$ independent of

$n\left(\hat{\boldsymbol{\Sigma}}_1 - \hat{\boldsymbol{\Sigma}}\right) : W_{m,r-q}(\boldsymbol{\Sigma}), \quad n\hat{\boldsymbol{\Sigma}}_1 = \left(\mathbf{Y} - \mathbf{Z}_1\hat{\boldsymbol{\beta}}_{(1)}\right)'\left(\mathbf{Y} - \mathbf{Z}_1\hat{\boldsymbol{\beta}}_{(1)}\right)$

Reject H_0 for large values of

$$-n\ln\Lambda = -n\ln\left(\frac{\left|\hat{\boldsymbol{\Sigma}}\right|}{\left|\hat{\boldsymbol{\Sigma}}_1\right|}\right) = -n\ln\frac{\left|n\hat{\boldsymbol{\Sigma}}\right|}{\left|n\hat{\boldsymbol{\Sigma}} + n\left(\hat{\boldsymbol{\Sigma}}_1 - \hat{\boldsymbol{\Sigma}}\right)\right|}$$

For n large, $-[n - r - 1 - (m - r + q + 1)/2]\ln\left(\frac{\left|\hat{\boldsymbol{\Sigma}}\right|}{\left|\hat{\boldsymbol{\Sigma}}_1\right|}\right) \sim \chi^2_{m(r-q)}$

100

Example 7.9

Example 7.5 data plus one more service - quality index.

$$n\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} 2977.39 & 1021.72 \\ 1021.72 & 2050.95 \end{bmatrix}$$

$$n\left(\hat{\boldsymbol{\Sigma}}_1 - \hat{\boldsymbol{\Sigma}}\right) = \begin{bmatrix} 441.76 & 246.16 \\ 246.16 & 366.12 \end{bmatrix}$$

$\boldsymbol{\beta}_{(2)}$: locatin – gender interaction parameters

$H_0 : \boldsymbol{\beta}_{(2)} = \mathbf{0}, \quad \alpha = 0.05,$

$r_1 = \text{rank}(\mathbf{Z}) - 1 = 5, \quad q_1 = \text{rank}(\mathbf{Z}_1) - 1 = 3$

$$-[n - r_1 - 1 - (m - r_1 + q_1 + 1)]\ln\left(\frac{|n\hat{\boldsymbol{\Sigma}}|}{|n\hat{\boldsymbol{\Sigma}} + n(\hat{\boldsymbol{\Sigma}}_1 - \hat{\boldsymbol{\Sigma}})|}\right) = 3.28$$

$< \chi^2_{m(r_1 - q_1)}(0.05) = 9.49$, do not reject H_0

101

Other Multivariate Test Statistics

$\mathbf{E} = n\hat{\boldsymbol{\Sigma}}, \quad \mathbf{H} = n\left(\hat{\boldsymbol{\Sigma}}_1 - \hat{\boldsymbol{\Sigma}}\right)$

$\eta_1 \ge \eta_2 \ge \cdots \ge \eta_s$: eigenvalues of $\mathbf{HE}^{-1}$, $s = \min(p, r - q)$

$$\text{Wilk's lambda} = \frac{|\mathbf{E}|}{|\mathbf{E} + \mathbf{H}|} = \prod_{i=1}^{s} \frac{1}{1 + \eta_i}$$

$$\text{Pillai's trace} = \text{tr}\left(\mathbf{H}(\mathbf{H} + \mathbf{E})^{-1}\right) = \sum_{i=1}^{s} \frac{1}{1 + \eta_i}$$

$$\text{Hotelling - Lawlay trace} = \text{tr}\left(\mathbf{HE}^{-1}\right) = \sum_{i=1}^{s} \eta_i$$

$$\text{Roy's greatest root} = \frac{\eta_1}{1 + \eta_1}$$

102

Predictions from Regressions

Result 7.10 $\Rightarrow \hat{\boldsymbol{\beta}}'\mathbf{z}_0 : N_m\left(\boldsymbol{\beta}'\mathbf{z}_0, \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\boldsymbol{\Sigma}\right)$

$n\hat{\boldsymbol{\Sigma}}$: independently distributed as $W_{n-r-1}(\boldsymbol{\Sigma})$

$$T^2 = \left(\frac{\hat{\boldsymbol{\beta}}'\mathbf{z}_0 - \boldsymbol{\beta}'\mathbf{z}_0}{\sqrt{\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}}\right)'\left(\frac{n\hat{\boldsymbol{\Sigma}}}{n - r - 1}\right)^{-1}\left(\frac{\hat{\boldsymbol{\beta}}'\mathbf{z}_0 - \boldsymbol{\beta}'\mathbf{z}_0}{\sqrt{\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}}\right)$$

100(1-α)% confidence ellipsoid for $\boldsymbol{\beta}'\mathbf{z}_0$:

$$\left(\hat{\boldsymbol{\beta}}'\mathbf{z}_0 - \boldsymbol{\beta}'\mathbf{z}_0\right)'\left(\frac{n\hat{\boldsymbol{\Sigma}}}{n - r - 1}\right)^{-1}\left(\hat{\boldsymbol{\beta}}'\mathbf{z}_0 - \boldsymbol{\beta}'\mathbf{z}_0\right)$$

$$\le \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\left[\left(\frac{m(n - r - 1)}{n - r - m}\right)F_{m,n-r-m}(\alpha)\right]$$

