Chapter 8 Transformers and mutual inductance

-optional-

8.1 Ideal transformers

ideal transformer model, referred source network, referred load network, maximum power transfer

8.2 Magnetic coupling and mutual inductance

self-inductance, mutual inductance, coupling coefficient

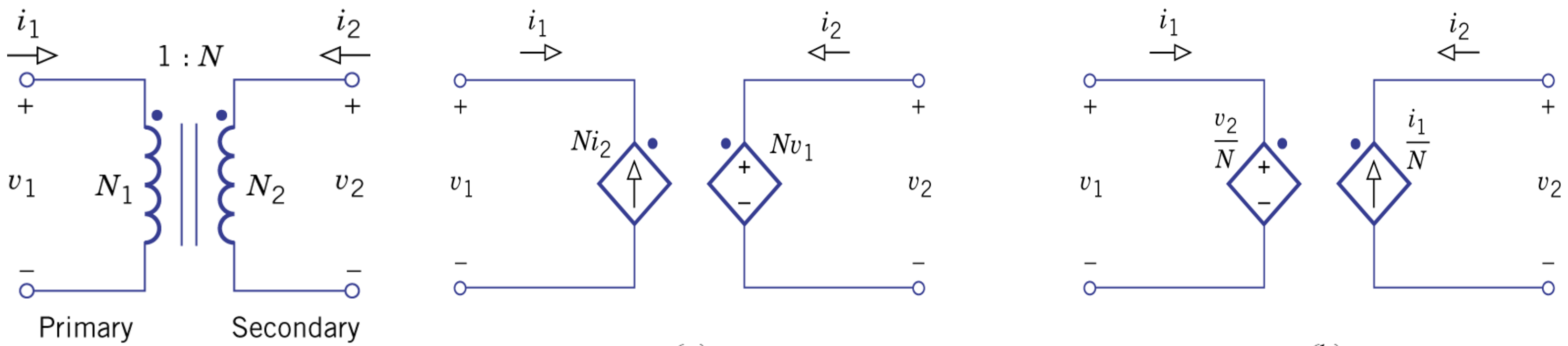
8.3 Circuits with mutual inductance

impedance analysis, T- and π -equivalent networks

8.1 Ideal transformers

<u>Basics</u>

1. Ideal transformer model



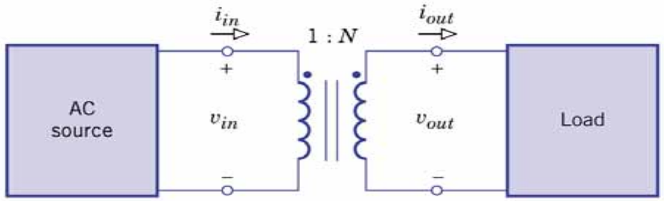
(a) (b)

$$\boxed{\frac{v_1(t)}{v_2(t)}=\frac{N_1}{N_2}=\frac{1}{N},\frac{i_1(t)}{i_2(t)}=-\frac{N_2}{N_1}=-N},N\equiv\frac{N_2}{N_1}$$

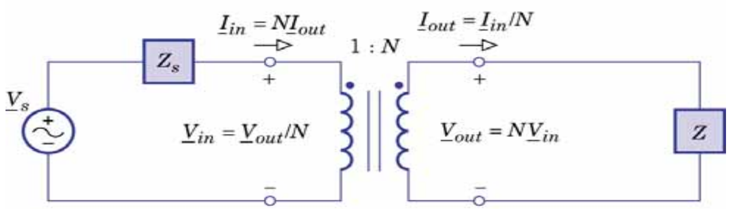
$$p(t)=v_1(t)i_1(t)+v_2(t)i_2(t)=v_1(t)i_1(t)+Nv_1(t)\frac{-i_1(t)}{N}=0$$

"•" means higher potential terminal.

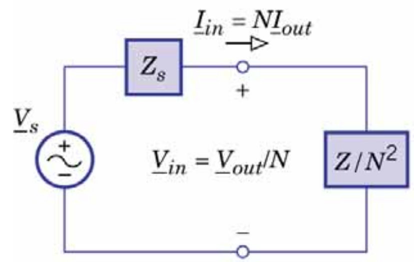
2. Referred load network and referred source network



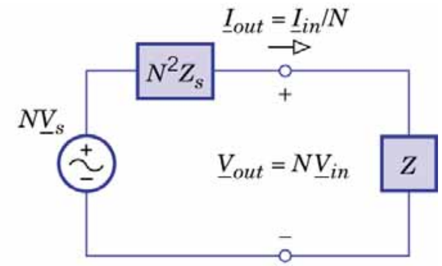
(a) Ideal transformer interfacing a source and a load



