1.0 “Probabilities” is considered an important background for analysis and design of communications systems

1.1 Introduction (Physical phenomenon)
   Deterministic model : No uncertainty about its time-dependent behavior at any instant of time . (e.g. cos(wt))
   Random model : The future value is subject to “chance” (probability) (e.g. cos(wt+θ), θ is a random variable)
   ↑in the interval(-T,T)

Example: Thermal noise, Random data stream
1.1.1 Example of Stochastic Models

Channel noise and interference
Source of information, such as voice
1.1.2 Relative Frequency

How to determine the probability of “head appearance” for a coin?

Answer: Relative frequency. Specifically, by carrying out \( n \) coin-tossing experiments, the relative frequency of head appearance is equal to \( \frac{N_n(A)}{n} \), where \( N_n(A) \) is the number of head appearance in these \( n \) random experiments.
A1.1 Relative Frequency

Is relative frequency close to the true probability (of head appearance)?

It could occur that 4-out-of-10 tossing results are “head” for a fair coin!

Can one guarantee that the true “head appearance probability” remains unchanged (i.e., time-invariant) in each experiment (performed at different time instance)?
A1.1 Relative Frequency

Similarly, the previous question can be extended to “In a communication system, can we estimate the noise by repetitive measurements at consecutive but different time instance?”

Some assumptions on the statistical models are necessary!
1.1.2 Axioms of Probability

Definition of a Probability System \((S, F, P)\) (also named Probability Space)

1. Sample space \(S\)
   
   All possible outcomes (sample points) of the experiment

2. Event space \(F\)
   
   Subset of sample space, which can be probabilistically measured.
   
   \[^\text{A} \in F \text{ and } B \in F \implies A \cup B \in F.\]

3. Probability measure \(P\)
1.1.2 Axioms of Probability

3. Probability measure $P$,

A probability measure satisfies:

† $P(S) = 1$ and $P(\text{EmptySet}) = 0$

For any $A$ in $F$, $0 \leq P(A) \leq 1$.

† For any two mutually exclusive events $A$ and $B$, $P(A \cup B) = P(A) + P(B)$

These Axioms coincide with the relative-frequency expectation.

1. $N_n(S) = n$ and $N_n(\text{emptySet}) = 0$.
2. $0 \leq \frac{N_n(A)}{n} \leq 1$.
3. $N_n(A \cup B) = N_n(A) + N_n(B)$
1.1.2 Axioms of Probability

Example. Rolling a dice.

- \( S = \{ \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \square & \square & \square & \square \\ \bullet & \bullet & \bullet & \bullet \end{array} \} \)

- \( F = \{ \text{EmptySet}, \{ \begin{array}{cccc} \bullet & \bullet & \bullet \end{array} \}, \{ \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \end{array} \}, S \} \)

- \( P \) satisfies
  - \( P(\text{EmptySet}) = 0 \)
  - \( P(\{ \begin{array}{cccc} \bullet & \bullet & \bullet \end{array} \}) = 0.4 \)
  - \( P(\{ \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \end{array} \}) = 0.6 \)
  - \( P(S) = 1 \).
1.1.3 Properties from Axioms

All the properties are induced from Axioms

Example 1. \( P(A^c) = 1 - P(A) \).

Proof. Since \( A \) and \( A^c \) are mutually exclusive events,
\[
P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1.
\]

Example 2. \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \).

Proof. \( P(A \cup B) = P(A/B) + P(B/A) + P(A \cap B) = [P(A/B) + P(A \cap B)] + [P(B/A) + P(A \cap B)] - P(A \cap B) = P(A) + P(B) - P(A \cap B) \).
1.1.4 Conditional Probability

Definition of conditional probability

\[ P(B \mid A) \approx \frac{N_n(A \cap B)}{N_n(A)} = \frac{P(A \cap B)}{P(A)} \]

Independence of events \( P(B \mid A) = P(B) \)

- A knowledge of occurrence of event \( A \) tells us no more about the probability of occurrence of event \( B \) than we knew without this knowledge.
- Hence, they are statistically independent.
1.1.5 Random Variable

A probability system \((S, F, P)\) can be “visualized” (or observed, recorded) through a real-valued random variable \(X\)

After mapping the sample point in the sample space to a real number, the cumulative distribution function (cdf) can be defined as:

\[
F_X(x) = \Pr[X \leq x] = P\{s \in S : X(s) \leq x\}
\]
1.1.5 Random Variable

Since the event \([X \leq x]\) has to be \textit{probabilistically measurable} for any real number \(x\), the event space \(F\) has to consist of all the elements of the form \([X \leq x]\).