103

Predictions from Regressions

100(1 − α)% simultaneous confidence intervals

for $E(Y_i) = \mathbf{z}_0'\boldsymbol{\beta}_{(i)}$:

$$\mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(i)} \pm \sqrt{\left(\frac{m(n - r - 1)}{n - r - m}\right)F_{m,n-r-m}(\alpha)}\sqrt{\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\left(\frac{n}{n - r - 1}\right)\hat{\sigma}_{ii}}$$

$\hat{\sigma}_{ii}$: the ith diagonal element of $\hat{\boldsymbol{\Sigma}}$

104

Predictions from Regressions

$\mathbf{Y}_0 = \boldsymbol{\beta}'\mathbf{z}_0 + \boldsymbol{\varepsilon}_0,$

$\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0 = \left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)\mathbf{z}_0 + \boldsymbol{\varepsilon}_0 : N_m(0, (1 + \mathbf{z}_0'\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{z}_0)\boldsymbol{\Sigma})$

independently of $n\hat{\boldsymbol{\Sigma}}$

$100(1-\alpha)\%$ prediction ellipsoid for $\mathbf{Y}_0$:

$$\left(\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0\right)\left(\frac{n}{n-r-1}\hat{\boldsymbol{\Sigma}}\right)^{-1}\left(\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0\right)$$

$$\leq (1 + \mathbf{z}_0'\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{z}_0)\left[\left(\frac{m(n-r-1)}{n-r-m}\right)F_{m,n-r-m}(\alpha)\right]$$

$100(1-\alpha)\%$ simultaneous prediction intervals for Y_{0i} :

$$\mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(i)} \pm \sqrt{\left[\left(\frac{m(n-r-1)}{n-r-m}\right)F_{m,n-r-m}(\alpha)\right]}\sqrt{(1 + \mathbf{z}_0'\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{z}_0)\left(\frac{n}{n-r-1}\hat{\sigma}_{ii}\right)}$$

105

Example 7.10

Example 7.6 data + Y_2 : disk I/O capacity

Fitted equation : $\hat{y}_2 = 14.14 + 2.25z_1 + 5.67z_2, \quad s = 1.812$

$\hat{\boldsymbol{\beta}}_{(2)} = [14{,}14 \quad 2.25 \quad 5.67]', \quad \mathbf{z}_0'\left(\mathbf{Z}'\mathbf{Z}\right)\mathbf{z}_0 = 0.34725$

From Example 7.6, $\hat{\boldsymbol{\beta}}_{(1)} = [8.42 \quad 1.08 \quad 42]$

$\mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(1)} = 151.97, \quad \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(2)} = 349.17$

$$n\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \left(\mathbf{y}_{(1)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(1)}\right)'\left(\mathbf{y}_{(1)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(1)}\right) & \left(\mathbf{y}_{(1)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(1)}\right)'\left(\mathbf{y}_{(2)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(2)}\right) \\ \left(\mathbf{y}_{(2)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(2)}\right)'\left(\mathbf{y}_{(1)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(1)}\right) & \left(\mathbf{y}_{(2)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(2)}\right)'\left(\mathbf{y}_{(2)} - \mathbf{Z}\hat{\boldsymbol{\beta}}_{(2)}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5.80 & 5.30 \\ 5.30 & 13.13 \end{bmatrix}$$

106

Example 7.10

$$\hat{\boldsymbol{\beta}}'\mathbf{z}_0 = \begin{bmatrix} \hat{\boldsymbol{\beta}}_{(1)}' \\ \hat{\boldsymbol{\beta}}_{(2)}' \end{bmatrix}\mathbf{z}_0 = \begin{bmatrix} \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(1)} \\ \mathbf{z}_0'\hat{\boldsymbol{\beta}}_{(2)} \end{bmatrix} = \begin{bmatrix} 151.97 \\ 349.17 \end{bmatrix}$$

$n = 7, \quad r = 2, \quad m = 2$

95% confidence ellipse for $\boldsymbol{\beta}'\mathbf{z}_0$:

$$4\left[\mathbf{z}_0'\boldsymbol{\beta}_{(1)} - 151.97, \mathbf{z}_0'\boldsymbol{\beta}_{(2)} - 349.17\right]\begin{bmatrix} 5.80 & 5.30 \\ 5.30 & 13.13 \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{z}_0'\boldsymbol{\beta}_{(1)} - 151.97 \\ \mathbf{z}_0'\boldsymbol{\beta}_{(2)} - 349.17 \end{bmatrix}$$

