Chapter 6 Passband Data Transmission

6.1 Introduction

![Waveforms for signaling binary information](image)

**Figure 6.1** Illustrative waveforms for the three basic forms of signaling binary information. (a) Amplitude-shift keying. (b) Phase-shift keying. (c) Frequency-shift keying with continuous phase.

Hierarchy of Digital Modulation Techniques

Coherent, Noncoherent

M-ary ASK, M-ary PSK, M-ary FSK

QAM, DPSK

Probability of Error Evaluation

a. exact formulas  b. approximate formulas (union bound)
Power Spectra

a. bandwidth

b. cochannel interference in Multiplexing systems

Given a modulated signal \( s(t) \)

\[
s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t)
\]

\[
= \text{Re}\left[\tilde{s}(t) \exp(j2\pi f_c t)\right]
\] (6.1)

The complex envelope is

\[
\tilde{s}(t) = s_I(t) + js_Q(t)
\] (6.2)

when \( \frac{1}{2}w < f_c \)

Let \( s_B(f) \) be the baseband power spectral density

The PSD of \( s(t) \) is given by

\[
S_S(t) = \frac{1}{4}[S_B(f - f_c) + S_B(f + f_c)]
\] (6.4)

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Bandwidth Efficiency

\[ \rho = \frac{\text{data rate}}{\text{bandwidth}} \]

\[ = \frac{R_b}{B} \text{ bits/s/Hz} \quad (6.5) \]

\( \rho \) is affected by levels of encoding and spectral shaping.

6.2 Passband Transmission Model

Let \( P(m_1), P(m_2), \ldots, P(m_M) \) be the a priori prob. of \( m_1, m_2, \ldots, m_M \)

When \( M \) symbols are equally likely.

\[ p_i = P(m_i) \]

\[ = \frac{1}{M} \quad \text{for all } i \quad (6.6) \]
The energy of $s_i(t)$ is

$$E_i = \int_0^T s_i^2(t) dt , i = 1,2,...,M \quad (6.7)$$

Channel Model

a. Linear channel with enough bandwidth, the transmission of $s_i(t)$ is no distortion

b. AWGN with zero mean and PSD $\frac{N_0}{2}$

The receiver recovers the original message by computing the estimate $\hat{m}$ with minimum noise effect
6.3 Coherent Phase-Shift Keying

BPSK

\[ s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \quad \text{for Symbol "1"} \quad (6.8) \]

\[ s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) \quad \text{for Symbol "0"} \]

\[ = -s_1(t) \quad (6.9) \]

where \( f_c = \frac{n_c}{T_b} \), \( n_c \) is an integer

\( s_1(t) \) and \( s_2(t) \) are called antipodal signals (180°)

From (6.8) and (6.9) we clearly have only one basis function

\[ \phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \quad , \quad 0 \leq t < T_b \quad (6.10) \]
Figure 6.3 Signal-space diagram for coherent binary PSK system. The waveforms depicting the transmitted signals $s_1(t)$ and $s_2(t)$, displayed in the inserts, assume $n_c = 2$. 

BER of BPSK
Figure 6.3 Signal-space diagram for coherent binary PSK system. The waveforms depicting the transmitted signals \( s_1(t) \) and \( s_2(t) \), displayed in the inserts, assume \( n_c = 2 \).

\[
\begin{align*}
  s_1(t) &= \sqrt{E_b} \phi_1(t) , \\
  s_2(t) &= -\sqrt{E_b} \phi_1(t)
\end{align*}
\]

\[
\begin{align*}
  s_{11} &= \int_0^{T_b} s_1(t) \phi_1(t) dt = \sqrt{E_b} \\
  s_{21} &= \int_0^{T} s_2(t) \phi_1(t) dt = -\sqrt{E_b}
\end{align*}
\]
Desicion Regions

\[ Z_1 : \ 0 < x_1 < \infty \]
\[ Z_2 : -\infty < x_1 < 0 \]

\[ x_1 = \int_0^{T_b} x(t)\phi_1(t)dt \quad (6.15) \]

\[
\begin{align*}
    f_{x_1}(x_1|0) &= \frac{1}{\sqrt{\pi N_0}} \exp \left[ -\frac{1}{N_0} (x_1 - s_{21})^2 \right] \\
    &= \frac{1}{\sqrt{\pi N_0}} \exp \left[ -\frac{1}{N_0} (x_1 + \sqrt{E_b})^2 \right] \quad (6.16)
\end{align*}
\]
\[ p_{10} = P(\text{estimated } 1 \mid 0 \text{ sent}) \]
\[ = \int_{0}^{\infty} f_{x_1}(x_1 \mid 0) dx_1 \]
\[ = \frac{1}{\sqrt{\pi} N_0} \int_{0}^{\infty} \exp \left[ -\frac{1}{N_0} (x_1 + \sqrt{E_b})^2 \right] dx_1 \]
\[ = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{E_b / N_0}} \exp(-z^2) dz \]
\[ = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) \quad (6.19) \]

where \( z = \frac{1}{\sqrt{N_0}} (x_1 + \sqrt{E_b}) \),

\( x_1 = 0, \ z = \sqrt{\frac{E_b}{N_0}}, \ x = \infty, \ z = \infty \)

Similarly \( P_{01} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) \)

\[ p_e = \frac{1}{2} p_{01} + \frac{1}{2} p_{10} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) \quad (6.20) \]
Generation and Detection of Coherent BPSK

Figure 6.4 Block diagrams for (a) binary PSK transmitter and (b) coherent binary PSK receiver.
Recall Examples 1.3 and 1.6

\[
R_x = \begin{cases} 
A^2 (1 - \frac{|\tau|}{T}) & |\tau| < T \\
0 & |\tau| \geq T 
\end{cases}
\]

\[
S_x(f) = A^2 T \sin^2 (fT) \quad (1.50)
\]

Let \( T = T_b \), \( A = \sqrt{\frac{2E_b}{T_b}} \)

The PSD of BPSK

\[
S_B(f) = \left( \frac{2E_b}{T_b} \right) T_b \frac{\sin^2 (\pi T_b f)}{(\pi T_b f)^2}
\]

\[
= 2E_b \sin^2 (T_b f) \quad (6.22)
\]
Figure 6.5 Power spectra of binary PSK and FSK signals.
Quadriphase - Shift Keying (QPSK)

\[ S_i(t) = \begin{cases} 
\sqrt{\frac{2E}{T}} \cos \left[ 2\pi f_c t + (2i-1)\frac{\pi}{4} \right], & 0 \leq t \leq T, \\ 0, & \text{elsewhere} \end{cases} \]

\[ i = 1, 2, 3, 4 \quad (6.23) \]

**Figure 6.6** Signal-space diagram of coherent QPSK system.
Signal-Space Diagram of QPSK

\[ S_i(t) = \sqrt{\frac{2E}{T}} \cos \left[ 2\pi f_c t + (2i-1)\frac{\pi}{4} \right] \]

\[ = \sqrt{\frac{2E}{T}} \cos \left( 2i-1 \frac{\pi}{4} \right) \cos (2\pi f_c t) - \sqrt{\frac{2E}{T}} \cos \left( 2i-1 \frac{\pi}{4} \right) \sin (2\pi f_c t) \] (6.24)

The basis functions are 

\[ \phi_i(t) = \sqrt{\frac{2}{T}} \cos (2\pi f_c t), \quad 0 \leq t < T \] (6.25)

\[ \phi_2(t) = \sqrt{\frac{2}{T}} \sin (2\pi f_c t), \quad 0 \leq t < T \] (6.26)

The signal vectors are 

\[ S_i = \begin{bmatrix} \sqrt{E} \cos \left( 2i-1 \frac{\pi}{4} \right) \\ -\sqrt{E} \sin \left( 2i-1 \frac{\pi}{4} \right) \end{bmatrix} \quad i = 1, 2, 3, 4 \] (6.27)

<table>
<thead>
<tr>
<th>TABLE 6.1</th>
<th>Signal-space characterization of QPSK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grey-encoded Input Digit</td>
<td>Phase of QPSK Signal (radians)</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>( \pi/4 )</td>
</tr>
<tr>
<td>00</td>
<td>3( \pi/4 )</td>
</tr>
<tr>
<td>01</td>
<td>5( \pi/4 )</td>
</tr>
<tr>
<td>11</td>
<td>7( \pi/4 )</td>
</tr>
</tbody>
</table>
Figure 6.7 illustrates the sequences and waveforms involved in the generation of a QPSK signal. The input binary sequence 01101000 is shown in Figure 6.7a. This sequence is divided into two other sequences, consisting of odd- and even-numbered bits of the input sequence. These two sequences are shown in the top lines of Figures 6.7b and 6.7c. The waveforms representing the two components of the QPSK signal, namely, \( s_1 \phi_1(t) \) and \( s_2 \phi_2(t) \), are also shown in Figures 6.7b and 6.7c, respectively. These two waveforms may individually be viewed as examples of a binary PSK signal. Adding them, we get the QPSK waveform shown in Figure 6.7d.