In previous example, the event space \(F\) must contain the intersections and unions of the following 6 sets.

\[
\begin{align*}
\{ \} &= \{s \in S : X(s) \leq 1 \} \\
\{ \bullet \} &= \{s \in S : X(s) \leq 2 \} \\
\{ \bullet \} &= \{s \in S : X(s) \leq 3 \} \\
\{ \bullet \} &= \{s \in S : X(s) \leq 4 \} \\
\{ \bullet \} &= \{s \in S : X(s) \leq 5 \} \\
\{ \bullet \} &= \{s \in S : X(s) \leq 6 \}
\end{align*}
\]

\[\text{Otherwise, the cdf is not well-defined.}\]
1.1.5 Random Variable

It can be proved that we can construct a well-defined probability system \((S, F, P)\) for any random variable \(X\) and its cdf \(F_X\).

- So to define a **real-valued** random variable by its cdf is *good* enough from engineering standpoint.
- In other words, it is not necessary to mention the probability system \((S, F, P)\) before defining a random variable.
1.1.5 Random Variable

It can be proved that any function satisfying:

1. \( \lim_{x \to -\infty} G(x) = 0 \) and \( \lim_{x \to \infty} G(x) = 1 \).
2. Right-continuous.
3. Non-decreasing.

is a legitimate cdf for some random variable.

It suffices to check the above three properties for \( F_X(x) \) to well-define a random variable.
1.1.5 Random Variable

A non-negative function $f_X(x)$ satisfies

$$\Pr(X \leq x) = \int_{-\infty}^{x} f_X(t) dt$$

is called the \textit{probability density function (pdf)} of random variable $X$.

If the pdf of $X$ exists, then

$$f_X(x) = \frac{\partial F_X(x)}{\partial x}$$
1.1.6 Random Vector

We can extend a random variable to a (multi-dimensional) random vector.

- For two random variables $X$ and $Y$, the joint cdf is defined as:

$$F_{X,Y}(x, y) = \Pr(X \leq x \text{ and } Y \leq y)$$

- Again, the event $[X \leq x \text{ and } Y \leq y]$ must be probabilistically measurable in some probability system $(S, F, P)$ for any real numbers $x$ and $y$. 
1.1.6 Random Vector

As continuing from previous example, the event space $F$ must contain the intersections and unions of the following 4 sets.

$$(X, Y) : S \rightarrow \mathbb{R} \times \mathbb{R}$$

- $\{ \bullet \} = \{ s \in S : X(s) \leq 1 \text{ and } Y(s) \leq 1 \} \rightarrow (1, 1)$
- $\{ \bullet \bullet \} = \{ s \in S : X(s) \leq 2 \text{ and } Y(s) \leq 1 \} \rightarrow (1, 2)$
- $\{ \bullet \bullet \bullet \} = \{ s \in S : X(s) \leq 1 \text{ and } Y(s) \leq 2 \} \rightarrow (2, 1)$
- $\{ \bullet \bullet \bullet \bullet \} = \{ s \in S : X(s) \leq 2 \text{ and } Y(s) \leq 2 \} \rightarrow (2, 2)$
1.1.6 Random Vector

- If its joint density $f_{X,Y}(x,y)$ exists, then

\[ f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \]

- The conditional density function of $Y$ given that $[X = x]$ is

\[ f_{X,Y}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \]

provided that $f_X(x) \neq 0$. 

18
1.2 **Mathematical Definition** of a Random Process (RP)

The properties of RP

a. Function of time.

b. Random in the sense that before conducting an experiment, not possible to define the waveform.

Sample space $S$ function of time, $X(t,s)\ (-T,T)$

A random variable (r.v) is a real-valued function defined on the elements of the Sample space.
\[ S \rightarrow X(t,s) \quad -T \leq t \leq T \] (1.1)

2T: The total observation interval
\[ s_j \rightarrow X(t, s_j) = x_j(t) \] (1.2)

\( x_j(t) \) sample function \( \lambda \leftarrow \) which is deterministic

At \( t = t_k \), the set \( \{ x_j(t_k) \} \) constitutes a random
variable (RV).

To simplify the notation, let \( X(t,s) = X(t) \)

\( X(t) \): Random process, an ensemble of time functions
together with a probability rule.