(b) Frequency–domain diagram



(a) Referred load impedance in the primary



(b) Referred source network in the secondary

Maximum power transfer by impedance matching

$$\frac{Z}{N^2} = Z_s^* = R_s - jX_s$$

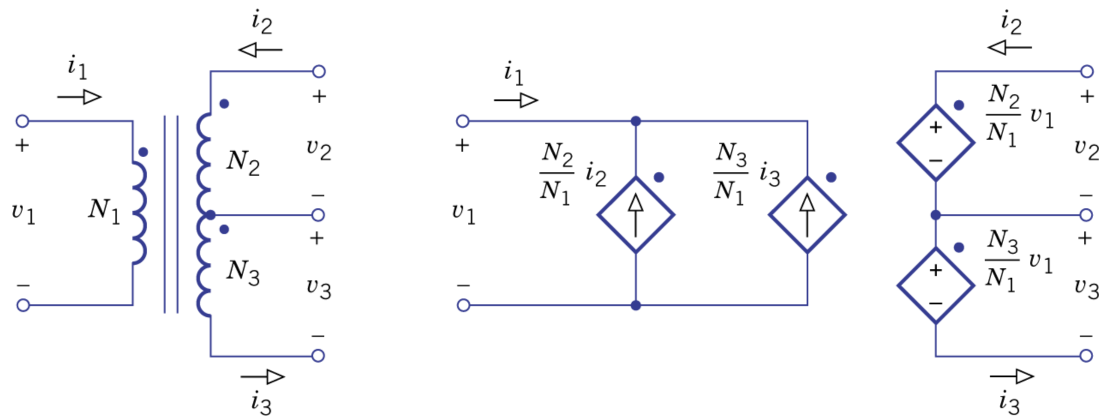
$$\frac{\underline{V}_{in}}{\underline{V}_{out}} = \frac{-\underline{I}_{out}}{\underline{I}_{in}} = \frac{1}{N}$$

referred to the primary $\rightarrow \underline{V}_{in} = \dfrac{\underline{V}_{out}}{N}, \underline{I}_{in} = N\underline{I}_{out}, Z_{in} = \dfrac{\underline{V}_{in}}{\underline{I}_{in}} = \dfrac{\underline{V}_{out}}{N}\dfrac{1}{N\underline{I}_{out}} = \dfrac{Z_{out}}{N^2}$

referred to the secondary $\rightarrow \underline{V}_{out} = N\underline{V}_{in}, \underline{I}_{out} = \dfrac{\underline{I}_{in}}{N}, Z_{out} = \dfrac{\underline{V}_{out}}{\underline{I}_{out}} = N\underline{V}_{in}\dfrac{N}{\underline{I}_{in}} = N^2 Z_{in}$

Discussion

1. Transformer with taps



(a) Ideal transformer with tapped secondary

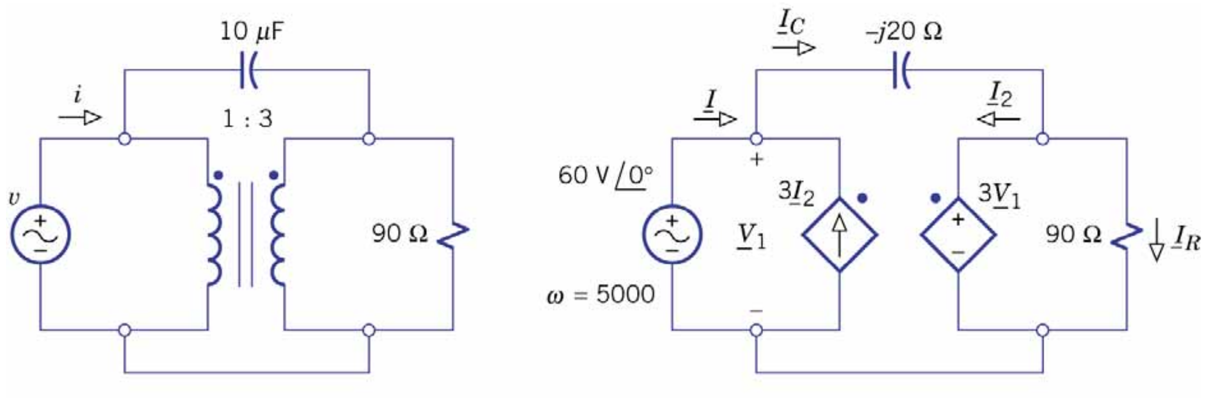
(b) Controled–source model

$$\frac{v_1(t)}{v_2(t)} = \frac{N_1}{N_2}, \frac{v_1(t)}{v_3(t)} = \frac{N_1}{N_3}$$

$$p(t) = v_1(t)i_1(t) + v_2(t)i_2(t) + v_3(t)i_3(t) = 0$$

$$\rightarrow i_1(t) = -\frac{v_2}{v_1} i_2 - \frac{v_3}{v_1} i_3 = -\frac{N_2}{N_1} i_2 - \frac{N_3}{N_1} i_3$$

2. Ex.8.1 find $\underline{I}, P$ from source



(a) Circuit with an ideal transformer

(b) Frequency–domain diagram

$$\underline{I} = \underline{I}_C - 3\underline{I}_2 = \underline{I}_C - 3(\underline{I}_C - \underline{I}_R) = -2\times\frac{\underline{V}_1 - 3\underline{V}_1}{-j20} + 3\times\frac{3\underline{V}_1}{90} = 6 + j12 = \sqrt{180}\angle 63.4^\circ$$

$$Z = \frac{\underline{V}_1}{\underline{I}} = 2 - j4 \rightarrow P = \frac{1}{2}\left|I\right|^2 \mathrm{Re}[Z] = 180$$

$$P_R = \frac{1}{2} I_R^2 R = 180 = P : \text{lossless}$$

3. Ex.8.2 use transformer to improve η

$$(a): P = 15k, P = \frac{1}{2}\left|\underline{I}_{out}\right|^2 R$$

$$\rightarrow \left|\underline{I}_{out}\right| = 100, \left|\underline{V}_{out}\right| = 3 \times \left|\underline{I}_{out}\right| = 300$$

$$\rightarrow \left|\underline{I}\right| = 100, \left|\underline{V}_s\right| = 5 \times \left|\underline{I}\right| = 500, \eta = \frac{3}{2+3} = 60\%$$