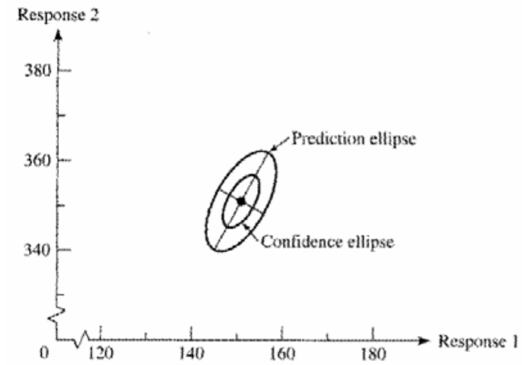
$$\leq 0.34725\left[\left(\frac{2\times 4}{3}\right)F_{2,3}(0.05)\right]$$

95% prediction ellipse : replace $\mathbf{z}_0'\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{z}_0 = 0.34725$ with

$1 + \mathbf{z}_0'\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\mathbf{z}_0 = 1.34725$

107

Example 7.10



108

Outline

- Model Checking and Other Aspects of Regression
- Multivariate Multiple Regression
- The Concept of Linear Regression
- Comparing the Two Formulations of the Regression Model
- Multiple Regression Models with Time Dependent Errors

Questions

- What are the results if the response Y is also treated as random in regression? (Result 7.12)
- What is the Population Multiple Correlation Coefficient?
- What is the maximum likelihood eatimator if the response Y is also treated as random in regression? (Result 7.13, 7.14)

Questions

- What is the partial correlation coefficient?

Linear Regression

$Y, Z_1, Z_2, \cdots, Z_r$: random variables

$f(y, z_1, z_2, \cdots, z_r)$: not necessarily normal

mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_Y \\ \boldsymbol{\mu}_Z \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{YY} & \boldsymbol{\sigma}'_{\mathbf{Z}Y} \\ \boldsymbol{\sigma}_{\mathbf{Z}Y} & \boldsymbol{\Sigma}_{\mathbf{ZZ}} \end{bmatrix}$$

linear predictor $= b_0 + b_1 Z_1 + b_2 Z_2 + \cdots + b_r Z_r = b_0 + \mathbf{b}'\mathbf{Z}$

prediction error $= Y - b_0 - \mathbf{b}'\mathbf{Z}$

mean square error $= E(Y - b_0 - \mathbf{b}'\mathbf{Z})^2$

Result 7.12

Linear predictor $\beta_0 + \boldsymbol{\beta}'\mathbf{Z}$ has minimum mean square among all linear predictors of the response Y

$$\beta_0 = \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}, \quad \beta_0 = \mu_Y - \boldsymbol{\beta}'\boldsymbol{\mu}_{\mathbf{Z}}$$

$$E(Y - \beta_0 - \boldsymbol{\beta}'\mathbf{Z})^2 = \sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}$$

Also, $\beta_0 + \boldsymbol{\beta}'\mathbf{Z}$ is the linear predictor having maximum correlation with Y

$$\mathrm{Corr}(Y, \beta_0 + \boldsymbol{\beta}'\mathbf{Z}) = \max_{b_0, \mathbf{b}} \mathrm{Corr}(Y, b_0 + \mathbf{b}'\mathbf{Z})$$

$$= \sqrt{\frac{\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}\boldsymbol{\beta}}{\sigma_{YY}}} = \sqrt{\frac{\boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}{\sigma_{YY}}}$$

Proof of Result 7.12

$$b_0 + \mathbf{b}'\mathbf{Z} = b_0 + \mathbf{b}'\mathbf{Z} - (\mu_Y + \mathbf{b}'\boldsymbol{\mu}_{\mathbf{Z}}) + (\mu_Y + \mathbf{b}'\boldsymbol{\mu}_{\mathbf{Z}})$$

$$E(Y - b_0 - \mathbf{b}'\mathbf{Z})^2$$

$$= E[Y - \mu_Y - \mathbf{b}'(\mathbf{Z} - \boldsymbol{\mu}_{\mathbf{Z}}) + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_{\mathbf{Z}})]^2$$

$$= E(Y - \mu_Y)^2 + E(\mathbf{b}'(\mathbf{Z} - \boldsymbol{\mu}_{\mathbf{Z}}))^2 + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_{\mathbf{Z}})^2$$

$$- 2E[\mathbf{b}'(\mathbf{Z} - \boldsymbol{\mu}_{\mathbf{Z}})(Y - \mu_Y)]$$

$$= \sigma_{YY} + \mathbf{b}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}\mathbf{b} + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_{\mathbf{Z}})^2 - 2\mathbf{b}'\boldsymbol{\sigma}_{\mathbf{Z}Y}$$

$$= \sigma_{YY} + (\mathbf{b} - \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y})'\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\mathbf{b} - \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}) + (\mu_Y - b_0 - \mathbf{b}'\boldsymbol{\mu}_{\mathbf{Z}})^2$$

$$- \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}$$

minimized at $\mathbf{b} = \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} = \boldsymbol{\beta}, \quad b_0 = \mu_Y - \mathbf{b}'\boldsymbol{\mu}_{\mathbf{Z}} = \mu_Y - (\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y})'\boldsymbol{\mu}_{\mathbf{Z}}$