To define the decision rule for the detection of the transmitted data sequence, we partition the signal space into four regions, in accordance with Equation (5.59) of Chapter 5. The individual regions are defined by the set of points closest to the message point represented by signal vectors \( s_1, s_2, s_3, \) and \( s_4 \). This is readily accomplished by constructing the perpendicular bisectors of the square formed by joining the four message points and then marking off the appropriate regions. We thus find that the decision regions are quadrants whose vertices coincide with the origin. These regions are marked \( Z_1, Z_2, Z_3, \) and \( Z_4 \), in Figure 6.6, according to the message point around which they are constructed.
Error Probability of QPSK

\[ x(t) = s_i(t) + w(t) \]  \hspace{1cm} (6.28)

\[ x_1 = \int_0^T x(t) \phi_1(t) \, dt \]

\[ = \sqrt{E} \cos[(2i - 1) \frac{\pi}{4}] + w_1 \]

\[ = \pm \sqrt{\frac{E}{2}} + w_1 \]  \hspace{1cm} (6.29)

\[ x_2 = \int_0^T x(t) \phi_2(t) \, dt \]

\[ = -\sqrt{E} \sin[(2i - 1) \frac{\pi}{4}] + w_2 \]

\[ = \mp \sqrt{\frac{E}{2}} + w_2 \]  \hspace{1cm} (6.30)

A QPSK system is equivalent to two BPSK systems multiplexed together. Each BPSK system has the signal energy E/2 and noise PSD N_0/2.
The $p_c$ of each BPSK channel for the QPSK is

$$P' = \frac{1}{2} \text{erfc}(\sqrt{\frac{E}{2N_0}}) = \frac{1}{2} \text{erfc}(\sqrt{\frac{E}{2N_0}})$$  \hspace{1cm} (6.31)$$

The average probability of a correct decision for QPSK is

$$P_c = (1 - P')^2$$

$$= \left[ 1 - \frac{1}{2} \text{erfc}(\sqrt{\frac{E}{2N_0}}) \right]^2$$

$$= 1 - \text{erfc}(\sqrt{\frac{E}{2N_0}}) + \frac{1}{4} \text{erfc}^2(\sqrt{\frac{E}{2N_0}})$$

$$P_c = 1 - P_c = \text{erfc}(\sqrt{\frac{E}{2N_0}})$$  \hspace{1cm} (6.34)$$

Since the symbol energy is twice the bit energy

$$E = 2E_b$$

$$P_c = \text{erfc}(\sqrt{\frac{E_b}{N_0}})$$

With Gray coding, BER

$$\text{BER} = \frac{1}{2} \text{erfc}(\sqrt{\frac{E_b}{N_0}})$$  \hspace{1cm} (6.38)$$
Figure 6.8 Block diagrams of (a) QPSK transmitter and (b) coherent QPSK receiver.
Power Spectra of QPSK Signals

1. The in-phase and quadrature components equal $+ g(t)$ or $- g(t)$, $g(t)$ denotes the symbol shaping function.

$$g(t) = \sqrt{\frac{E}{T}} \quad 0 \leq t \leq T$$  \hspace{1cm} (6.39)

They have a common PSD, $E \sin c^2 (Tf)$

2. They are statistically independent

$$S_B (f) = 2E \sin c^2 (Tf) \quad \text{The baseband QPSK PSD}$$

$$= 4E_b \sin c^2 (2T_b f) \quad \text{equals the sum of the inphase and quadrature PSD}$$
Figure 6.9 Power spectra of QPSK and MSK signals.
The ordinary QPSK

1. The carrier changes ±180°, when both components change sign (e.g. 01→10, Fig 6.7)
2. The carrier changes ±90°, when one component changes sign (e.g. 10→00, Fig 6.7)
3. The carrier does not change when no component changes (e.g. 10→10, Fig 6.7)

⇒ The carrier changes may cause the carrier amplitude changes, after filtered.

**Figure 6.7** (a) Input binary sequence. (b) Odd-numbered bits of input sequence and associated binary PSK wave. (c) Even-numbered bits of input sequence and associated binary PSK wave. (d) QPSK waveform defined as $s(t) = s_{11}f_1(t) + s_{12}f_2(t)$. 
Offset QPSK (Reducing Carrier Amplitude Change)

Figure 6.10 Possible paths for switching between the message points in (a) QPSK and (b) offset QPSK.
The basis functions of offset QPSK

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) , \ 0 \leq t \leq T \quad (6.41) \]

\[ \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) , \ \frac{T}{2} \leq t \leq \frac{3T}{2} \quad (6.42) \]

(compare to 6.25, 6.26)

The QPSK carrier change is confined to ±90°

⇒ Smaller amplitude fluctuation

The offset QPSK has the same error performance as QPSK

\[ P_e \approx \text{erfc}\left( \sqrt{\frac{E}{2N_0}} \right) \quad (6.34) \]
$\pi/4$-shifted QPSK

Two ordinary QPSK constellations

Figure 6.11 Two commonly used signal constellations of QPSK; the arrows indicate the paths along which the QPSK modulator can change its state.
\(\pi/4\)-shifted QPSK picks the carrier phase for successive symbols alternately from one of these two costellations.

![Diagram of phase states for \(\pi/4\)-shifted QPSK modulator.]

**Figure 6.12** Eight possible phase states for the \(\pi/4\)-shifted QPSK modulator.

<table>
<thead>
<tr>
<th>GrayEncoded Input Dibit</th>
<th>Phase Change, (\Delta \theta) (radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>(\pi/4)</td>
</tr>
<tr>
<td>01</td>
<td>(3\pi/4)</td>
</tr>
<tr>
<td>11</td>
<td>(-3\pi/4)</td>
</tr>
<tr>
<td>10</td>
<td>(-\pi/4)</td>
</tr>
</tbody>
</table>
Attractive features of the $\frac{\pi}{4}$-shifted QPSK

1. Phase changes $\pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$

2. Can be noncoherently detected

Generation of $\frac{\pi}{4}$-shifted QPSK

$$S(t) = \sqrt{\frac{2E}{T}} \cos(i \frac{\pi}{4}) \cos(2\pi f_c t)$$

$$- \sqrt{\frac{2E}{T}} \sin(i \frac{\pi}{4}) \sin(2\pi f_c t), \ i = 0, 1, ..., 7$$

The phase states of the $\frac{\pi}{4}$-shifted QPSK modulator are divided into two QPSK groups
Define

\[ S_I(t) = \sqrt{\frac{2E}{T}} \cos(i \frac{\pi}{4}) \quad , \quad S_Q(t) = \sqrt{\frac{2E}{T}} \sin(i \frac{\pi}{4}) \]

The basis functions for \( \frac{\pi}{4} \)-shifted QPSK are

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad , \quad \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \]

\[ S(t) = \sqrt{E} \cos(i \frac{\pi}{4}) \phi_1(t) - \sqrt{E} \sin(i \frac{\pi}{4}) \phi_2(t) \]
When $\frac{\pi}{4}$-shifted QPSK is differentially encoded, we call DQPSK.

Let $\Delta \theta_k$ denote the differentially encoded phase change.

\begin{align*}
S(t) &= \sqrt{\frac{2E}{T}} \cos(\theta_{k-1} + \Delta \theta_k) \cos(2\pi f_c t) \quad \theta_k = \theta_{k-1} + \Delta \theta_k \\
&\quad - \sqrt{\frac{2E}{T}} \sin(\theta_{k-1} + \Delta \theta_k) \sin(2\pi f_c t) \\
&= \sqrt{\frac{2E}{T}} \left[ \cos \theta_{k-1} \cos \Delta \theta_k - \sin \theta_{k-1} \sin \Delta \theta_k \right] \cos(2\pi f_c t) \\
&\quad - \sqrt{\frac{2E}{T}} \left[ \sin \theta_{k-1} \cos \Delta \theta_k + \cos \theta_{k-1} \sin \Delta \theta_k \right] \sin(2\pi f_c t)
\end{align*}

let

\begin{align*}
I_k &= \cos(\theta_{k-1} + \Delta \theta_k) = \cos \theta_k \quad (6.43) \\
Q_k &= \sin(\theta_{k-1} + \Delta \theta_k) = \sin \theta_k \quad (6.44)
\end{align*}

\[s(t) = \sqrt{\frac{2E}{T}} I_k \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} Q_k \sin(2\pi f_c t)\]
\[ I_k = \cos \Theta_{k-1} \cos \Delta \Theta_k - \sin \Theta_{k-1} \sin \Delta \Theta_k \]
\[ = \cos(\Theta_{k-2} + \Delta \Theta_{k-1}) \cos \Delta \Theta_k - \sin(\Theta_{k-2} + \Delta \Theta_{k-1}) \sin \Delta \Theta_k \]
From which we have the recursive form
\[ I_k - I_{k-1} \cos \Delta \Theta_k = Q_{k-1} \sin \Delta \Theta_k \]
Similarly
\[ Q_k = Q_{k-1} \cos \Delta \Theta_k + I_{k-1} \sin \Delta \Theta_k \]
From the definition of \( I_k \) and \( Q_k \)
\[ I_k = \cos \Theta_k, Q_k = \sin \Theta_k \]

<table>
<thead>
<tr>
<th>TABLE 6.2 Correspondence between input dibilit and phase change for ( \pi/4 )-shifted DQPSK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gray-Encoded Input Dibit</strong></td>
</tr>
<tr>
<td>00</td>
</tr>
<tr>
<td>01</td>
</tr>
<tr>
<td>11</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>
EXAMPLE 6.2

Continuing with the input binary sequence of Example 6.1, namely, 01101000, suppose that the phase angle $\theta_0 = \pi/4$ in the constellation of Figure 6.11b is assigned as the initial phase state of the $\pi/4$-shifted DQPSK modulator. Then, arranging the input binary sequence as a sequence of dibits and following the convention of Table 6.2, we get the results presented in Table 6.3 for the example at hand.