(b): both ends referred to the middle

$$\begin{cases} \underline{V}_s = 500 \\ \underline{I}_{in} \end{cases} \rightarrow \begin{matrix} 500\,N \\ \underline{I}_{in} / N \end{matrix}, \underline{V}_a = N\underline{V}_{in}$$

$$\begin{cases} 3\Omega \\ \underline{I}_{out} = 100 \end{cases} \rightarrow \begin{matrix} 4^2 \times 3\Omega \\ \underline{I}_{out} / 4 = 25 \end{matrix}, \underline{V}_b = 4\underline{V}_{out} = 1200$$

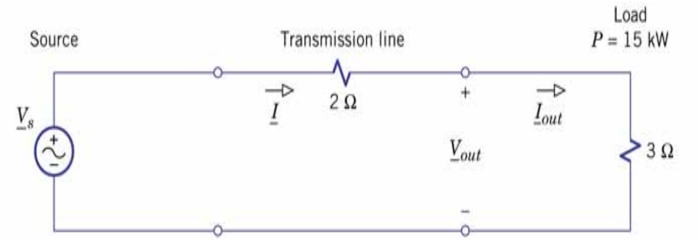
$$\eta = \frac{48}{48+2} = 96\%$$ increases

$$\underline{I} = \frac{500\,N}{2+48} = 25 \rightarrow N = 2.5$$

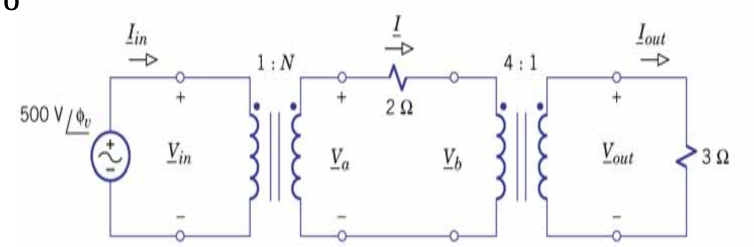
decreases

$$\underline{V}_a = 2\underline{I} + \underline{V}_b = 1250$$

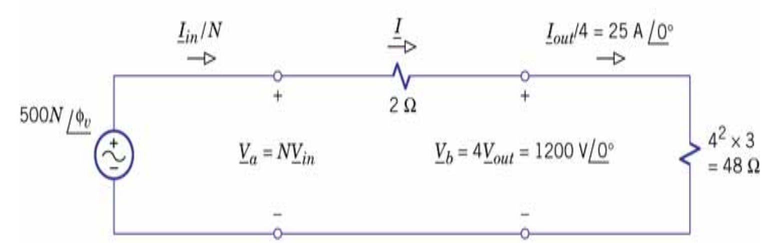
increases



(a) Power transfer via transmission line



(b) Circuit with transformers at each end



(c) Frequency–domain diagram with both ends referred into the middle

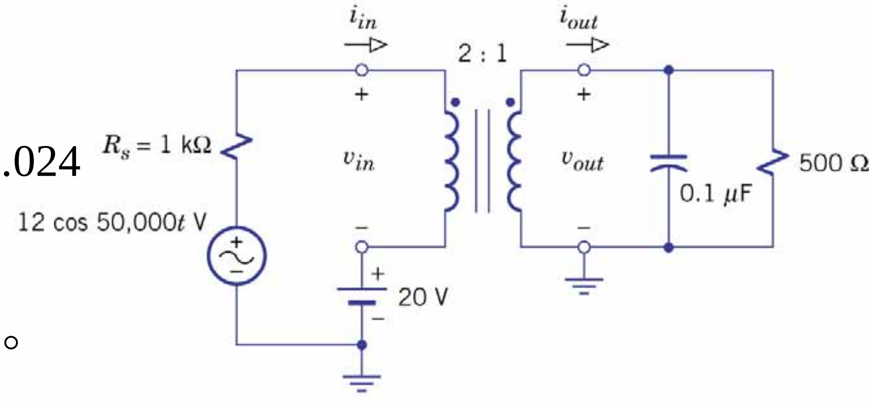
4. Ex. 8.3 oscillator application for ac signal coupling

$$N = \frac{1}{2} \rightarrow \text{referred source network}$$

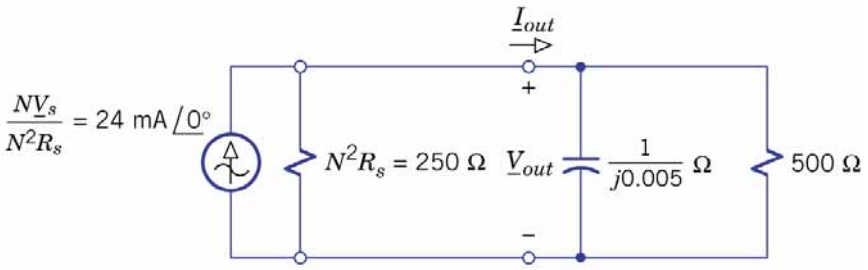
$$KCL \rightarrow \left(\frac{1}{250} + j0.005 + \frac{1}{500}\right)\underline{V}_{out} = 0.024$$

$$\rightarrow \underline{V}_{out} = 3\angle -40°$$

$$\underline{I}_{out} = \left(j0.005 + \frac{1}{500}\right)\underline{V}_{out} = 0.016\angle 20°$$