**Difference between RV and RP**

RV: The outcome is mapped into a real number
RP: The outcome is mapped into a function of time
Figure 1.1 An ensemble of sample functions:
\[ \{ x_j(t) \mid j = 1, 2, \ldots, n \} \]

Note: \( X(t_1), X(t_2) \ldots X(t_n) \) are statistically independent for any \( t_1, t_2, \ldots, t_n \)
1.3 Stationary Process

Stationary Process:

The statistical characterization of a process is independent of the time at which observation of the process is initiated.

Nonstationary Process:

Not a stationary process (unstable phenomenon)

Consider $X(t)$ which is initiated at $t = -\infty$, $X(t_1), X(t_2), \ldots, X(t_k)$ denote the RV obtained at $t_1, t_2, \ldots, t_k$

For the RP to be stationary in the strict sense (strictly stationary)

The joint distribution function independent of $\tau$

$$F_{X(t_1+\tau), \ldots, X(t_k+\tau)}(x_1, \ldots, x_k) = F_{X(t_1), \ldots, X(t_k)}(x_1, \ldots, x_k) \quad (1.3)$$

For all time shift $\tau$, all $k$, and all possible choice of $t_1, t_2, \ldots, t_k$
1.3 Random Process

Question: Can we map $s$ to two or more real-valued deterministic functions?

Answer: Yes, such as $(X(t), Y(t))$.

Then, we can discuss any finite-dimensional joint distribution of $X(t)$ and $Y(t)$, such as, the joint distribution of

$$(X(t_1), X(t_2), X(t_3), Y(t_1), Y(t_4))$$
$X(t)$ and $Y(t)$ are jointly strictly stationary if the joint finite-dimensional distribution of \( \{X(t_1)\cdots X(t_k)\} \) and \( \{Y(t'_1)\cdots Y(t'_j)\} \) are invariant w.r.t. the origin $t = 0$.

Special cases of Eq.(1.3)

1. $k = 1$, $F_X(t)(x) = F_X(t+\tau)(x) = F_X(x)$ for all $t$ and $\tau$ (1.4)
2. $k = 2$, $\tau = -t_1$

\[
F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2) \tag{1.5}
\]

which only depends on $t_2-t_1$ (time difference)
1.3 Stationary

Example 1.1. Suppose that any finite-dimensional cdf of a random process $X(t)$ is defined. Find the probability of the joint event.

$$A = \left[ a_1 < X(t_1) \leq b_1 \text{ and } a_2 < X(t_2) \leq b_2 \right]$$

Answer:

$$P(A) = F_{X(t_1),X(t_2)}(b_1, b_2) - F_{X(t_1),X(t_2)}(b_1, a_2) - F_{X(t_1),X(t_2)}(a_1, b_2) + F_{X(t_1),X(t_2)}(a_1, a_2)$$
1.3 (Strictly) Stationary
1.3 (Strictly) Stationary

Example 1.1. Further assume that $X(t)$ is strictly stationary.

Then, $P(A)$ is also equal to:

$$P(A) = F_{X(t_1+\tau),X(t_2+\tau)}(b_1, b_2) - F_{X(t_1+\tau),X(t_2+\tau)}(b_1, a_2)$$

$$- F_{X(t_1+\tau),X(t_2+\tau)}(a_1, b_2) + F_{X(t_1+\tau),X(t_2+\tau)}(a_1, a_2)$$
1.3 (Strictly) Stationary

Why introducing “stationarity”?

With stationarity, we can certain that the observations made at different time instances have the same distributions!
1.4 Mean, Correlation, and Covariance Function

Let $X(t)$ be a strictly stationary RP

The mean of $X(t)$ is

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} xf_{X(t)}(x) \, dx$$

$$= \mu_X \quad \text{(indep. of t) for all } t$$

$f_{X(t)}(x)$: the first order pdf which is independent of time.

The autocorrelation function of $X(t)$ is

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1x_2 f_{X(t_1)X(t_2)}(x_1, x_2) \, dx_1 \, dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1x_2 f_{X(0)X(t_2-t_1)}(x_1, x_2) \, dx_1 \, dx_2$$

$$= R_X(t_2 - t_1) \quad \text{stationary for all } t_1 \text{ and } t_2$$

$f_{X(t_1)X(t_2)}(x_1, x_2)$ is the second order pdf
The autocovariance function

\[ C_X(t_1, t_2) = E[(X(t_1) - \mu_X)(X(t_2) - \mu_X)] \]

\[ = R_X(t_2 - t_1) - \mu_X^2 \]  \hspace{1cm} (1.10)

Which is of function of time difference \((t_2-t_1)\).

