

(Transformation of Multivariate Gaussian Distribution)

The population in *Exercise 1* is indeed a multivariate Gaussian distribution. From the statistics derived in *Exercise 1*, we can guess the true parameters of the distribution assuming the estimation error is sufficiently small. Now according to the random sample from *Exercise 1*, please answer the following questions.

- (a) Calculate the generalized sample variance and the total sample variance of the random sample in *Exercise 1*.

- Solution: The sample covariance matrix is

$$S = \begin{pmatrix} 0.8012 & 0.7485 \\ 0.7485 & 1.0858 \end{pmatrix}.$$

Hence the determinant (generalized sample variance) of  $S$  is

$$|S| = 0.3097,$$

and the trace (total sample variance) is 1.8870.

- (b) The true covariance matrix of the population in *Exercise 1* is

$$\Sigma = \begin{pmatrix} 0.8000 & 0.7505 \\ 0.7505 & 1.1000 \end{pmatrix}.$$

Use the computational program to derive the eigenvalues of the covariance matrix. Calculate the product, sum of the eigenvalues and compare it to the generalized sample variance and the total sample variance.

- Solution: The eigenvalues of  $\Sigma$  are 0.1847 and 1.7153. The product  $0.1847 \times 1.7153 = 0.3168$ . The error is -0.0224. The sum  $0.1847 + 1.7153 = 1.9$ . The error is -0.0068.

- (c) Linearly transform the original random sample according to the matrix

$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Plot the transformed random sample and carefully observe the difference between the original random sample. Calculate the new generalized sample variance and the total sample variance.

- Solution: See Fig. 1, 2. From the properties of multivariate Gaussian distribution, we know the covariance of the transformed data is

$$S' = ASA'.$$

However, because the transformation matrix  $A$  is an orthogonal matrix (reflection matrix in  $\mathbb{R}^2$ ,

$$S' = ASA' = ASA^{-1}.$$

Hence the new covariance matrix  $S'$  is similar to the original covariance matrix  $S$ . The eigenvalues are invariant under similar transformation. The generalized sample variance and the total sample variance are both unchanged.

- (d) Linearly transform the original random sample according to the matrix

$$B = \begin{pmatrix} 1000.8012 & 0.7485 \\ 0.7485 & 1001.0858 \end{pmatrix}.$$

Calculate the new generalized sample variance and the total sample variance. (Hint: The *Cayley-Hamilton* says that, for each matrix  $A$  with eigenvalues  $\{\lambda_i\}_1^n$ , then the transformed matrix  $f(A)$  has eigenvalues  $\{f(\lambda_i)\}_1^n$ , where  $f(x)$  is an analytic function.)

- Solution: It is obviously that

$$B = S + 1000I,$$

where  $I$  is a  $2 \times 2$  identity matrix. The new covariance matrix is

$$\begin{aligned} S' &= (S + 1000I)S(S + 1000I)' \\ &= (S + 1000I)S(S + 1000I) \quad S \text{ is symmetric} \\ &= (S^2 + 1000S)(S + 1000I) \\ &= S^3 + 2000S^2 + 1000000S. \end{aligned}$$

According to the *Cayley-Hamilton* theorem, the eigenvalues of the new covariance matrix is  $(0.1816)^3 + 2000(0.1816)^2 + 1000000(0.1816) = 1.8166 \times 10^5$  and  $(1.7054)^3 + 2000(1.7054)^2 + 1000000(1.7054) = 1.7112 \times 10^6$ . Hence the generalized variance is  $(1.8166 \times 10^5) \times (1.7112 \times 10^6) = 3.1086 \times 10^{11}$  and the total variance is  $(1.8166 \times 10^5) + (1.7112 \times 10^6) = 1.8929 \times 10^6$

*Note:* This problem is a toy problem that illustrates the properties of matrix algebra and the transformation of multivariate Gaussian distribution.