Proof of Result 7.12

$$\mathrm{Cov}(b_0 + \mathbf{b}'\mathbf{Z}, Y) = \mathbf{b}'\sigma_{\mathbf{Z}Y}$$

$$[\mathrm{Corr}(b_0 + \mathbf{b}'\mathbf{Z}, Y)]^2 = \frac{|\mathbf{b}'\boldsymbol{\sigma}_{\mathbf{Z}Y}|^2}{\sigma_{YY}(\mathbf{b}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}\mathbf{b})}$$

Extended Cauchy-Schwartz inequality

$$(\mathbf{b}'\boldsymbol{\sigma}_{\mathbf{Z}Y})^2 \le \mathbf{b}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}\mathbf{b}\boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}$$

$$[\mathrm{Corr}(b_0 + \mathbf{b}'\mathbf{Z}, Y)]^2 \le \frac{\boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}{\sigma_{YY}}$$

with equality for $\mathbf{b} = \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} = \boldsymbol{\beta}$

$$\boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} = \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\beta} = \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\Sigma}_{\mathbf{ZZ}}\boldsymbol{\beta} = \boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}\boldsymbol{\beta}$$

Population Multiple Correlation Coefficient

Population multiple correlation coefficient :

$$\rho_{Y(\mathbf{Z})} = +\sqrt{\frac{\boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}{\sigma_{YY}}}$$

Population coefficient of determination : $\rho_{Y(\mathbf{Z})}^2$

Mean square error in using $\beta_0 + \boldsymbol{\beta}'\mathbf{Z}$ to forecast Y :

$$\sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} = \sigma_{YY}(1 - \rho_{Y(\mathbf{Z})}^2)$$

$\rho_{Y(\mathbf{Z})}^2 = 0$: no predictive power in $\mathbf{Z}$

$\rho_{Y(\mathbf{Z})}^2 = 1$: Y can be predicted with no error

Example 7.11

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_Y \\ -- \\ \boldsymbol{\mu}_{\mathbf{Z}} \end{bmatrix} = \begin{bmatrix} 5 \\ -- \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{YY} & | & \boldsymbol{\sigma}_{\mathbf{Z}Y}' \\ -- & + & -- \\ \boldsymbol{\sigma}_{\mathbf{Z}Y} & | & \boldsymbol{\Sigma}_{\mathbf{ZZ}} \end{bmatrix} = \begin{bmatrix} 10 & | & 1 & -1 \\ -- & + & -- & -- \\ 1 & | & 7 & 3 \\ -1 & | & 3 & 2 \end{bmatrix}$$

$\boldsymbol{\beta} = \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} = [1 \quad -2]', \quad \beta_0 = \mu_Y - \boldsymbol{\beta}'\boldsymbol{\mu}_{\mathbf{Z}} = 3$

best linear predictor : $\beta_0 + \boldsymbol{\beta}'\mathbf{Z} = 3 + Z_1 - 2Z_2$

mean square error $= \sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y} = 7$

$$\rho_{Y(\mathbf{Z})} = \sqrt{\frac{\boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}}{\sigma_{YY}}} = \sqrt{\frac{3}{10}} = 0.548$$

117

Linear Predictors and Normality

$[Y \quad Z_1 \quad Z_2 \quad \cdots \quad Z_r]' : N_{r+1}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Conditional distribution of Y with $\mathbf{Z} = \mathbf{z}$:

$N\left(\mu_Y + \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \boldsymbol{\mu}_{\mathbf{z}}), \sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}\right)$

$E(Y \mid \mathbf{z}) = \mu_Y + \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \boldsymbol{\mu}_{\mathbf{z}}) = \beta_0 + \boldsymbol{\beta}'\mathbf{z}$

(linear regression function)

When the population is not normal, $E(Y \mid \mathbf{z})$ need not be linear. Nevertheless, it still predicts Y with the smallest mean square error.

118

Result 7.13

Joint distribution of Y and $\mathbf{Z}$: $N_{r+1}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} \bar{Y} \\ \bar{\mathbf{Z}} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} s_{YY} & \mathbf{s}_{\mathbf{Z}Y}' \\ \mathbf{s}_{\mathbf{Z}Y} & \mathbf{S}_{\mathbf{ZZ}} \end{bmatrix}$$

maximum likelihood estimator of the coefficients

$\hat{\boldsymbol{\beta}} = \mathbf{S}_{\mathbf{ZZ}}^{-1}\mathbf{s}_{\mathbf{Z}Y}, \quad \hat{\beta}_0 = \bar{Y} - \mathbf{s}_{\mathbf{Z}Y}'\mathbf{S}_{\mathbf{ZZ}}^{-1}\bar{\mathbf{Z}} = \bar{Y} - \hat{\boldsymbol{\beta}}'\bar{\mathbf{Z}}$

maximum likelihood estimator

$\hat{\beta}_0 + \hat{\boldsymbol{\beta}}'\mathbf{z} = \bar{Y} + \mathbf{s}_{\mathbf{Z}Y}'\mathbf{S}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \bar{\mathbf{Z}})$

maximum likelihood estimator of $E[Y - \beta_0 - \boldsymbol{\beta}'\mathbf{Z}]^2$

$$\hat{\sigma}_{YY\bullet\mathbf{Z}} = \frac{n-1}{n}\left(s_{YY} - \mathbf{s}_{\mathbf{Z}Y}'\mathbf{S}_{\mathbf{ZZ}}^{-1}\mathbf{s}_{\mathbf{Z}Y}\right)$$