(a) Transformer circuit with ac and dc sources



(b) Frequency–domain diagram with referred source network in the secondary

5. Ex. 8.4 use transformer for maximum power transfer, find N and X for P_{max}

$$|\underline{V}_s| = |\underline{I}_s Z_s| = 18$$

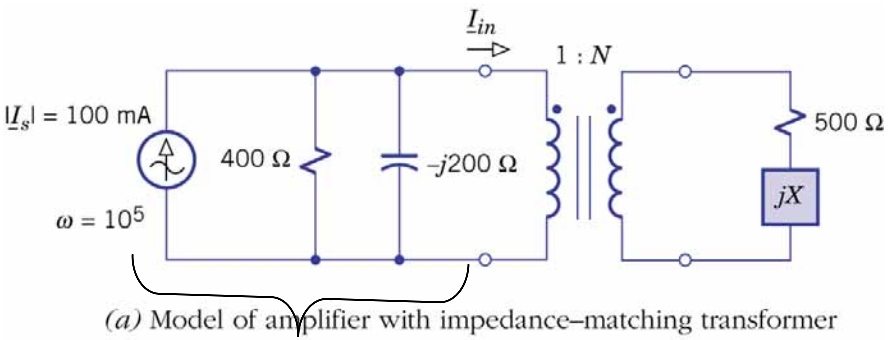
$$Z_s = 400 // -j200 = 80 - j160$$

$$\frac{Z}{N^2} = \frac{500 + jX}{N^2} = Z_s^* = 80 + j160$$

$$\rightarrow N = \sqrt{\frac{500}{80}} = 2.5$$

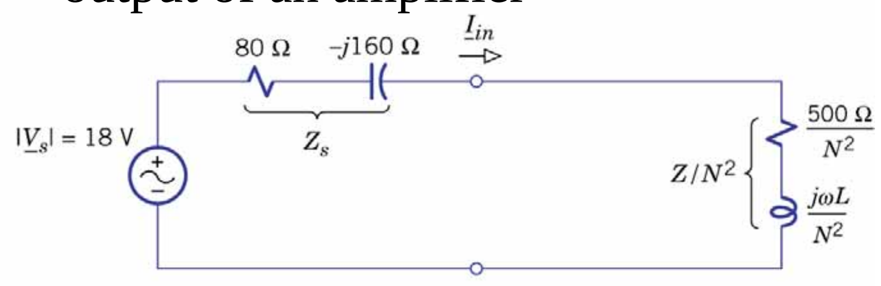
$$\frac{wL}{2.5^2} = 160, L = 0.01$$

$$P_{\max} = \frac{1}{4}\frac{V_{rms}^{\ 2}}{R} = \frac{1}{4}\frac{\left(\frac{18}{\sqrt{2}}\right)^2}{80} = 0.51$$



(a) Model of amplifier with impedance–matching transformer

output of an amplifier



(b) Referred load in the primary

Transformer is a practical component as a lossless impedance transformer.

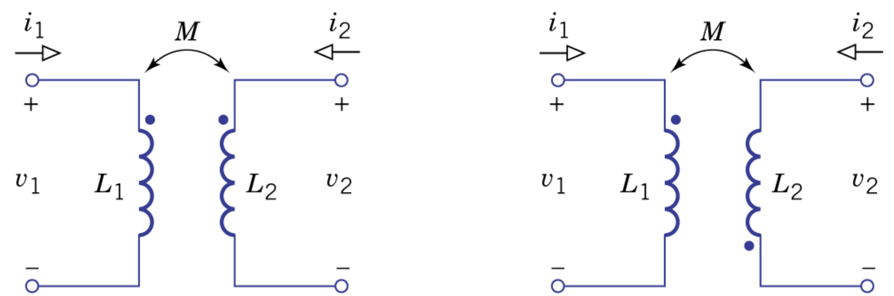
8.2 Magnetic coupling and mutual inductance

Basics

1.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} L_1 & \pm M \\ \pm M & L_2 \end{bmatrix} \begin{bmatrix} \dfrac{di_1}{dt} \\ \dfrac{di_2}{dt} \end{bmatrix}$$

$$\begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \end{bmatrix} = jw \begin{bmatrix} L_1 & \pm M \\ \pm M & L_2 \end{bmatrix} \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$$



(a) The same winding sense (b) Opposite winding sense

$$\boxed{\frac{L_2}{L_1} = \left(\frac{N_2}{N_1}\right)^2, M = k\sqrt{L_1 L_2}, 0 \le k \le 1, k : \text{coupling coefficien t}}$$

$$p(t) = v_1 i_1 + v_2 i_2 = L_1 i_1 \frac{di_1}{dt} + M i_1 \frac{di_2}{dt} + M i_2 \frac{di_1}{dt} + L_2 i_2 \frac{di_2}{dt}$$