We can determine \(C_X(t_1, t_2)\) if \(\mu_X\) and \(R_X(t_2-t_1)\) are known.

Note that:

1. \(\mu_X\) and \(R_X(t_2-t_1)\) only provide a partial description.
2. If \(\mu_X(t) = \mu_X\) and \(R_X(t_1, t_2)=R_X(t_2-t_1)\),
   then \(X(t)\) is wide-sense stationary (stationary process).
3. The class of strictly stationary processes with finite second-order moments is a subclass of the class of all stationary processes.
4. The first- and second-order moments may not exist.

\[(e.g. f(x) = \frac{1}{\pi \frac{1}{1+x^2}}, -\infty < x < \infty)\]
1.4 Wide-Sense Stationary (WSS)

Since in most cases of practical interest, only the first two moments ($\mu_X(t)$ and $C_X(t_1, t_2)$) are concerned, an alternative definition of stationarity is introduced.

Definition (Wide-Sense Stationarity) A random process $X(t)$ is WSS if

\[
\begin{aligned}
&\mu_X(t) = \text{constant;} \\
&C_X(t_1, t_2) = C_X(t_1 - t_2)
\end{aligned}
\]

or

\[
\begin{aligned}
&\mu_X(t) = \text{constant;} \\
&R_X(t_1, t_2) = R_X(t_1 - t_2)
\end{aligned}
\]
1.4 Wide-Sense Stationary (WSS)

Alternative names for WSS include:
- Weakly stationary
- Stationary in the weak sense
- Second-order stationary

If the first two moments of a random process exists, then strictly stationary implies weakly stationary (but not vice versa).
1.4 Cyclostationarity

Definition (Cyclostationarity) A random process $X(t)$ is cyclostationary if there exists a constant $T$ such that

$$
\begin{align*}
\mu_X(t + T) &= \mu_X(t); \\
C_X(t_2 + T, t_1 + T) &= C_X(t_1, t_2).
\end{align*}
$$
Properties of the autocorrelation function

For convenience of notation, we redefine

\[ R_X(\tau) = E[X(t-\tau)X(t)], \quad \text{for all } t \quad (1.11) \]

1. The mean-square value

\[ R_X(0) = E[X^2(t)], \quad \tau = 0 \quad (1.12) \]

\[ \therefore R(-\tau) = E[x(t+\tau)x(t)] = E[x(\nu)x(\nu-\tau)] \quad \nu = t + \tau = R(\tau) \quad (1.13) \]

2. \[ R_X(\tau) = R(-\tau) \quad (1.14) \]

3. \[ |R_X(\tau)| \leq R_X(0) \]
Proof of property 3:

Consider \( E[(X(t + \tau) \pm X(t))^2] \geq 0 \)

\[ \Rightarrow \quad E[X^2(t + \tau)] \pm 2E[X(t + \tau)X(t)] + E[X^2(t)] \geq 0 \]

\[ \Rightarrow \quad 2E[X^2(t)] \pm 2R_X(\tau) \geq 0 \]

\[ \Rightarrow \quad 2R_X(0) \pm 2R_X(\tau) \geq 0 \]

\[ \Rightarrow \quad -R_X(0) \leq R_X(\tau) \leq R_X(0) \]

\[ \therefore \quad |R_X(\tau)| \leq R_X(0) \]
The $R_x(\tau)$ provides the interdependence information of two random variables obtained from $X(t)$ at times $\tau$ seconds apart.
Example 1.2  \( X(t) = A\cos(2\pi f_c t + \Theta) \) \hspace{1cm} (1.15)

\[
f_\Theta(\theta) = \begin{cases} 
\frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\
0, & \text{elsewhere}
\end{cases}
\hspace{1cm} (1.16)
\]

\[
R_X(\tau) = E[X(t + \tau)X(t)] = \frac{A^2}{2} \cos(2\pi f_c \tau) \hspace{1cm} (1.17)
\]
Appendix 2.1 Fourier Transform