119

Proof of Result 7.13

Apply the invariance property of maximum likelihood, and

$\beta_0 = \mu_Y - \left(\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}\right)\boldsymbol{\mu}_{\mathbf{Z}}, \quad \boldsymbol{\beta} = \boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y},$

$\beta_0 + \boldsymbol{\beta}'\mathbf{z} = \mu_Y + \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \boldsymbol{\mu}_{\mathbf{Z}})$

mean square error $= \sigma_{YY\bullet\mathbf{Z}} = \sigma_{YY} - \boldsymbol{\sigma}_{\mathbf{Z}Y}'\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\sigma}_{\mathbf{Z}Y}$

to get the conclusion by substitution of the maximum likelihood estimators $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}} = \left(\frac{n-1}{n}\right)\mathbf{S}$ for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

Unbiased estimator for $\sigma_{YY\bullet\mathbf{Z}}$:

$$\left(\frac{n-1}{n-r-1}\right)\left(s_{YY} - \mathbf{s}_{\mathbf{Z}Y}'\mathbf{S}_{\mathbf{ZZ}}^{-1}\mathbf{s}_{\mathbf{Z}Y}\right) = \frac{\sum_{j=1}^{n}\left(Y_j - \hat{\beta}_0 - \hat{\boldsymbol{\beta}}'\mathbf{Z}_j\right)^2}{n-r-1}$$

120

Invariance Property

$\hat{\theta}$: maximum likelihood estimator of θ

$h(\hat{\theta})$: maximum likelihood estimator of $h(\theta)$

Examples :

MLE of $\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$

MLE of $\sqrt{\sigma_{ii}} = \sqrt{\hat{\sigma}_{ii}}$

$$\hat{\sigma}_{ii} = \frac{1}{n}\sum_{j=1}^{n}\left(X_{ji} - \overline{X}_i\right)^2 = \text{MLE of } \mathrm{Var}(X_i)$$

121

Example 7.12

Example 7.6 data, $n = 7$ observations on Y, Z_1, Z_2

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} \bar{y} \\ -- \\ \bar{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} 150.44 \\ ---- \\ 130.24 \\ 3.547 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} s_{YY} & | & \mathbf{s}_{\mathbf{Z}Y}' \\ -- & + & -- \\ \mathbf{s}_{\mathbf{Z}Y} & | & \mathbf{S}_{\mathbf{ZZ}} \end{bmatrix} = \begin{bmatrix} 467.913 & | & 418.763 & 35.983 \\ ---- & + & ---- & ---- \\ 418.763 & | & 377.200 & 28.034 \\ 35.983 & | & 28.034 & 13.657 \end{bmatrix}$$

122

Example 7.12

$$\hat{\boldsymbol{\beta}} = \mathbf{S}_{\mathbf{ZZ}}^{-1}\mathbf{s}_{\mathbf{Z}Y} = \begin{bmatrix} 1.079 \\ 0.420 \end{bmatrix}, \quad \hat{\beta}_0 = \bar{y} - \hat{\boldsymbol{\beta}}'\bar{\mathbf{z}} = 8.421$$

estimated regression function

$$\hat{\beta}_0 + \hat{\boldsymbol{\beta}}'\mathbf{z} = 8.42 - 1.08z_1 + 0.42z_2$$

maximum likelihood estimate of the mean square error

$$\left(\frac{n\text{-}1}{n}\right)\left(s_{YY} - \mathbf{s}_{\mathbf{Z}Y}'\mathbf{S}_{\mathbf{ZZ}}^{-1}\mathbf{s}_{\mathbf{Z}Y}\right) = 0.894$$

123

Prediction of Several Variables

$$\begin{bmatrix} \underset{(m\times1)}{\mathbf{Y}} \\ -- \\ \underset{(r\times1)}{\mathbf{Z}} \end{bmatrix} : N_{m+r}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \quad \boldsymbol{\mu} = \begin{bmatrix} \underset{(m\times1)}{\boldsymbol{\mu}_{\mathbf{Y}}} \\ -- \\ \underset{(r\times1)}{\boldsymbol{\mu}_{\mathbf{Z}}} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \underset{(m\times m)}{\boldsymbol{\Sigma}_{\mathbf{YY}}} & | & \underset{(m\times r)}{\boldsymbol{\Sigma}_{\mathbf{YZ}}} \\ -- & + & -- \\ \underset{(r\times m)}{\boldsymbol{\Sigma}_{\mathbf{ZY}}} & | & \underset{(r\times r)}{\boldsymbol{\Sigma}_{\mathbf{ZZ}}} \end{bmatrix}$$

multivariate regression of $\mathbf{Y}$ and $\mathbf{Z}$:

$$E[\mathbf{Y} \mid \mathbf{z}] = \boldsymbol{\mu}_{\mathbf{Y}} + \boldsymbol{\Sigma}_{\mathbf{YZ}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \boldsymbol{\mu}_{\mathbf{Z}})$$

composed of m univariate regressions. For example,

$$E[Y_1 \mid \mathbf{z}] = \mu_{Y_1} + \boldsymbol{\Sigma}_{Y_1\mathbf{Z}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \boldsymbol{\mu}_{\mathbf{Z}})$$

minimizes the mean square error for the prediction of Y_1

$\boldsymbol{\beta} = \boldsymbol{\Sigma}_{\mathbf{YZ}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}$: matrix of regression coefficients