<table>
<thead>
<tr>
<th>TABLE 6.2 Correspondence between input dibit and phase change for $\pi/4$-shifted DQPSK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gray-Encoded Input Dibit</td>
</tr>
<tr>
<td>-------------------------</td>
</tr>
<tr>
<td>00</td>
</tr>
<tr>
<td>01</td>
</tr>
<tr>
<td>11</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TABLE 6.3 $\pi/4$-shifted DQPSK results for Example 6.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step $k$</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

where $\theta_{k-1}$ is the absolute phase angle of symbol $k - 1$, and $\Delta \theta_k$ is the differentially encoded phase change defined in accordance with Table 6.2.
Let \( \Delta \theta_k = \theta_k - \theta_{k-1} \)

The correction logic operates as

If \( \Delta \theta_k < -180^\circ \) then \( \Delta \theta_k = \Delta \theta_k + 360^\circ \)

If \( \Delta \theta_k > 180^\circ \) then \( \Delta \theta_k = \Delta \theta_k - 360^\circ \)

Example \( \theta_{k-1} = 350^\circ \uparrow \) and \( \theta_k = 60^\circ \uparrow \)

\[ \Delta \theta_k = 60^\circ - 350^\circ = -290^\circ \]

The phase change is \( 70^\circ \) not \( 290^\circ \)

\[ \Delta \theta_k = -290^\circ + 360^\circ = 70^\circ \]

**Figure 6.14** Illustrating the possibility of phase angles wrapping around the positive real axis.
Detection of $\pi/4$-shifted DQPSK Signals

The detector has three features:

a. Extracting $\theta = \tan^{-1}\left(\frac{Q_k}{I_k}\right)$, $I_k = \cos \theta_k$, $Q_k = \sin \theta_k$

b. Determining phase change

c. Correcting errors due to phase wrapping

Figure 6.13 Block diagram of the $\pi/4$-shifted DQPSK detector.
M - ary PSK

\[ S_i(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M}(i - 1)\right), \quad i = 1, 2, \ldots, M \quad (6.46) \]

The Euclidean distance between two adjacent messages

\[ d_{\text{min}} = 2\sqrt{E} \sin\left(\frac{\pi}{M}\right) \]

Recall (5.92)

\[ P_e \leq \frac{1}{2} \sum_{k=1 \atop k \neq i}^{M} \text{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right), \quad \text{for all } i \]

The symbol error for coherent M - ary PSK is

\[ P_e \approx \text{erfc}\left(\sqrt{\frac{E}{N_0}} \sin\left(\frac{\pi}{M}\right)\right) \quad (6.47) \]

只計算鄰近兩個 points 所產生的 errors, 所以

\[ \frac{1}{2} \sum_{k=1 \atop k \neq i}^{M} \text{erfc}(\bullet) \approx \text{erfc}(\bullet) \]
Figure 6.15 (a) Signal-space diagram for octaphase-shift keying (i.e., $M = 8$). The decision boundaries are shown as dashed lines. (b) Signal-space diagram illustrating the application of the union bound for octaphase-shift keying.
Power Spectra of M-ary PSK

\[ T = T_b \log_2 M \]  \hspace{1cm} (6.48)

\[ S_B(f) = 2E \text{sinc}^2(Tf) \]

\[ = 2E_b \log_2 M \text{sinc}^2(T_b f \log_2 M) \]  \hspace{1cm} (6.49)

Figure 6.16 Power spectra of \( M \)-ary PSK signals for \( M = 2, 4, 8 \).
6.4 Hybrid Amplitude/Phase Modulation Schemes

M-ary QAM

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T \]  \hspace{1cm} (6.53)

\[ \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \leq t \leq T \]  \hspace{1cm} (6.54)

Let \( d_{\min}/2 = \sqrt{E_0} \)

\[ S_k(t) = \sqrt{\frac{2E_0}{T}} a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}} b_k \sin(2\pi f_c t), \quad k = 0, \pm 1, \pm 2, \ldots \]  \hspace{1cm} (6.55)

**QAM Square Constellation**

With even number of bits/symbol

\[ L = \sqrt{M} \]  \hspace{1cm} (6.56)

\( L \): one dimension constellation

\[ \left\{ a_i b_j \right\} = \begin{bmatrix} (-L+1,L-1), \cdots, (L-1,L-1) \\ (-L+1,L-3), \cdots, (L-1,L-3) \\ (-L+1,-L+1), \cdots, (L-1,-L+1) \end{bmatrix} \]  \hspace{1cm} (6.57)
Consider a 16-QAM whose signal constellation is depicted in Figure 6.17a. The encoding of the message points shown in this figure is as follows:

- Two of the four bits, namely, the left-most two bits, specify the quadrant in the $(\phi_1, \phi_2)$-plane in which a message point lies. Thus, starting from the first quadrant and proceeding counterclockwise, the four quadrants are represented by the dibits 11, 10, 00, and 01.
- The remaining two bits are used to represent one of the four possible symbols lying within each quadrant of the $(\phi_1, \phi_2)$-plane.

Note that the encoding of the four quadrants and also the encoding of the symbols in each quadrant follow the Gray coding rule.

![Figure 6.17](image)

**Figure 6.17** (a) Signal-space diagram of $M$-ary QAM for $M = 16$; the message points in each quadrant are identified with Gray-encoded quadbits. (b) Signal-space diagram of the corresponding 4-PAM signal.

For the example at hand, we have $L = 4$. Thus the square constellation of Figure 6.17a is the Cartesian product of the 4-PAM constellation shown in Figure 6.17b with itself. Moreover, the matrix of Equation (6.57) has the value

$$
\begin{bmatrix}
-3 & 3 & 1 & 3 \\
-3 & 1 & -1 & 1 \\
-3 & -1 & -1 & -1 \\
-3 & -3 & -1 & -3 \\
\end{bmatrix}
$$
Signal-constellation of M-ary QAM for $M = 16$. (The message points are identified with 4-bit Gray codes)
quadrature Gray mapping

in-phase Gray mapping

\( \phi_2 \)

\( \phi_1 \)

10 11 01 00

0010 0011 0001 0000 00

0110 0111 0101 0100 01

1110 1111 1101 1100 11

1010 1011 1001 1000 10
$P_e$ of $M$-ary QAM

1. The prob. of correction

$$P_c = (1 - P_e')^2$$

$P_e'$ is the prob. of symbol error of one-dim $L$-ary PAM

2. $P_e' = (1 - \frac{1}{\sqrt{M}}) \text{erfc}(\frac{A}{2\sqrt{2\sigma}})$ $\leftarrow$ problem 4.27

$$= (1 - \frac{1}{\sqrt{M}}) \text{erfc}(\sqrt{\frac{E_0}{N_0}}), \quad \begin{cases} A = 2\sqrt{E_0} \\ \sigma = \sqrt{N_0/2} \end{cases}$$

3. $P_e = 1 - P_c$

$$= 1 - (1 - P_e')^2$$

$$\approx 2P_e' \quad (6.60)$$
Symbol error probability $P_e$ as a function of $E_s/N_0$ for M-PSK: approximate error probability (4.261) and (4.269) in solid lines, and exact error probability (4.267) in dashed lines. The two sets of lines overlap.
Symbol error probability $P_e$ as a function of $E_s/N_0$ for $M$-QAM: approximated error probability (4.248) in solid lines, and exact error probability (4.247) in dashed lines.
\[ P_e \approx 2(1 - \frac{1}{\sqrt{M}}) \text{erfc}(\sqrt{\frac{E_0}{N_0}}) \]  

(6.61)

Assuming the levels of In-phase and quadrature components are equally likely.