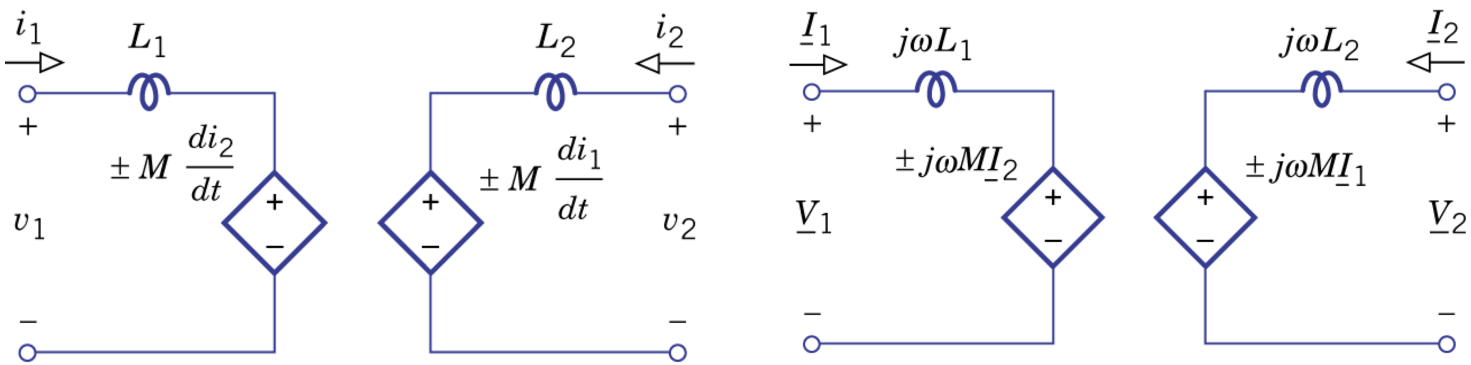
$$= L_1 i_1 \frac{di_1}{dt} + M \frac{di_1 i_2}{dt} + L_2 i_2 \frac{di_2}{dt}$$

$$w(t) = \int p(t)dt = \frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2$$

8.3 Circuits with mutual inductance

<u>Basics</u>

1. Frequency-domain model



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} L_1 & \pm M \\ \pm M & L_2 \end{bmatrix} \begin{bmatrix} \dfrac{di_1}{dt} \\ \dfrac{di_2}{dt} \end{bmatrix} \rightarrow \begin{bmatrix} \underline{V}_1 \\ \underline{V}_2 \end{bmatrix} = \begin{bmatrix} jwL_1 & \pm jwM \\ \pm jwM & jwL_2 \end{bmatrix} \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$$

2. Impedance analysis

$\underline{I}_2 = -\underline{I}_{out}$

loop 1: $jwL_1\underline{I}_{in} - jwM\,\underline{I}_{out} = \underline{V}_{in}$

loop 2: $(jwL_2 + Z)\underline{I}_{out} = jwM\,\underline{I}_{in}$

$$\rightarrow \frac{\underline{I}_{out}}{\underline{I}_{in}} = \frac{jwM}{jwL_2 + Z}$$

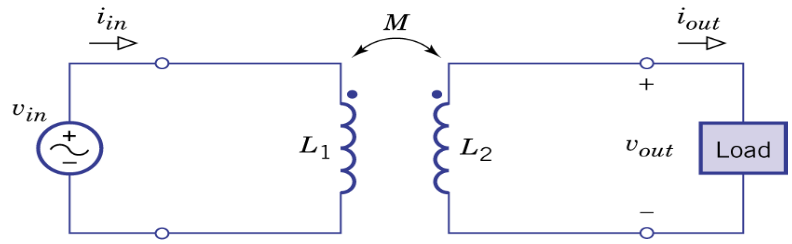
$$Z_{in} = \frac{\underline{V}_{in}}{\underline{I}_{in}} = jwL_1 - jwM\,\frac{\underline{I}_{out}}{\underline{I}_{in}}$$

$$= jwL_1 + \frac{(wM)^2}{jwL_2 + Z}$$

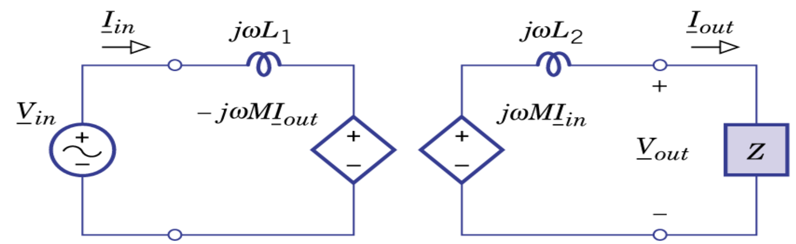
$$= \frac{-w^2L_1L_2 + jwL_1Z + (wM)^2}{jwL_2 + Z}$$

$$\frac{\underline{V}_{out}}{\underline{V}_{in}} = \frac{Z\,\underline{I}_{out}}{Z_{in}\,\underline{I}_{in}}$$

$$= \frac{jwMZ}{-w^2L_1L_2 + jwL_1Z + (wM)^2}$$



(a) Transformer interfacing a source and a load



(b) Frequency–domain diagram with controlled sources

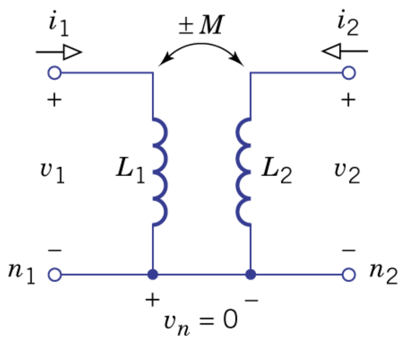
$$\text{if } k = 1, \frac{L_2}{L_1} = \left(\frac{N_2}{N_1}\right)^2 = N^2 \rightarrow M = \sqrt{L_1L_2} = NL_1 = \frac{L_2}{N}$$

$$\rightarrow \frac{\underline{I}_{out}}{\underline{I}_{in}} = \frac{M}{L_2}\frac{1}{1 + Z/jwL_2} = \frac{1}{N}\frac{1}{1 + Z/jwL_2} \approx \frac{1}{N}, \text{if } wL_2 >> Z$$