124

Result 7.14

$\mathbf{Y}$ and $\mathbf{Z}: N_{m+r}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

regression of $\mathbf{Y}$ and $\mathbf{Z}: \boldsymbol{\beta}_0 + \boldsymbol{\beta}\mathbf{z} = \mu_Y + \boldsymbol{\Sigma}_{\mathbf{YZ}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \boldsymbol{\mu}_{\mathbf{Z}})$

$E(\mathbf{Y} - \boldsymbol{\beta}_0 - \boldsymbol{\beta}\mathbf{Z})(\mathbf{Y} - \boldsymbol{\beta}_0 - \boldsymbol{\beta}\mathbf{Z})' = \boldsymbol{\Sigma}_{\mathbf{YY\bullet Z}} = \boldsymbol{\Sigma}_{\mathbf{YY}} - \boldsymbol{\Sigma}_{\mathbf{YZ}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\Sigma}_{\mathbf{ZY}}$

maximum likelihood estimators

$\hat{\boldsymbol{\beta}}_0 + \hat{\boldsymbol{\beta}}\mathbf{z} = \overline{\mathbf{Y}} + \mathbf{S}_{\mathbf{YZ}}\mathbf{S}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \overline{\mathbf{Z}})$

$\hat{\boldsymbol{\Sigma}}_{\mathbf{YY\bullet Z}} = \left(\frac{n-1}{n}\right)\left(\mathbf{S}_{\mathbf{YY}} - \mathbf{S}_{\mathbf{YZ}}\mathbf{S}_{\mathbf{ZZ}}^{-1}\mathbf{S}_{\mathbf{ZY}}\right)$

125

Example 7.13

Data of Example 7.6 and 7.10.

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} \bar{\mathbf{y}} \\ -- \\ \bar{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} 150.44 \\ 327.79 \\ --- \\ 130.24 \\ 3.547 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 467.913 & 1148.556 & | & 418.763 & 35.983 \\ 1148.556 & 3072.491 & | & 1008.976 & 140.558 \\ ----- & ----- & + & ----- & ----- \\ 418.763 & 1008.976 & | & 377.200 & 28.034 \\ 35.983 & 140.558 & | & 28.034 & 13.657 \end{bmatrix}$$

126

Example 7.13

$$\hat{\boldsymbol{\beta}}_0 + \hat{\boldsymbol{\beta}}\mathbf{z} = \bar{\mathbf{y}} + \mathbf{S}_{\mathbf{YZ}}\mathbf{S}_{\mathbf{ZZ}}^{-1}(\mathbf{z} - \bar{\mathbf{z}})$$

$$= \begin{bmatrix} 150.44 \\ 327.79 \end{bmatrix} + \begin{bmatrix} 1.079(z_1 - 130.24) + 0.420(z_2 - 3.547) \\ 2.254(z_1 - 130.24) + 5.665(z_2 - 3.547) \end{bmatrix}$$

maximum likelihood estimate of $\boldsymbol{\Sigma}_{\mathbf{YY\bullet Z}}$

$$\left(\frac{n-1}{n}\right)\left(\mathbf{S}_{\mathbf{YY}} - \mathbf{S}_{\mathbf{YZ}}\mathbf{S}_{\mathbf{ZZ}}^{-1}\mathbf{S}_{\mathbf{ZY}}\right) = \begin{bmatrix} 0.894 & 0.893 \\ 0.893 & 2.205 \end{bmatrix}$$

127

Partial Correlation Coefficient

Pair of errors:

$$Y_1 - \mu_{Y_1} - \boldsymbol{\Sigma}_{Y_1\mathbf{Z}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{Z} - \boldsymbol{\mu}_{\mathbf{Z}}),\ Y_2 - \mu_{Y_2} - \boldsymbol{\Sigma}_{Y_2\mathbf{Z}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}(\mathbf{Z} - \boldsymbol{\mu}_{\mathbf{Z}})$$

Error covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{YY\bullet Z}} = \boldsymbol{\Sigma}_{\mathbf{YY}} - \boldsymbol{\Sigma}_{\mathbf{YZ}}\boldsymbol{\Sigma}_{\mathbf{ZZ}}^{-1}\boldsymbol{\Sigma}_{\mathbf{ZY}}$$