Average energy per symbol is

\[ E_{av} = 2 \left[ \frac{2E_0}{L} \sum_{i=1}^{L/2} (2i-1)^2 \right] \]

\[ = \frac{2(L^2 - 1)E_0}{3} \]

(6.62)

\[ = \frac{2(M - 1)E_0}{3} \]

(6.63)

\[ P_e \approx 2(1 - \frac{1}{\sqrt{M}}) \text{erfc}(\sqrt{\frac{3E_{av}}{2(M - 1)N_0}}) \]

(6.64)
QAM Cross Constellation

$M=2^n$ where $n$ is odd (e.g., 5, 7, …)

Figure 6.18 Illustrating how a square QAM constellation can be expanded to form a QAM cross-constellation.

$$p_e \approx 2 \left(1 - \frac{1}{\sqrt{2M}}\right) \text{erfc} \left(\sqrt{\frac{E_b}{N_0}}\right)$$

(6.65)
6.5 Coherent FSK

BFSK

\[ s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) & 0 \leq t \leq T_b, \quad i = 1,2 \\ 0 & \text{elsewhere} \end{cases} \]  \hspace{1cm} (6.86)

\[ f_i = \frac{n_c + i}{T_b}, \quad n_c : \text{integer}, \quad i = 1,2 \]  \hspace{1cm} (6.87)

It is called Sunde's FSK, an example of continuous-phase FSK (CPFSK)

\[ \phi_i(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) \quad , \quad 0 \leq t \leq T_b , \quad i = 1,2 \]  \hspace{1cm} (6.88)

\[ s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)dt \]

\[ = \sqrt{\frac{2E_b}{T_b}} \sqrt{\frac{2}{T_b}} \int_0^{T_b} \cos(2\pi f_i t) \cos(2\pi f_j t)dt \]  \hspace{1cm} (6.89)

\[ = \sqrt{E_b} \delta_{ij} \]
BFSK has a two-dimensional signal space

\[ s_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix} \]

(6.90), (6.91)

**Figure 6.25** Signal-space diagram for binary FSK system. The diagram also includes two inserts showing example waveforms of the two modulated signals \( s_1(t) \) and \( s_2(t) \).
Error Probability of BFSK

\[ x_1 = \int_0^{T_b} x(t) \phi_1(t) \, dt \] 
\[ x_2 = \int_0^{T_b} x(t) \phi_2(t) \, dt \]  

(6.92)

(6.93)

Define a new Gaussian R.V. \( Y \).

\[ y = x_1 - x_2 \]  

(6.94)

Given sent 1, \( x_1 \) and \( x_2 \) have mean values equal to \( \sqrt{E_b} \) and zero

\[ E[Y|1] = E[x_1|1] - E[x_2|1] \]

\[ = +\sqrt{E_b} \]  

(6.95)

Similarly,

\[ E[Y|0] = E[x_1|0] - E[x_2|0] \]

\[ = -\sqrt{E_b} \]  

(6.96)
Since $x_1$ and $x_2$ are independent

$$
\sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 = N_0
$$

(6.97)

$$
f_Y(y \mid 0) = \frac{1}{\sqrt{2\pi N_0}} \exp \left[ - \frac{(y + \sqrt{E_b})^2}{2N_0} \right]
$$

(6.98)

$x_1 \perp x_2 \Rightarrow y \perp 0$

$$
p_{10} = P(y \mid 0 \mid 0 \text{ was sent})
$$

$$
= \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{\infty} \exp \left[ - \frac{(y + \sqrt{E_b})^2}{2N_0} \right] dy
$$

(6.99)

let $z = \frac{y + \sqrt{E_b}}{\sqrt{2N_0}}$

(6.100)

$$
p_{10} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-z^2)dz = \frac{1}{2} \text{erfc}(\sqrt{\frac{E_b}{2N_0}})
$$

(6.101)

For symmetric channels

$$
p_e = \frac{1}{2} p_{10} + \frac{1}{2} p_{01} = \frac{1}{2} \text{erfc}(\sqrt{\frac{E_b}{2N_0}})
$$

(6.102)
Generation and Detection of Coherent BFSK

Figure 6.26 Block diagrams for (a) binary FSK transmitter and (b) coherent binary FSK receiver.
Power Spectra of Binary FSK Signals

Consider the Sunde's FSK

\[ s(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t \pm \frac{\pi t}{T_b}), \quad 0 \leq t \leq T \]

\[ = \sqrt{\frac{2E_b}{T_b}} \cos(\pm \frac{\pi t}{T_b}) \cos(2\pi f_c t) \mp \sqrt{\frac{2E_b}{T_b}} \sin(\pm \frac{\pi t}{T_b}) \sin(2\pi f_c t) \]

\[ = \sqrt{\frac{2E_b}{T_b}} \cos(\frac{\pi t}{T_b}) \cos(2\pi f_c t) \mp \sqrt{\frac{2E_b}{T_b}} \sin(\frac{\pi t}{T_b}) \sin(2\pi f_c t) \]

1. The in-phase component is indep. of the input signal

⇒ two delta functions at \( f = \pm \frac{1}{2} T_b \), \( \cos(\frac{\pi t}{T}) \) for all \( t \)

2. The symbol shape function, \( g(t) \), is

\[ g(t) = \sqrt{\frac{2E_b}{T_b}} \sin(\frac{\pi t}{T_b}), \quad 0 \leq t \leq T_b \text{, (注意)} \]

The energy SD of \( g(t) \leftrightarrow \psi_g(f) = \frac{8E_b T_b \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2} \)
The Power Spectra of BFSK is

$$S_B(f) = \frac{E_b}{2T_b} \left[ \delta(f - \frac{1}{2T_b}) + \delta(f + \frac{1}{2T_b}) \right] + \frac{8E_b \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2} \quad (6.107)$$

Figure 6.5 Power spectra of binary PSK and FSK signals.
Minimum Shift Keying

In the coherent BFSK detection, the phase information is not fully exploited. We may use the phase information to improve the performance.

Consider a CPFSK signal

\[
s(t) = \begin{cases} 
\sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_1 t + \theta(0)] & \text{for "1"} \\
\sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_2 t + \theta(0)] & \text{for "0"}
\end{cases}
\]  

(6.108)

where \(\theta(0) = \theta(t)\big|_{t=0}\)

We may also express the CPFSK signal as

\[
s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_c t + \theta(t)]
\]

\(\theta(t)\) increase or decrease linearly with time in \(T_b\)

\[
\theta(t) = \theta(0) \pm \frac{\pi h}{T_b} t \quad , \quad 0 \leq t \leq T_b
\]  

(6.110)
+ and – correspond to "1" and "0", \( h \) is to be defined

\[
f_1 = f_c + \frac{h}{2T_b} \quad (6.111)
\]

\[
f_2 = f_c - \frac{h}{2T_b} \quad (6.112)
\]

\[
f_c = \frac{1}{2} (f_1 + f_2) \quad (6.113)
\]

\[
h = T_b (f_1 - f_2) \quad (6.114)
\]

where \( h \) is the deviation ratio

From (6.110), at \( t = T_b \)

\[
\theta(T_b) - \theta(0) = \begin{cases} 
\pi h & \text{for } "1" \\
-\pi h & \text{for } "0"
\end{cases} \quad (6.115)
\]

⇒ For "1" the phase increases \( \pi h \), for "0" the phase reduces \( \pi h \)
Note that \( f = \frac{1}{2\pi} \frac{d\theta}{dt} \)

**SPECIAL CASE 1**, \( h = 1 \) (fig 6.27) \((f_1 - f_2 = \frac{1}{T_b}, 6.87)\)

\[
\theta(T_b) - \theta(0) = \pm \pi
\]  

(6.115)

\( \therefore \) A change of \( \pi \) is the same as a change of \( -\pi \) for modulo \( 2\pi \)

Knowing the change in the previous bit provides no information in the current bits \( \Rightarrow \) memoryless

**SPECIAL CASE 2**, \( h = \frac{1}{2} \)

In odd bits, the change are \( \pm \frac{\pi}{2} \)

In even bits, the change are 0, \( \pi \)
Figure 6.27 Phase tree. \((h=1)\)
Figure 6.28 Phase trellis; boldfaced path represents the sequence 1101000. \((h=1/2)\)
Signal - Space Diagram of MSK

\[ s(t) = \frac{2E_b}{T_b} \cos[2\pi f_c t + \theta(t)] \]

\[ = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] \sin(2\pi f_c t) \]  

(6.109)

Consider the in-phase component and \( h = \frac{1}{2} \)

from (6.110)

\[ \theta(t) = \theta(0) \pm \frac{\pi}{2T_b} t \quad , \quad 0 \leq t \leq T_b \]  

(6.117)

"+" corresponds to "I"

For interval \(-T_b \leq t \leq 0\)

\[ \theta(t) = \theta(-T_b) \pm (\mp) \frac{\pi}{2T_b} t \quad , \quad \pm (\mp) \] depending on history

From Fig6.28, \( \theta(0) \) is 0 or \( \pi \)
The polarity of $\cos[\theta(t)]$ depends only on $\theta(0)$ regardless the data sequence before or after $t = 0$.