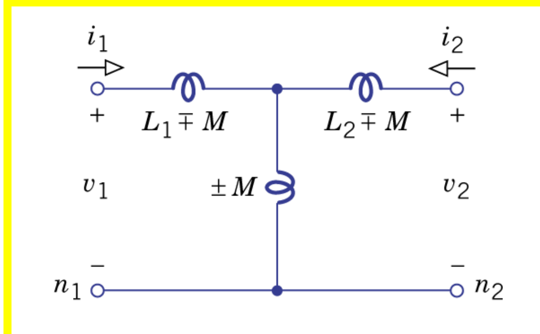
$$Z_{in} = \frac{jwL_1Z}{jwL_2 + Z} = \frac{L_1Z/L_2}{1 + Z/jwL_2} = \frac{1}{N^2}\frac{Z}{1 + Z/jwL_2} \approx \frac{Z}{N^2}$$

$$\frac{\underline{V}_{out}}{\underline{V}_{in}} = \frac{jwMZ}{jwL_1Z} = \frac{M}{L_1} = \frac{N_2}{N_1} = N, \text{"ideal transform er"}$$

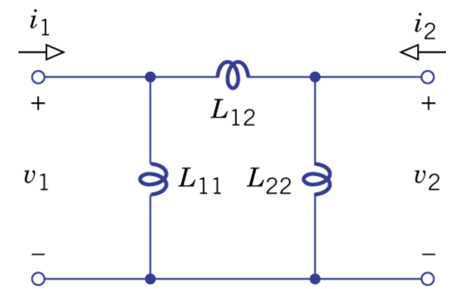
3. T- and π -equivalent networks



(a) Magnetically coupled coils



(b) Equivalent tee (T) network



T – network :

$$KVL \rightarrow \begin{array}{l} \underline{V}_1 = jw(L_1 \mp M)\underline{I}_1 \pm jwM(\underline{I}_1 + \underline{I}_2) = jwL_1\underline{I}_1 \pm jwM\,\underline{I}_2 \\ \underline{V}_2 = jw(L_2 \mp M)\underline{I}_2 \pm jwM(\underline{I}_1 + \underline{I}_2) = \pm jwM\,\underline{I}_1 \pm jwL_2\underline{I}_2 \end{array}$$

$$Y - \Delta \text{ transformation}(4.37): R_{ab} = \frac{R_a R_b + R_b R_c + R_c R_a}{R_c}$$

π – network :

$$(L_1 \mp M)(\pm M) + (\pm M)(L_2 \mp M) + (L_2 \mp M)(L_1 \mp M) = L_1 L_2 - M^2$$

$$L_{11} = \frac{L_1 L_2 - M^2}{L_2 \mp M}, L_{12} = \frac{L_1 L_2 - M^2}{\pm M}, L_{22} = \frac{L_1 L_2 - M^2}{L_1 \mp M}$$

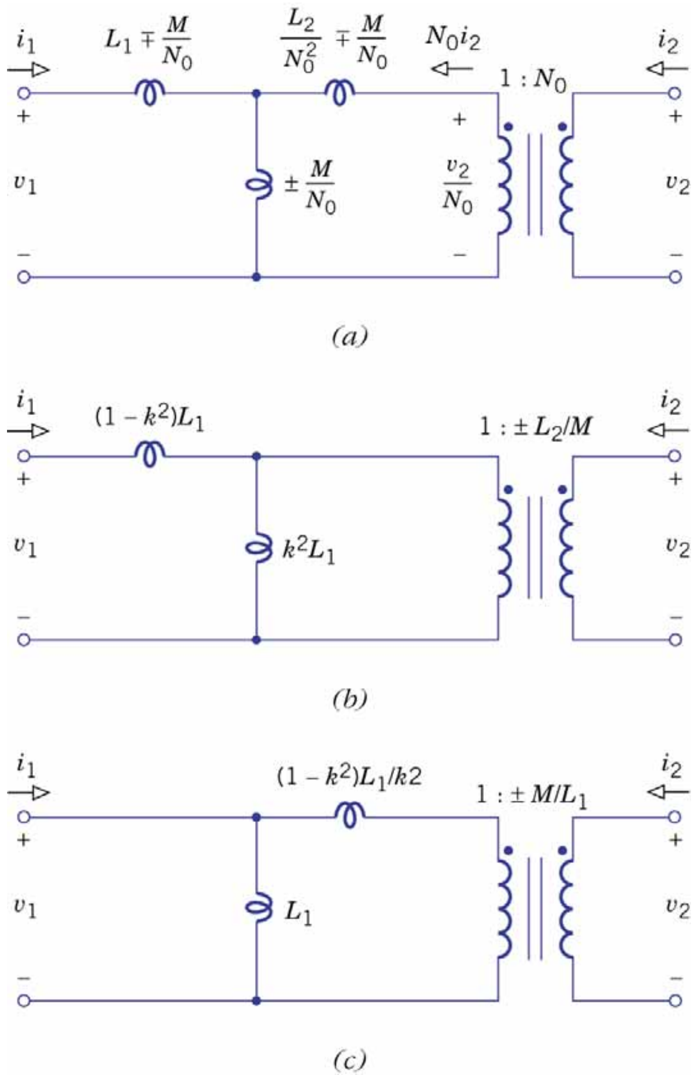
4. Equivalent circuits

if $N_o = \pm \dfrac{L_2}{M}$

$$\rightarrow \begin{array}{c} \mp \dfrac{M}{N_o} = -\dfrac{M^2}{L_2} = -k^2 L_1 \\ \dfrac{L_2}{N_o^2} \mp \dfrac{M}{N_o} = 0 \end{array} \Rightarrow$$