Let \( g(t) \) denote a nonperiodic deterministic signal, expressed as some function of time \( t \). By definition, the Fourier transform of the signal \( g(t) \) is given by the integral

\[
G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi ft) \, dt \tag{A2.1}
\]

where \( j = \sqrt{-1} \), and the variable \( f \) denotes frequency. Given the Fourier transform \( G(f) \), the original signal \( g(t) \) is recovered exactly using the formula for the inverse Fourier transform:

\[
g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft) \, df \tag{A2.2}
\]

For the Fourier transform of a signal \( g(t) \) to exist, it is sufficient but not necessary that \( g(t) \) satisfies three conditions known collectively as Dirichlet's conditions:

1. The function \( g(t) \) is single-valued, with a finite number of maxima and minima in any finite time interval.
2. The function \( g(t) \) has a finite number of discontinuities in any finite time interval.
3. The function \( g(t) \) is absolutely integrable, that is,

\[
\int_{-\infty}^{\infty} |g(t)| \, dt < \infty
\]
We refer to $|G(f)|$ as the magnitude spectrum of the signal $g(t)$, and refer to arg \{G(f)\} as its phase spectrum.

Indeed, we may go one step further and state that all energy signals, that is, signals $g(t)$ for which

$$
\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty
$$

are Fourier transformable.
Strictly speaking, the theory of the Fourier transform is applicable only to time functions that satisfy the Dirichlet conditions. Such functions include energy signals. However, it would be highly desirable to extend this theory in two ways:

1. To combine the Fourier series and Fourier transform into a unified theory, so that the Fourier series may be treated as a special case of the Fourier transform.

2. To include power signals (i.e., signals for which the average power is finite) in the list of signals to which we may apply the Fourier transform.
The Dirac delta function or just delta function, denoted by \( \delta(t) \), is defined as having zero amplitude everywhere except at \( t = 0 \), where it is infinitely large in such a way that it contains unit area under its curve; that is

\[
\delta(t) = 0, \quad t \neq 0
\]  

(A2.3)

and

\[
\int_{-\infty}^{\infty} \delta(t) \, dt = 1
\]  

(A2.4)

\[
\int_{-\infty}^{\infty} g(t) \delta(t-t_0) \, dt = g(t_0)
\]  

(A2.5)

\[
\int_{-\infty}^{\infty} g(\tau) \delta(t-\tau) \, d\tau = g(t)
\]  

(A2.6)
### Table A6.2  Summary of properties of the Fourier transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Mathematical Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Linearity</td>
<td>( ag_1(t) + bg_2(t) \Leftrightarrow aG_1(f) + bG_2(f) ) where ( a ) and ( b ) are constants</td>
</tr>
<tr>
<td>2. Time scaling</td>
<td>( g(at) \Leftrightarrow \frac{1}{</td>
</tr>
<tr>
<td>3. Duality</td>
<td>If ( g(t) \Leftrightarrow G(f) ), then ( G(t) \Leftrightarrow g(-f) )</td>
</tr>
<tr>
<td>4. Time shifting</td>
<td>( g(t - t_0) \Leftrightarrow G(f) \exp(-j2\pi ft_0) )</td>
</tr>
<tr>
<td>5. Frequency shifting</td>
<td>( \exp(j2\pi ft_c)g(t) \Leftrightarrow G(f - f_c) )</td>
</tr>
<tr>
<td>6. Area under ( g(t) )</td>
<td>( \int_{-\infty}^{\infty} g(t) , dt = G(0) )</td>
</tr>
<tr>
<td>7. Area under ( G(f) )</td>
<td>( g(0) = \int_{-\infty}^{\infty} G(f) , df )</td>
</tr>
<tr>
<td>8. Differentiation in the time domain</td>
<td>( \frac{d}{dt} g(t) \Leftrightarrow j2\pi f G(f) )</td>
</tr>
<tr>
<td>9. Integration in the time domain</td>
<td>( \int_{-\infty}^{t} g(\tau) , d\tau \Leftrightarrow \frac{1}{j2\pi f} , G(f) + \frac{G(0)}{2} \delta(f) )</td>
</tr>
<tr>
<td>10. Conjugate functions</td>
<td>If ( g(t) \Leftrightarrow G(f) ), then ( g^<em>(t) \Leftrightarrow G^</em>(-f) )</td>
</tr>
<tr>
<td>11. Multiplication in the time domain</td>
<td>( g_1(t)g_2(t) \Leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda) , d\lambda )</td>
</tr>
<tr>
<td>12. Convolution in the time domain</td>
<td>( \int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau) , d\tau \Leftrightarrow G_1(f)G_2(f) )</td>
</tr>
</tbody>
</table>
### Table A6.3 Fourier-transform pairs