The in-phase component (Fig 6.27)

$$S_I(t) = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)]$$

$$= \sqrt{\frac{2E_b}{T_b}} \cos[\theta(0)] \cos\left(\frac{\pi}{2T_b} t\right)$$

$$= \pm \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi}{2T_b} t\right), \quad -T_b \leq t \leq T_b \quad (6.118)$$

$\therefore \sin 0 = \sin \pi = 0, \cos 0 = 1, \cos \pi = -1$

Similarly in $0 \leq t \leq 2T_b$, $\theta(T_b) = \pm \frac{\pi}{2}$ (Fig 6.28)

$$S_Q(t) = \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)]$$

$$= \sqrt{\frac{2E_b}{T_b}} \sin[\theta(T_b)] \sin\left(\frac{\pi}{2T_b} t\right)$$

$$= \pm \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right), \quad 0 \leq t \leq 2T_b \quad (6.119)$$
We have $\theta(0) = 0, \pi$, $\theta(T_b) = \pm \frac{\pi}{2}$

1. $\theta(0) = 0$, $\theta(T_b) = \frac{\pi}{2}$ for "1"

2. $\theta(0) = \pi$, $\theta(T_b) = \frac{\pi}{2}$ for "0"

3. $\theta(0) = \pi$, $\theta(T_b) = -\frac{\pi}{2}$ for "1"
   (equivalently $\frac{3\pi}{2}$ modulo $2\pi$)

4. $\theta(0) = 0$, $\theta(T_b) = -\frac{\pi}{2}$ for "0"

⇒ The MSK signal has one of four possible forms, depending on $\theta(0)$ and $\theta(T_b)$
(We will see it in Fig 6.29)
Figure 6.28 Phase trellis; boldfaced path represents the sequence 1101000. \((h=1/2)\)
Recall \( s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] \cos(2\pi f_c t) \)
\[ -\sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] \sin(2\pi f_c t) \] (6.116)

From (6.116), \( s_I(t) \) and \( s_Q(t) \), we have (6.118), (6.119)

\[ \phi_1(t) = \sqrt{\frac{2}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right) \cos(2\pi f_c t), \ 0 \leq t \leq T_b \] (6.120)

\[ \phi_2(t) = \sqrt{\frac{2}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right) \sin(2\pi f_c t), \ 0 \leq t \leq T_b \] (6.121)

\[ s(t) = s_1 \phi_1(t) + s_2 \phi_2(t), \ \ 0 \leq t \leq T_b \] (6.122)

where \( s_1 \) and \( s_2 \) are related to the phase states of \( \theta(0) \) and \( \theta(T_b) \)
\begin{align*}
S_i &= \int_{T_b} \left\{ \sqrt{\frac{2E_b}{T_b}} \cos[\theta(0) + \frac{\pi}{2T_b} t] \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin[\theta(0) + \frac{\pi}{2T_b} t] \sin(2\pi f_c t) \right\} \times \sqrt{\frac{2}{T_b}} \cos(\frac{\pi}{2T_b} t) \cdot \cos(2\pi f_c t) dt \\
&= \int_{T_b} \left\{ \sqrt{\frac{2E_b}{T_b}} \cos[\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t] \right\} \times \sqrt{\frac{2}{T_b}} \cos(\frac{\pi}{2T_b} t) \cdot \cos(2\pi f_c t) \cdot dt \\
&= \int_{T_b} \frac{2}{T_b} \sqrt{E_b} \cdot \cos[\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t] \times \frac{1}{2} \left[ \cos(\frac{\pi}{2T_b} t + 2\pi f_c t) + \cos(\frac{\pi}{2T_b} t - 2\pi f_c t) \right] dt \\
&= \int_{T_b} \frac{\sqrt{E_b}}{T_b} \left\{ \cos[\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t] \cdot \cos(\frac{\pi}{2T_b} t + 2\pi f_c t) + \cos[\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t] \cos(\frac{\pi}{2T_b} t - 2\pi f_c t) \right\} dt \\
&\quad - \int_{T_b} \frac{\sqrt{E_b}}{T_b} \left\{ \frac{1}{2} \left[ \cos(\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t + \frac{\pi}{2T_b} t + 2\pi f_c t) + \cos(\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t - \frac{\pi}{2T_b} t - 2\pi f_c t) \right] \right\} dt \\
&\quad + \frac{1}{2} \left[ \cos(\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t + \frac{\pi}{2T_b} t - 2\pi f_c t) + \cos(\theta(0) + \frac{\pi}{2T_b} t + 2\pi f_c t - \frac{\pi}{2T_b} t + 2\pi f_c t) \right] dt \\
&= \int_{T_b} \frac{\sqrt{E_b}}{T_b} \left[ \frac{1}{2} \cos(\theta(0) + \frac{\pi}{T_b} t + 4\pi f_c t) + \frac{1}{2} \cos(\theta(0) + \frac{\pi}{T_b} t) + \frac{1}{2} \cos(\theta(0) + 4\pi f_c t) \right] dt \\
&= \frac{\sqrt{E_b}}{T_b} \frac{1}{2} \cos \theta(0) \times 2T_b \\
&= \sqrt{E_b} \cos \theta(0) \\
\end{align*}

\text{(}: f_c T_b = n_c + i, i = 1, 2, (6.87), f_c T_b = \frac{f + h}{2} T_b = n_c + \frac{3}{2})
\[ s_1 = \int_{-T_b}^{T_b} s(t) \phi_1(t) \, dt \]
\[ = \sqrt{E_b} \cos[\theta(0)] , \quad -T_b \leq t \leq T_b \]  (6.123)

**Recall**
\[ s_Q(t) = \pm \sqrt{\frac{2E_b}{T_b}} \sin \left( \frac{\pi}{2T_b} t \right) , \quad 0 \leq t \leq 2T_b \]  (6.119)

\[ s_2 = \int_{0}^{2T_b} s(t) \phi_2(t) \, dt \]
\[ = -\sqrt{E_b} \sin[\theta(T_b)] , \quad 0 \leq t \leq 2T_b \]  (6.124)

1. The integral interval = \( 2T_b \)
2. The upper and lower bounds of integration of \( s_2 \) shifted \( T_b \)
3. The phase states \( \theta(0) \) and \( \theta(T_b) \) are defined in \( 0 \leq t \leq T_b \) which is common to both

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### Table 6.5 Signal-space characterization of MSK

<table>
<thead>
<tr>
<th>Transmitted Binary Symbol, $0 \leq t \leq T_b$</th>
<th>Phase States (radians)</th>
<th>Coordinates of Message Points</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta(0)$</td>
<td>$\theta(T_b)$</td>
</tr>
<tr>
<td><strong>0</strong></td>
<td>0</td>
<td>$\pm \sqrt{E_b}$</td>
</tr>
<tr>
<td><strong>1</strong></td>
<td>$\pi$</td>
<td>$\pm \sqrt{E_b}$</td>
</tr>
<tr>
<td><strong>0</strong></td>
<td>$\pi$</td>
<td>$\mp \sqrt{E_b}$</td>
</tr>
<tr>
<td><strong>1</strong></td>
<td>0</td>
<td>$\pm \sqrt{E_b}$</td>
</tr>
</tbody>
</table>

\[
s_1 = \sqrt{E_b} \cos[\theta(0)]
\]

\[
s_2 = \sqrt{E_b} \sin[\theta(T_b)]
\]
The constellation of MSK is similar to that of QPSK.

Figure 6.29 Signal-space diagram for MSK system.
Example 6.5

Figure 6.30 shows the sequences and waveforms involved in the generation of an MSK signal for the binary sequence 1101000. The input binary sequence is shown in Figure 6.30a. The two modulation frequencies are: \( f_1 = 5/4T_b \) and \( f_2 = 3/4T_b \). Assuming that, at time \( t = 0 \) the phase \( \theta(0) \) is zero, the sequence of phase states is as shown in Figure 6.30, modulo \( 2\pi \). The polarities of the two sequences of factors used to scale the time functions \( \phi_1(t) \) and \( \phi_2(t) \) are shown in the top lines of Figures 6.30b and 6.30c. Note that these two sequences are offset relative to each other by an interval equal to the bit duration \( T_b \). The waveforms of the resulting two components of \( s(t) \), namely, \( s_1\phi_1(t) \) and \( s_2\phi_2(t) \), are also shown in Figures 6.30b and 6.30c. Adding these two modulated waveforms, we get the desired MSK signal \( s(t) \) shown in Figure 6.30d.
**Figure 6.30** (a) Input binary sequence. (b) Waveform of scaled time function $s_1 \Phi_1(t)$. (c) Waveform of scaled time function $s_2 \Phi_2(t)$. (d) Waveform of the MSK signal $s(t)$ obtained by adding $s_1 \Phi_1(t)$ and $s_2 \Phi_2(t)$ on a bit-by-bit basis.
with \( h = \frac{1}{2} \), \( \frac{1}{2} = T_b (f_1 - f_2) \)  

(6.114)

\[ f_1 - f_2 = \frac{1}{2T_b} \]

The difference is half the bit rate

\[ f_1 = \frac{n_c}{T_b}, f_2 = \frac{n_c}{T_b} + \frac{1}{2T_b} = f_1 + \frac{1}{2T_b} \]

The two FSK signals with spacing \( \frac{1}{2T_b} \) are coherently orthogonal ⇒ minimum shift keying (MSK)

→ 以後證明
Error Prob. of MSK

\[ x(t) = s(t) + w(t) \]

\[ x_1 = \int_{-T_b}^{T_b} x(t)\phi_1(t)dt \]
\[ = s_1 + w_1 \quad -T_b \leq t \leq T_b \quad (6.125) \]

\[ x_2 = \int_{0}^{2T_b} x(t)\phi_2(t)dt \]
\[ = s_2 + w_2 \quad 0 \leq t \leq 2T_b \quad (6.126) \]

The observation interval is \( 2T_b \).