Partial correlation coefficient between Y_1 and Y_2, eliminating the effects of $\mathbf{Z}$:

$$\rho_{Y_1Y_2\bullet\mathbf{Z}} = \frac{\sigma_{Y_1Y_2\bullet\mathbf{Z}}}{\sqrt{\sigma_{Y_1Y_1\bullet\mathbf{Z}}}\sqrt{\sigma_{Y_2Y_2\bullet\mathbf{Z}}}}$$

Sample partial correlation coefficient

$$r_{Y_1Y_2\bullet\mathbf{Z}} = \frac{s_{Y_1Y_2\bullet\mathbf{Z}}}{\sqrt{s_{Y_1Y_1\bullet\mathbf{Z}}}\sqrt{s_{Y_2Y_2\bullet\mathbf{Z}}}}$$

128

Example 7.14

Example 7.13 data

$$\mathbf{S_{YY}} - \mathbf{S_{YZ}}\mathbf{S_{ZZ}^{-1}}\mathbf{S_{ZY}} = \begin{bmatrix} 1.043 & 1.042 \\ 1.042 & 2.572 \end{bmatrix}$$

$$r_{Y_1Y_2 \bullet \mathbf{Z}} = \frac{s_{Y_1Y_2 \bullet \mathbf{Z}}}{\sqrt{s_{Y_1Y_1 \bullet \mathbf{Z}}}\sqrt{s_{Y_2Y_2 \bullet \mathbf{Z}}}} = \frac{1.042}{\sqrt{1.043}\sqrt{2.572}} = 0.64$$

$$r_{Y_1Y_2} = 0.96$$

Correlation between Y_1 and Y_2 has been sharply reduced after eliminating the effects of $\mathbf{Z}$ on both responses

129

Outline

- Model Checking and Other Aspects of Regression
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130

Questions

- What is the mean corrected form for multivariate multiple regressions?
- Compare the classical regression model and the approach that treats the result as a conditional expectation?

131

Mean Corrected Form of the Regression Model

$$\begin{aligned} Y_j &= \beta_0 + \beta_1 z_{j1} + \cdots + \beta_r z_{jr} + \varepsilon_j \\ &= \beta_* + \beta_1(z_{j1} - \bar{z}_1) + \cdots + \beta_r(z_{jr} - \bar{z}_r) + \varepsilon_j \end{aligned}$$

mean corrected design matrix

$$\mathbf{Z}_c = \begin{bmatrix} 1 & z_{11} - \bar{z}_1 & \cdots & z_{1r} - \bar{z}_r \\ 1 & z_{21} - \bar{z}_1 & \cdots & z_{2r} - \bar{z}_r \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_{n1} - \bar{z}_1 & \cdots & z_{nr} - \bar{z}_r \end{bmatrix} = [\mathbf{1} \,|\, \mathbf{Z}_{c2}], \quad \mathbf{Z}_{c2}'\mathbf{1} = 0$$

$$\mathbf{Z}_c'\mathbf{Z}_c = \begin{bmatrix} \mathbf{1'1} & \mathbf{1'Z}_{c2} \\ \mathbf{Z}_{c2}'\mathbf{1} & \mathbf{Z}_{c2}'\mathbf{Z}_{c2} \end{bmatrix} = \begin{bmatrix} n & \mathbf{0'} \\ \mathbf{0} & \mathbf{Z}_{c2}'\mathbf{Z}_{c2} \end{bmatrix}$$

132

Mean Corrected Form of the Regression Model

$$\begin{bmatrix}\hat{\beta}_* \\ \hat{\boldsymbol{\beta}}_c\end{bmatrix} = \left(\mathbf{Z}_c^{'}\mathbf{Z}_c\right)^{-1}\mathbf{Z}_c^{'}\mathbf{y}$$

$$= \begin{bmatrix}1/n & \mathbf{0}' \\ \mathbf{0} & \left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{1}'\mathbf{y} \\ \mathbf{Z}_{c2}^{'}\mathbf{y}\end{bmatrix} = \begin{bmatrix}\bar{y} \\ \left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1}\mathbf{Z}_{c2}^{'}\mathbf{y}\end{bmatrix}$$

$$\hat{y} = \hat{\beta}_* + \hat{\boldsymbol{\beta}}_c^{'}(\mathbf{z}-\bar{\mathbf{z}}) = \bar{y} + \mathbf{y}'\mathbf{Z}_{c2}\left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1}(\mathbf{z}-\bar{\mathbf{z}})$$

$$\begin{bmatrix}\mathrm{Var}(\hat{\beta}_*) & \mathrm{Cov}(\hat{\beta}_*, \hat{\boldsymbol{\beta}}_c) \\ \mathrm{Cov}(\hat{\boldsymbol{\beta}}_c, \hat{\beta}_*) & \mathrm{Cov}(\hat{\boldsymbol{\beta}}_c)\end{bmatrix} = \left(\mathbf{Z}_c^{'}\mathbf{Z}_c\right)^{-1}\sigma^2$$

$$= \begin{bmatrix}\sigma^2/n & \mathbf{0}' \\ \mathbf{0} & \left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1}\sigma^2\end{bmatrix}$$