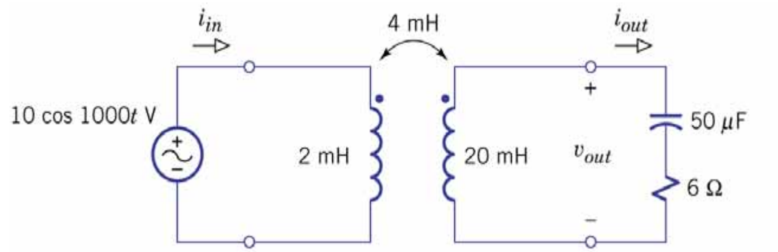
if $N_o = \pm \dfrac{M}{L_1}$

$$\rightarrow \begin{array}{c} L_1 \mp \dfrac{M}{N_o} = 0 \\ \dfrac{L_2}{N_o^2} = \dfrac{L_1}{k^2} \end{array} \Rightarrow$$

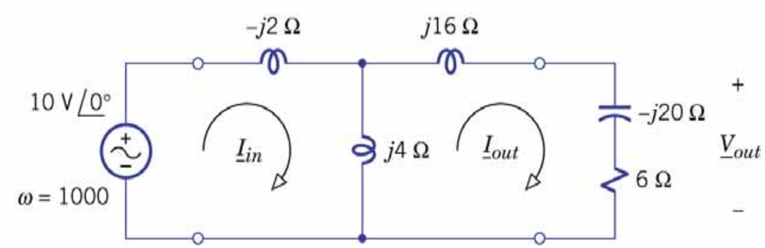


Discussion

1. Ex. 8.7 find $\underline{I}_{in}, \underline{I}_{out}, \underline{V}_{out}$



(a) Circuit with magnetic coupling



(b) Frequency–domain diagram with the equivalent tee network

$$L_1 - M = -2mH, L_2 - M = 16mH$$

mesh equation, $\begin{bmatrix} -j2 + j4 & -j4 \\ -j4 & j4 + j16 - j20 + 6 \end{bmatrix} \begin{bmatrix} \underline{I}_{in} \\ \underline{I}_{out} \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

$$\underline{I}_{in} = 3\angle -36.9°, \underline{I}_{out} = 2\angle 53.1°, \underline{V}_{out} = (6 - j20)\underline{I}_{out} = 41.8\angle -20.2°$$

2. Step-up autotransformer, find $\dfrac{\underline{I}_{out}}{\underline{I}_{in}}, \dfrac{\underline{V}_{out}}{\underline{V}_{in}}$

mesh equation

$$\begin{bmatrix} jwL_1 & -jw(L_1+M) \\ -jw(L_1+M) & jw(L_1+2M+L_2)+Z \end{bmatrix}\begin{bmatrix} \underline{I}_{in} \\ \underline{I}_{out} \end{bmatrix} = \begin{bmatrix} \underline{V}_{in} \\ 0 \end{bmatrix}$$

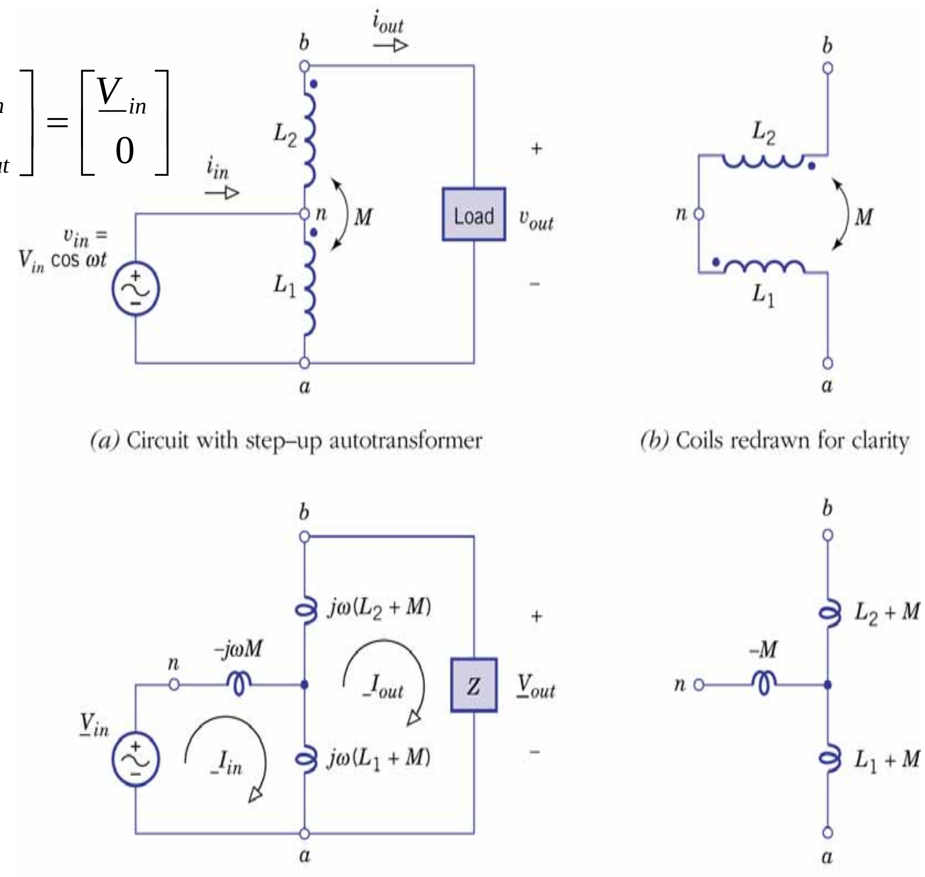
$$\frac{\underline{I}_{out}}{\underline{I}_{in}} = \frac{jw(L_1+M)}{jw(L_1+2M+L_2)+Z}$$