From Fig 6.29, if \( x_1 > 0 \) (in \( Z_1 \) or \( Z_4 \)), \( \hat{\theta}(0) = 0 \), if \( x_1 < 0 \), \( \hat{\theta}(0) = \pi \), if \( x_2 > 0 \) (in \( Z_1 \) or \( Z_2 \)), we have \( \hat{\theta}(T_b) = -\pi/2 \), otherwise \( x_2 < 0 \), \( \hat{\theta}(T_b) = \pi/2 \).
<table>
<thead>
<tr>
<th>Transmitted Binary Symbol, (0 \leq t \leq T_b)</th>
<th>Phase States (radians)</th>
<th>Coordinates of Message Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{z}_1)</td>
<td>(0)</td>
<td>(-\pi/2) (+\sqrt{E_b}) (+\sqrt{E_b})</td>
</tr>
<tr>
<td>(\hat{z}_2)</td>
<td>(\pi)</td>
<td>(-\pi/2) (-\sqrt{E_b}) (+\sqrt{E_b})</td>
</tr>
<tr>
<td>(\hat{z}_3)</td>
<td>(0)</td>
<td>(+\pi/2) (-\sqrt{E_b}) (-\sqrt{E_b})</td>
</tr>
<tr>
<td>(\hat{z}_4)</td>
<td>(\pi)</td>
<td>(+\pi/2) (+\sqrt{E_b}) (-\sqrt{E_b})</td>
</tr>
</tbody>
</table>

For \(\hat{\theta}(0) = 0\) and \(\hat{\theta}(T_b) = -\pi/2\) \[\Rightarrow \hat{m} = 0\]

or \(\hat{\theta}(0) = \pi\) and \(\hat{\theta}(T_b) = \pi/2\) \[\Rightarrow \hat{m} = 0\]

For \(\hat{\theta}(0) = \pi\) and \(\hat{\theta}(T_b) = -\pi/2\) \[\Rightarrow \hat{m} = 1\]

or \(\hat{\theta}(0) = 0\) and \(\hat{\theta}(T_b) = \pi/2\) \[\Rightarrow \hat{m} = 1\]

The BER for MSK is

\[
p_e = \frac{1}{2} \text{erfc}(\sqrt{\frac{E_b}{N_0}})
\]

which is exactly the same as that for BPSK and QPSK.

Reference: R. Ziemer and R. Peterson

"Introduction to Digital Communication"

Page 170 ~ 187
Figure 6.31 Block diagrams for (a) MSK transmitter and (b) coherent MSK receiver.
Power Spectra of MSK

1. The in-phase component is

\[ g(t) = \pm \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right), \quad -T_b \leq t \leq T_b \]  \hfill (6.128)

\[ g(t) \leftrightarrow \frac{4\sqrt{2E_bT_b}}{\pi} \frac{\cos(2\pi T_b f)}{1 - (4T_b f)^2} \]

The energy spectra of \( g(t) \)

\[ \psi_g(f) = \frac{32E_bT_b}{\pi^2} \left[ \frac{\cos(2\pi T_b f)}{1 - (4T_b f)^2} \right]^2 \]

2. The quadrature component

\[ g'(t) = \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right), \quad 0 \leq t \leq 2T_b \]  \hfill (6.130)

\[ \psi_g'(f) = \psi_g(f) \]
The PSD of MSK

\[ S_B(f) = \frac{\psi_g(f)}{2T_b} + \frac{\psi_g'(f)}{2T_b} \]

\[ = 2 \left[ \frac{\psi_g(f)}{2T_b} \right] \]

\[ = \frac{32E_b}{\pi^2} \left[ \frac{\cos(2\pi T_b f)}{16T_b^2 f^2 - 1} \right]^2 \] (6.131)

1. Constant envelope
2. Relatively narrow bandwidth
3. Performance ~ QPSK

However the out-of-band spectra may interfere the adjacent channel in wireless communication ⇒ Gaussian-Filtered MSK (used in GSM)
Let $W$ denote the 3dB baseband bandwidth of the pulse-shaping filter.

$$H(f) = \exp\left(-\frac{\log 2}{2} \left(\frac{f}{W}\right)^2\right)$$  \hspace{1cm} (6.132)

**Figure 6.33** Power spectra of MSK and GMSK signals for varying time-bandwidth product. (Reproduced with permission from Dr. Gordon Stüber, Georgia Tech.)
Figure 6.34  Theoretical $E_b/N_0$ degradation of GMSK for varying time-bandwidth product. (Taken from Murata and Hirade, 1981, with permission of the IEEE.)
$M$ - ary FSK

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos \left[ \frac{\pi}{T} (n_e + i)t \right], \quad 0 \leq t \leq T$$

$$i = 1, 2, \ldots, M$$

$T$: symbol duration

$E$: symbol energy

$$\int_0^T s_i(t)s_j(t)dt = \delta_{ij}$$

$$\phi_i(t) = \frac{1}{\sqrt{E}} s_i(t), \quad 0 \leq t \leq T, \quad i = 1, 2, \ldots, M$$

$$d_{\text{min}} = \sqrt{2E}$$

Recall (5.95, page 335)

$$P_e \leq \frac{(M-1)}{2} \text{erfc} \left( \frac{d_{\text{min}}}{2}\sqrt{\frac{E}{N_0}} \right) = \frac{M-1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right)$$

(6.140)
Figure 6.36 Power spectra of $M$-ary FSK signals for $M = 2, 4, 8$. 

$$f_c - f_{c-1} = \frac{1}{2T} \text{ (Hz)}$$
Bandwidth Efficiency of MFSK

Define the bandwidth required for MFSK

\[ B = \frac{M}{2T} \]  \hspace{1cm} (6.141)

\[ T = T_b \log_2 M, \quad R_b = \frac{1}{T_b} \]

\[ B = \frac{R_b M}{2 \log_2 M} \]  \hspace{1cm} (6.142)

\[ \rho = \frac{R_b}{B} = \frac{2 \log_2 M}{M} \]  \hspace{1cm} (6.143)

<table>
<thead>
<tr>
<th>Table 6.6</th>
<th>Bandwidth efficiency of M-ary FSK signals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>2</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1</td>
</tr>
<tr>
<td>( \rho ) (bits/s/Hz)</td>
<td>1</td>
</tr>
</tbody>
</table>

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6.6 Detection of Signals with Unknown Phase (FSK)

Optimum Quadratic Receiver

\[ S_i(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_i t) \quad , \quad i = 1, 2 \quad (6.144) \]

\[ f_i = \frac{n_i}{2T} \quad , \quad n_i = \text{integer} \]

Assume noncoherent

\[ x(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_i t + \theta) + w(t) \quad , \quad i = 1, 2 \quad (6.145) \]

\( \theta \): unknown carrier phase with distribution (Difference of (t) and VCO)

\[ f_\Theta(\theta) = \frac{1}{2\pi} \quad , \quad -\pi < \theta \leq \pi \quad (6.146) \]
To design an optimum receiver, we may formulate the conditional likelihood function as

$$L(s_i(\theta)) = \exp\left(\sqrt{\frac{E}{N_0 T}} \int_{0}^{T} x(t) \cos(2\pi f_i t + \theta)dt\right) \quad (6.147)$$

To remove dependence of $L(s_i(\theta))$ on $\theta$, we have

$$L(s_i) = \int_{-\pi}^{\pi} L(s_i(\theta)) f_\theta(\theta) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\sqrt{\frac{E}{N_0 T}} \int_{0}^{T} x(t) \cos(2\pi f_i t + \theta)dt\right) d\theta \quad (6.148)$$

$$\int_{0}^{T} x(t) \cos(2\pi f_i t + \theta)dt = \cos \theta \int_{0}^{T} x(t) \cos(2\pi f_i t)dt$$

$$- \sin \theta \int_{0}^{T} x(t) \sin(2\pi f_i t)dt \quad (6.149)$$
Define
\[
l_i = \left[ \left( \frac{\int_0^t x(t) \cos(2\pi f_i t) dt}{y_1^2} \right)^2 + \left( \frac{\int_0^t x(t) \sin(2\pi f_i t) dt}{y_0^2} \right)^2 \right]^{\frac{1}{2}} \tag{6.150}
\]
\[
\beta_i = \tan^{-1}\left( \frac{\int_0^t x(t) \sin(2\pi f_i t) dt}{\int_0^t x(t) \cos(2\pi f_i t) dt} \right)
\]

From (6.149)
\[
\int_0^t x(t) \cos(2\pi f_i t + \theta) dt
\]
\[
= l_i (\cos \theta \cos \beta_i - \sin \theta \sin \beta_i)
\]
\[
= l_i \cos(\theta + \beta_i) \tag{6.152}
\]

Using (6.152) and (6.148)
\[
L(s_i) = \frac{1}{2\pi} \int_0^\pi \exp\left( \sqrt{\frac{E}{N_0 T}} l_i \cos(\theta + \beta_i) \right) d\theta
\]
\[
= \frac{1}{2\pi} \int_{\pi - q + \beta_i}^{\pi + q} \exp\left( \sqrt{\frac{E}{N_0 T}} l_i \cos \theta \right) d\theta
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left( \sqrt{\frac{E}{N_0 T}} l_i \cos \theta \right) d\theta
\]
Recall the modified Bessel function of zero order

\[ I_0(\sqrt{\frac{E}{N_0 T}}l_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\sqrt{\frac{E}{N_0 T}}l_i \cos \theta) d\theta \]  \hspace{1cm} (6.154) (A3.21)

\[ L(s_i) = I_0(\sqrt{\frac{E}{N_0 T}}l_i) \] \hspace{1cm} (6.155)