133

Mean Corrected Form for Multivariate Multiple Regressions

least square estimates of the coefficient vectors for the ith response:

$$\hat{\boldsymbol{\beta}}_{(i)} = \begin{bmatrix}\bar{y}_{(i)} \\ \left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1}\mathbf{Z}_{c2}^{'}\mathbf{y}_{(i)}\end{bmatrix}$$

standardized input variables

$$(z_{ji} - \bar{z}_i)/\sqrt{(n-1)s_{z_iz_i}}$$

$$\tilde{\beta}_i = \beta_i\sqrt{(n-1)s_{z_iz_i}}$$

$$\hat{\tilde{\beta}}_i = \hat{\beta}_i\sqrt{(n-1)s_{z_iz_i}}$$

134

Relating the Formulations

Result 7.13: $\hat{\beta}_0 + \hat{\boldsymbol{\beta}}'\mathbf{z} = \bar{y} + \mathbf{s}_{ZY}^{'}\mathbf{S}_{\mathbf{ZZ}}^{-1}(\mathbf{z}-\bar{\mathbf{z}})$

mean corrected form:

$$\hat{y} = \hat{\beta}_* + \hat{\boldsymbol{\beta}}_c^{'}(\mathbf{z}-\bar{\mathbf{z}}) = \bar{y} + \mathbf{y}'\mathbf{Z}_{c2}\left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1}(\mathbf{z}-\bar{\mathbf{z}})$$

$$\hat{\beta}_* = \bar{y} = \hat{\beta}_0, \quad \hat{\boldsymbol{\beta}}_c^{'} = \mathbf{y}'\mathbf{Z}_{c2}\left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1} = \mathbf{s}_{ZY}^{'}\mathbf{S}_{\mathbf{ZZ}}^{-1} = \hat{\boldsymbol{\beta}}'$$

$$\because \mathbf{y}'\mathbf{Z}_{c2} = (\mathbf{y}-\bar{y}\mathbf{1})'\mathbf{Z}_{c2} + \bar{y}\mathbf{1}'\mathbf{Z}_{c2} = (\mathbf{y}-\bar{y}\mathbf{1})'\mathbf{Z}_{c2}$$

$$\mathbf{y}'\mathbf{Z}_{c2}\left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1} = (\mathbf{y}-\bar{y}\mathbf{1})'\mathbf{Z}_{c2}\left(\mathbf{Z}_{c2}^{'}\mathbf{Z}_{c2}\right)^{-1}$$

$$= (n-1)\mathbf{s}_{ZY}^{'}\left[(n-1)\mathbf{S}_{\mathbf{ZZ}}\right]^{-1} = \mathbf{s}_{ZY}^{'}\mathbf{S}_{\mathbf{ZZ}}^{-1}$$

135

Example 7.15

- Example 7.6, classical linear regression model
- Example 7.12, joint normal distribution, best predictor as the conditional mean
- Both approaches yielded the same predictor of Y_1

$$\hat{y} = 8.42 + 1.08z_1 + 0.42z_2$$

136

Remarks on Both Formulation

- Conceptually different
- Classical model
  - Input variables are set by experimenter
  - Optimal among linear predictors
- Conditional mean model
  - Predictor values are random variables observed with the response values
  - Optimal among all choices of predictors

137

Outline

- Model Checking and Other Aspects of Regression
- Multivariate Multiple Regression
- The Concept of Linear Regression
- Comparing the Two Formulations of the Regression Model
- Multiple Regression Models with Time Dependent Errors

138

Example 7.16 Natural Gas Data

Y Sendout	Z_1 DHD	Z_2 DHDLag	Z_3 Windspeed	Z_4 Weekend
227	32	30	12	1
236	31	32	8	1
228	30	31	8	0
252	34	30	8	0
238	28	34	12	0
⋮	⋮	⋮	⋮	⋮
333	46	41	8	0
266	33	46	8	0
280	38	33	18	0
386	52	38	22	0
415	57	52	18	0

139

Example 7.16 : First Model

$$\text{Sendout} = 1.858 + 5.874\ \text{DHD} + 1.405\ \text{DHDLag} + 1.315\ \text{Windspeed} - 15.857\ \text{Weekend}$$

$R^2 = 0.952$

All coefficients are significant, except the intercept

But,

$$\text{lag1 autocorrelation} = r_1(\hat{\varepsilon}) = \frac{\sum_{j=2}^{n} \hat{\varepsilon}_j \hat{\varepsilon}_{j-1}}{\sum_{j=1}^{n} \hat{\varepsilon}_j^2} = 0.52$$

140

Example 7.16 : Second Model

Replace the independent errors with an autoregressive noise

$$N_j = \phi_j N_{j-1} + \phi_7 N_{j-7} + \varepsilon_7$$

Apply SAS to get a fitted model as

$$\text{Sendout} = 2.130 + 5.810\ \text{DHD} + 1.426\ \text{DHDLag}$$
$$+ 1.207\ \text{Windspeed} - 10.109\ \text{Weekend}$$

$$N_j = 0.470 N_{j-1} + 0.240 N_{j-7} + \varepsilon_j$$

$$\hat{\sigma}^2 = 228.89$$

auto correlations of the residuals are all negligible

141