$$\frac{\underline{V}_{out}}{\underline{V}_{in}} = \frac{\underline{I}_{out}Z}{\underline{V}_{in}} \frac{jw(L_1+M)}{jwL_1Z - w^2(L_1L_2-M^2)}$$

if $\left|jw(L_1+2M+L_2)\right| >> |Z|, k=1$

$$\frac{\underline{I}_{out}}{\underline{I}_{in}} = \frac{L_1+M}{L_1+2M+L_2} = \frac{N_1}{N_1+N_2} \equiv \frac{1}{N}$$

$$\frac{\underline{V}_{out}}{\underline{V}_{in}} \approx \frac{L_1+M}{L_1} = \frac{N_1+N_2}{N_1} \equiv N$$



(a) Circuit with step–up autotransformer

(b) Coils redrawn for clarity

(d) Frequency–domain diagram

(c) Equivalent tee network

3. Tuned amplifier design

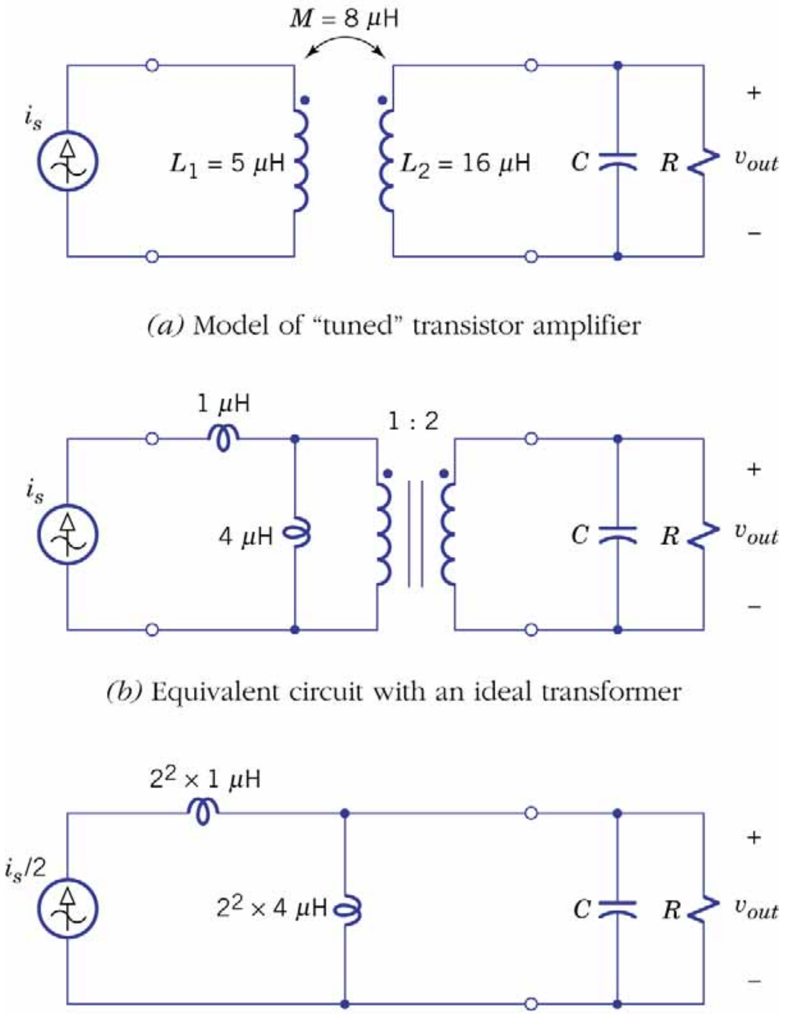
find R,C to give $w_o = 10^6, Q = 25$

$$k^2 = \frac{M^2}{L_1 L_2} = 0.8$$

$$\rightarrow (1-k^2)L_1 = 1, k^2 L_1 = 4, \frac{L_2}{M} = 2$$

$$w_o = \frac{1}{\sqrt{LC}} = 10^6 \rightarrow C = 62.5\,pF$$

$$Q_{par} = \frac{R}{w_o L} = 25 \rightarrow R = 400$$



(a) Model of "tuned" transistor amplifier

(b) Equivalent circuit with an ideal transformer

(c) Referred source network in the secondary