The binary hypothesis test can be written as

\[ I_0(\sqrt{\frac{E}{N_0 T}}l_1) \begin{cases} \mathcal{H}_1 \quad & I_0(\sqrt{\frac{E}{N_0 T}}l_2) \end{cases} \] \hspace{1cm} (6.156)

\[ I(\bullet) \text{ is a monotonically increasing function} \]

\[ l_1 \begin{cases} \mathcal{H}_1 \quad & l_2 \end{cases} \]

For convenience (Fig 6.3a) \[ l_1^2 \begin{cases} \mathcal{H}_1 \quad & l_2^2 \end{cases} \] \hspace{1cm} (6.157)
Suppose we have a filter that is matched to \( \cos(2 \pi f_i t + \theta) \)  

Because the output is unaffected by \( \theta \), we may choose a matched filter with impulse response \( \cos[2 \pi f_i (T - t)](\theta = 0) \)

\[
y(t) = \int_0^T x(\tau) \cos[2 \pi f_i (T - t + \tau)] d\tau \\
= \cos[2 \pi f_i (T - t)] \int_0^T x(\tau) \cos(2 \pi f_i \tau) d\tau \\
- \sin[2 \pi f_i (T - t)] \int_0^T x(\tau) \sin(2 \pi f_i \tau) d\tau \\
= a \cos[2 \pi f_i (T - t)] - b \sin[2 \pi f_i (T - t)]
\]

\[\text{Fig6.37(c)} \quad a = \int_0^T x(\tau) \cos(2 \pi f_i \tau) d\tau
\]

\[\text{b} = \int_0^T x(\tau) \sin(2 \pi f_i \tau) d\tau
\]

The envelope of \( y(t) \) is

\[
l_i = \left\{ a^2 + b^2 \right\}^{\frac{1}{2}} \\
= \left\{ \left[ \int_0^T x(\tau) \cos(2 \pi f_i \tau) d\tau \right]^2 + \left[ \int_0^T x(\tau) \sin(2 \pi f_i \tau) d\tau \right]^2 \right\}^{\frac{1}{2}}
\]

(6.159) \[\text{Fig6.37(a)} \]

which is the same as the output of the quadrature receiver  

\[\text{Fig6.37(b)} \quad y_Q = \int_0^T x(\tau) \sin[2 \pi f_c (T - t + \tau)] d\tau
\]

\[
= \sin(2 \pi f_c (T - \tau)) \int_0^T x(\tau) \cos(2 \pi f_c \tau) d\tau + \cos(2 \pi f_c (T - \tau)) \int_0^T x(\tau) \sin(2 \pi f_c \tau) d\tau
\]

\[= a \sin[2 \pi f_c (T - \tau)] + b \cos[2 \pi f_c (T - \tau)]
\]

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Two Equivalent Forms of the Quadratic Receiver

(a) Quadrature receiver using correlators.

Replace each correlator with a corresponding equivalent matched filter, we obtain Fig 6.37(b)

(b) Quadrature receiver using matched filters.
Figure 6.38 Output of matched filter for a rectangular RF wave: (a) $\theta = 0$, and (b) $\theta = 180$ degrees.
6.7 Noncoherent Orthogonal Modulation

$s_1(t)$ and $s_2(t)$ denote two orthogonal signals with equal energy. Let $g_1(t)$ and $g_2(t)$ represent the phase shifted versions of $s_1(t)$ and $s_2(t)$

$$x(t) = \begin{cases} g_1(t) + w(t), & s_1(t) \text{ sent} \\ g_2(t) + w(t), & s_2(t) \text{ sent} \end{cases} \quad (6.160)$$

---

**Figure 6.39** (a) Generalized binary receiver for noncoherent orthogonal modulation. (b) Quadrature receiver equivalent to either one of the two matched filters in part (a); the index $i = 1, 2$. 

---

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Receiver for noncoherent FSK.
Because phase is unknown, amplitude is the only discriminant. Fig 6.39a may be viewed as a quadrature receiver.

Let $\psi_i(t)$ represent a scaled version of $s_1(t)$ or $s_2(t)$ and $\hat{\psi}_i(t)$ be the shifted $-90^\circ$ version of $\psi_i(t)$ (Table A6.4, page 765)

$$\psi_i(t) \xrightarrow{\text{Hilbert transform}} \hat{\psi}_i(t)$$

If $\psi(t) = m(t)\cos(2\pi f_i t)$ \hspace{1cm} (6.161)

then $\hat{\psi}(t) = m(t)\sin(2\pi f_i t)$ \hspace{1cm} (6.162)

The $P_e$ of Fig 6.39a is

$$P_e = \frac{1}{2}\exp\left(-\frac{E}{2N_0}\right) \hspace{1cm} (6.163)$$
Derivation of 6.163
Recall Fig 6.39

1. There are four noise parameters $X_{11}, X_{Q1}, X_{12},$ and $X_{Q2}$, which are conditionally independent and identically distributed.

2. The receiver Fig. 6.39a is symmetric. The average prob. of error may be obtained by transmitting $S_1(t)$ and calculating the prob. of choosing $S_2(t)$, or vice versa, assuming that $S_1(t)$ and $S_2(t)$ are equiprobable.

\[ P_e = \frac{1}{2} P_{10} + \frac{1}{2} P_{01} = P_{10}, \quad (P_{10} = P_{01}) \]
Because $S_1(t)$ and $S_2(t)$ are orthogonal, when $S_i(t)$ is transmitted, $I_2$ is due to noise alone.

(Fig 6.39b, $i = 2$)

\[ I_2 = \sqrt{X_{I2}^2 + X_{Q2}^2} \]  \hfill (6.164)

Where $X_{I2}$ and $X_{Q2}$ are both Gaussian with zero mean and variance $\frac{N_0}{2}$ given $\theta$
\[ f_{X_{r2}}(x_{r2}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{r2}^2}{N_0}\right) \quad (6.165) \]

and

\[ f_{X_{q2}}(x_{q2}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{q2}^2}{N_0}\right) \quad (6.166) \]

Recall Section 1.12 (Rayleigh distribution) (page 67)

\[ r(t) = \left[n_I^2(t) + n_Q^2(t)\right]^{\frac{1}{2}} \quad (1.106) \]

\[ f_R(r) = \begin{cases} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), & r \geq 0 \\ 0, & \text{elsewhere} \end{cases} \quad (1.115) \]

\[ f_{L2}(l_2) = \begin{cases} \frac{2 l_2}{N_0} \exp\left(-\frac{l_2^2}{N_0}\right), & l_2 \geq 0 \\ 0, & \text{elsewhere} \end{cases} \quad (6.167) \]
Figure 6.41 Calculation of the conditional probability that \( l_2 > l_1 \), given \( l_1 \).

The conditional prob. that \( l_2 > l_1 \), given \( l_1 \), is

\[
P(l_2 > l_1 \mid l_1) = \int_{l_1}^{\infty} f_{L_2}(l_2) \, dl_2 \tag{6.168}
\]

\[
= \int_{l_1}^{\infty} \frac{2l_2}{N_0} \exp\left(-\frac{l_2^2}{N_0}\right) \, dl_2 
\]

\[
= \exp\left(-\frac{l_1^2}{N_0}\right) \tag{6.169}
\]
Consider \( l_1 \),

\[
l_1^2 = x_{I1}^2 + x_{Q1}^2
\]  (6.170)

\( x_{I1} \) is due to signal plus noise and \( x_{Q1} \) is due to noise only.

1. \( x_{I1} \) is Gaussian with mean \( \sqrt{E} \) and variance \( N_0/2 \)
2. \( x_{Q1} \) is Gaussian with zero mean and variance \( N_0/2 \)

\[
f_{x_{I1}}(x_{I1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_{I1}-\sqrt{E})^2}{N_0}\right)
\]  (6.171)

\[
f_{x_{Q1}}(x_{Q1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{Q1}^2}{N_0}\right)
\]  (6.172)

The joint pdf of \( x_{I1} \) and \( x_{Q1} \) is \( f_{x_{I1}}(x_{I1})f_{x_{Q1}}(x_{Q1}) \)
\[ l_1^2 = x_{l1}^2 + x_{q1}^2 \quad (6.173) \]

**The random variable** \( L_1 \) **is Rician distributed** (page 70)

\[
f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) I_0\left(\frac{Ar}{\sigma^2}\right)
\]

\[
f_{L_1}(l_1) = \frac{u}{N_0} \exp\left(-\frac{l_1^2 + F}{N_0}\right) I_0\left(\frac{2\sqrt{F}l_1}{N_0}\right) \quad (1.128)
\]

\( I_a(\cdot) \) **is the modified Bessel function of the first kind of zero order**.

\[
P_e = \int_0^\infty P(l_2 > l_1 \mid l_1) f_{L_1}(l_1) dl_1
\]

which is complicated. We will compute \( P_e \) by a simple way.

