

Diagonalization

$$\mathbf{B}^{-1} = \mathbf{B}'$$

$$\mathbf{A}\mathbf{B}' = \mathbf{B}'\mathbf{\Lambda}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}$$

$$\mathbf{A}\mathbf{b}_i = \lambda_i \mathbf{b}_i$$

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Eigenvalues

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 3$$

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Eigenvectors

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = 0$$

$$\lambda_1 = 1$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3$$

$$\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Matrix B

$$\mathbf{B} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = -\frac{\sqrt{5}}{2} \begin{bmatrix} 1 & -\frac{1}{\sqrt{5}} \\ 0 & -\frac{2}{\sqrt{5}} \end{bmatrix} \neq \mathbf{B}'$$

A Symmetric Matrix

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{vmatrix} = (8-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 8, \quad \lambda_2 = 3$$

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Eigenvectors

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = 0$$

$$\lambda_1 = 8$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

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Matrix B

$$\mathbf{B} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \mathbf{B}'$$

Spectral Decomposition

$$\begin{aligned} \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} &= 8 \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} + 3 \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\ &= 8 \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} + 3 \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \end{aligned}$$