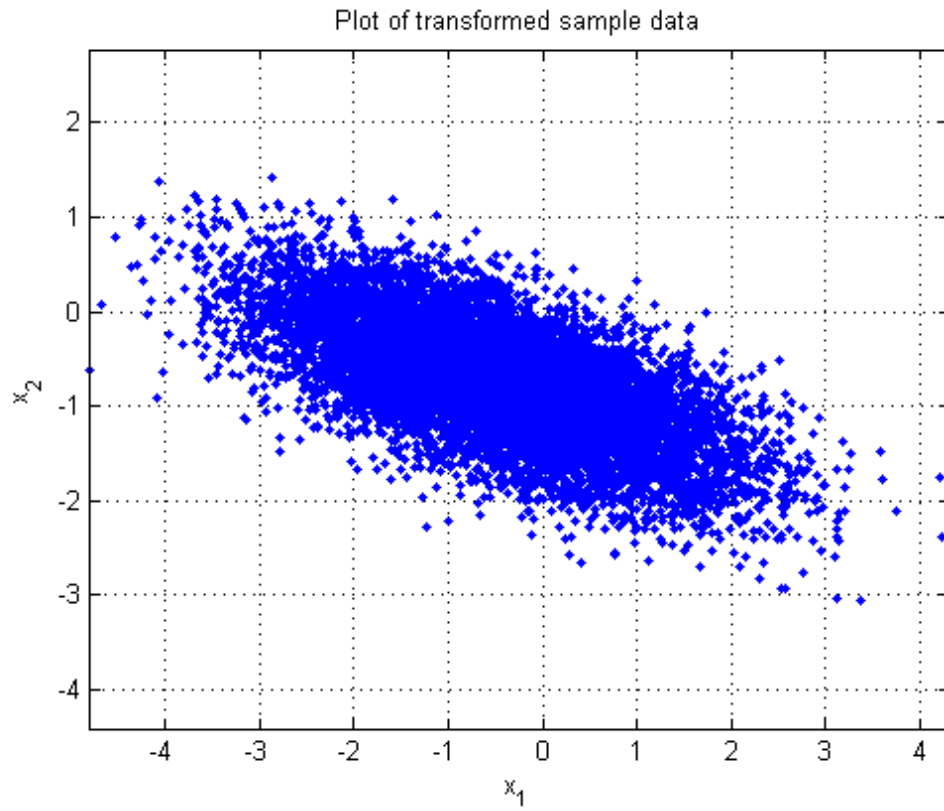


Figure 1: Distribution of The Transformed Random Sample in  $\mathbb{R}^2$

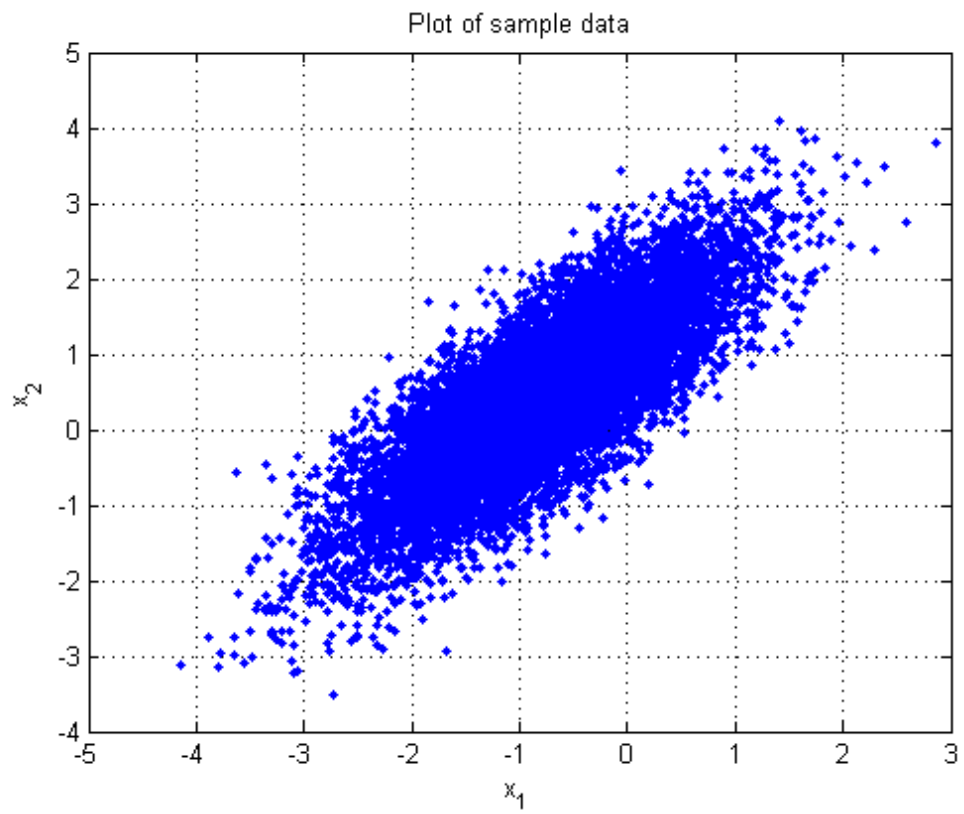


Figure 2: Distribution of The Random Sample in  $\mathbb{R}^2$