Recall \( P(\text{error} \mid l_1) = P(l_2 > l_1 \mid l_1) = \exp\left(-\frac{l_1^2}{N_0}\right) \quad (6.169) \)

\[ = \exp\left(-\frac{x_{l1}^2 + x_{q1}^2}{N_0}\right) \quad (6.174) \]
Modified Bessel function $I_n(x)$ of varying order $n$. 
Consider
\[
P(error \mid x_{I1}, x_{Q1}) f_{x_{I1}}(x_{I1}) f_{x_{Q1}}(x_{Q1})
\]
\[
= \frac{1}{\pi N_0} \exp\left\{-\frac{1}{N_0} [x_{I1}^2 + x_{Q1}^2 + (x_{I1} - \sqrt{E})^2 + x_{Q1}^2]\right\}
\]
\[
x_{I1}^2 + x_{Q1}^2 + (x_{I1} - \sqrt{E})^2 + x_{Q1}^2 = 2(x_{I1} - \frac{\sqrt{E}}{2})^2 + 2x_{Q1}^2 + \frac{E}{2}
\]
\[
P_e = \int_0^\infty \int_0^\infty P(error \mid x_{I1}, x_{Q1}) f_{x_{I1}}(x_{I1}) f_{x_{Q1}}(x_{Q1}) \, dx_{I1} \, dx_{Q1}
\]
\[
= \frac{1}{\pi N_0} \exp(-\frac{E}{2N_0}) \int_0^\infty \exp\left\{-\frac{2}{N_0} (x_{I1} - \frac{\sqrt{E}}{2})^2\right\} dx_{I1} \int_0^\infty \exp\left\{-\frac{2x_{Q1}^2}{N_0}\right\} dx_{Q1}
\]
We have two identities:
\[
\int_0^\infty \exp\left\{-\frac{2}{N_0} (x_{I1} - \frac{\sqrt{E}}{2})^2\right\} dx_{I1} = \sqrt{\frac{N_0 \pi}{2}}
\]
\[
\int_0^\infty \exp\left\{-\frac{2x_{Q1}^2}{N_0}\right\} dx_{Q1} = \sqrt{\frac{N_0 \pi}{2}}
\]
\[
\implies P_e = \frac{1}{2} \exp(-\frac{E}{2N_0})
\]
6.8 Noncoherent Binary FSK

\[ s_i(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_it), \quad 0 \leq t \leq T_b, \quad i = 1,2 \]  \hspace{1cm} (6.180)

The noncoherent FSK is a special case of noncoherent orthogonal modulation with \( T = T_b, E = E_b \) (Fig 6.37c)

\[ P_e = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right) \]  \hspace{1cm} (6.181)

**Figure 6.42** Noncoherent receiver for the detection of binary FSK signals.
6.9 Differential PSK (DPSK)

For "1"

\[
s_1(t) = \begin{cases} 
\sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t) & 0 \leq t \leq T_b \\
\sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t + \pi) & T_b \leq t \leq 2T_b
\end{cases}
\] (6.182)

For "0"

\[
s_2(t) = \begin{cases} 
\sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t) & 0 \leq t \leq T_b \\
\sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t + \pi) & T_b \leq t \leq 2T_b
\end{cases}
\] (6.183)

\(s_1(t)\) and \(s_2(t)\) are orthogonal over \(0 \leq t \leq 2T_b\). DPSK is a special case of noncoherent orthogonal modulation with \(T = 2T_b\), \(E = 2E_b\) (6.163)

\[
P_e = \frac{1}{2} \exp\left(-\frac{E_b}{N_0}\right)
\] (6.184)
Generation and Detection of DPSK

<table>
<thead>
<tr>
<th>TABLE 6.7</th>
<th>Illustrating the generation of DPSK signal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [b_k] )</td>
<td>1 0 0 1 0 0 1 1</td>
</tr>
<tr>
<td>( [d_{k-1}] )</td>
<td>1 1 0 1 1 0 1 1</td>
</tr>
<tr>
<td>Differentially encoded sequence ( {d_k} )</td>
<td>1 1 0 1 1 0 1 1</td>
</tr>
<tr>
<td>Transmitted phase (radians)</td>
<td>0 0 ( \pi ) 0 0 ( \pi ) 0 0 0</td>
</tr>
</tbody>
</table>

1. If \( b_k = 1 \), \( d_k \) is unchanged

2. If \( b_k = 0 \), \( d_k \) is changed w.r.t the previous bit

![Diagram of DPSK generation](image-url)
Detection of DPSK Signal
The receiver equips with an in-phase and a quadrature channel. The receiver signal points are \((A \cos \theta, A \sin \theta)\) and \((-A \cos \theta, -A \sin \theta)\) where \(\theta\) is the unknown phase and \(A\) is the amplitude.

\[ T_b \leq t \leq 2T_b, \]

sent \(s_2(t)\)

**Figure 6.44** Signal-space diagram of received DPSK signal.
The receiver measures \((x_{I_0}, x_{Q_0})\) at \(t = T_b\) and \((x_{I_1}, x_{Q_1})\) at \(t = 2T_b\).

We have to decide whether these two points map to the same signal point or different ones:

\[
\begin{align*}
    x_{I_0} x_{I_1} + x_{Q_0} x_{Q_1} &\quad \text{say } 1 \\
    &\quad \text{say } 0 \\
\end{align*}
\]

\[
x_{I_0} x_{I_1} + x_{Q_0} x_{Q_1} = \frac{1}{4} \left[ (x_{I_0} + x_{I_1})^2 - (x_{I_0} - x_{I_1})^2 \\
    + (x_{Q_0} + x_{Q_1})^2 - (x_{Q_0} - x_{Q_1})^2 \right] \\
\]

\[
(x_{I_0} + x_{I_1})^2 + (x_{Q_0} + x_{Q_1})^2 - (x_{I_0} - x_{I_1})^2 - (x_{Q_0} - x_{Q_1})^2 \quad \text{say } 1 \\
\]

\[
\begin{align*}
    &\quad \text{say } 0 \\
\end{align*}
\]

\[
\Rightarrow \text{testing whether the point } (x_{I_0}, x_{Q_0}) \text{ is closer to } (x_{I_1}, x_{Q_1}) \text{ or} \\
\quad \text{its image } (-x_{I_1}, -x_{Q_1})
\]

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6.10 Comparison of Digital Modulation Schemes

**Table 6.8** Summary of formulas for the bit error rate of different digital modulation schemes

<table>
<thead>
<tr>
<th>Signaling Scheme</th>
<th>Bit Error Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Coherent binary PSK</td>
<td>( \frac{1}{2} \text{erfc}(\sqrt{E_b/N_0}) )</td>
</tr>
<tr>
<td>Coherent QPSK</td>
<td></td>
</tr>
<tr>
<td>Coherent MSK</td>
<td></td>
</tr>
<tr>
<td>(b) Coherent binary FSK</td>
<td>( \frac{1}{2} \text{erfc}(\sqrt{E_b/2N_0}) )</td>
</tr>
<tr>
<td>(c) DPSK</td>
<td>( \frac{1}{2} \exp(-E_b/N_0) )</td>
</tr>
<tr>
<td>(d) Noncoherent binary FSK</td>
<td>( \frac{1}{2} \exp(-E_b/2N_0) )</td>
</tr>
</tbody>
</table>
Figure 6.45 Comparison of the noise performance of different PSK and FSK schemes.
### Table 6.9  Comparison of power-bandwidth requirements for M-ary PSK with binary PSK. Probability of symbol error $= 10^{-4}$

<table>
<thead>
<tr>
<th>Value of M</th>
<th>$(\text{Bandwidth})_{\text{M-ary}}$</th>
<th>$(\text{Average power})_{\text{M-ary}}$</th>
<th>$(\text{Bandwidth})_{\text{Binary}}$</th>
<th>$(\text{Average power})_{\text{Binary}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.5</td>
<td>0.34 dB</td>
<td>0.5</td>
<td>0.34 dB</td>
</tr>
<tr>
<td>8</td>
<td>0.333</td>
<td>3.91 dB</td>
<td>0.333</td>
<td>3.91 dB</td>
</tr>
<tr>
<td>16</td>
<td>0.25</td>
<td>8.52 dB</td>
<td>0.25</td>
<td>8.52 dB</td>
</tr>
<tr>
<td>32</td>
<td>0.2</td>
<td>13.52 dB</td>
<td>0.2</td>
<td>13.52 dB</td>
</tr>
</tbody>
</table>

From Shanmugan (1979, p. 424).
Comparison of 16 PSK and 16 QAM

For QAM receiver

\[ P_e \approx 2 \left( 1 - \frac{1}{\sqrt{M}} \right) \text{erfc} \left( \sqrt{\frac{3E_{av}}{2(M-1)N_0}} \right) \]  \hspace{1cm} (6.64)

\[ M = 16 \), \( P_e \approx \frac{3}{2} \text{erfc} \left( \frac{E_{av}}{10N_0} \right) \]

At \( P_e \approx 10^{-4} \), \( \frac{E_{av}}{N_0} \approx 13 \text{db} \)

For 16 PSK receiver

\[ P_e \approx \text{erfc} \left( \frac{E}{N_0} \sin \left( \frac{\pi}{M} \right) \right) \]  \hspace{1cm} (6.47)

\[ M = 16 \), \( P_e \approx \text{erfc} \left( \frac{E}{N_0} \sin \frac{\pi}{16} \right) \]

At \( P_e \approx 10^{-4} \), \( \frac{E}{N_0} \approx 16.5 \text{dB} \)

Figure 6.46 Signal constellation for (a) \( M \)-ary PSK and (b) corresponding \( M \)-ary QAM, for \( M = 16 \).