<table>
<thead>
<tr>
<th>Time Function</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>rect(\left(\frac{t}{T}\right))</td>
<td>(T \text{sinc}(fT))</td>
</tr>
<tr>
<td>sinc((2Wt))</td>
<td>(\frac{1}{2W} \text{rect}\left(\frac{f}{2W}\right))</td>
</tr>
<tr>
<td>exp((-at)u(t), a &gt; 0)</td>
<td>(\frac{1}{a + j2\pi f})</td>
</tr>
<tr>
<td>exp((-a</td>
<td>t</td>
</tr>
<tr>
<td>exp((-\pi t^2))</td>
<td>(\text{exp}\left(-\pi f^2\right))</td>
</tr>
<tr>
<td>(\begin{cases} 1 - \frac{</td>
<td>t</td>
</tr>
<tr>
<td>(\delta(t))</td>
<td>(\delta(f))</td>
</tr>
<tr>
<td>(\delta(t - t_0))</td>
<td>(\text{exp}\left(-j2\pi ft_0\right))</td>
</tr>
<tr>
<td>(\exp(j2\pi fc t))</td>
<td>(\delta(f - fc))</td>
</tr>
<tr>
<td>(\cos(2\pi fc t))</td>
<td>(\frac{1}{2j} [\delta(f - fc) + \delta(f + fc)])</td>
</tr>
<tr>
<td>(\sin(2\pi fc t))</td>
<td>(\frac{1}{2j} [\delta(f - fc) - \delta(f + fc)])</td>
</tr>
<tr>
<td>(\text{sgn}(t))</td>
<td>(\frac{1}{j\pi f})</td>
</tr>
<tr>
<td>(\frac{1}{\pi t})</td>
<td>(-j \text{sgn}(f))</td>
</tr>
<tr>
<td>(u(t))</td>
<td>(\frac{1}{2} \delta(f) + \frac{1}{j2\pi f})</td>
</tr>
<tr>
<td>(\sum_{i=-\infty}^{\infty} \delta(t - iT_0))</td>
<td>(\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right))</td>
</tr>
</tbody>
</table>

**Notes:**
- \(u(t)\) = unit step function
- \(\delta(t)\) = delta function, or unit impulse
- \(\text{rect}(t)\) = rectangular function of unit amplitude and unit duration centered on the origin
- \(\text{sgn}(t)\) = signum function
- \(\text{sinc}(t)\) = sinc function
Example 1.3 Random Binary Wave / Pulse

1. The pulses are represented by $\pm A$ volts (mean=0).
2. The first complete pulse starts at $t_d$.

$$f_{T_d} (t_d) = \begin{cases} \frac{1}{T}, & 0 \leq t_d \leq T \\ 0, & \text{elsewhere} \end{cases}$$

3. During $(n - 1)T < t - t_d < nT$, the presence of $+A$ or $-A$ is random.
4. When $|t_k - t_i| > T$, $T_k$ and $T_i$ are not in the same pulse interval, hence, $X(t_k)$ and $X(t_i)$ are independent.

$$\Rightarrow E[X(t_k)X(t_i)] = E[X(t_k)]E[X(t_i)] = 0$$
Figure 1.6
Sample function of random binary wave.

\[ x(t) \]

\[ \begin{align*}
+A \\
0 \\
-A \\
\end{align*} \]

\[ t_d \quad T \]
5. For $|t_k - t_i| < T$, $t_k = 0$, $t_i < t_k$, $t_i < 0$, $X(t_k)$ and $X(t_i)$ occur in the same pulse interval 

iff $t_d < T - |t_k - t_i|$

i.e., $t_d + |-t_i| < T$

$E[X(t_k)X(t_i) \mid t_d] = \begin{cases} 
A^2, & t_d < T - |t_k - t_i| \\
0, & \text{elsewhere}
\end{cases}$

$E[X(t_k)X(t_i)] = \int_0^{T - |t_k - t_i|} A^2 f_{T_d}(t_d) dt_d$

$= \int_0^{T - |t_k - t_i|} \frac{A^2}{T} dt_d$

$= A^2 \left(1 - \frac{|t_k - t_i|}{T}\right) \quad |t_k - t_i| < T$
6. Similar reason for any other value of \( t_k \)

\[
R_X(\tau) = \begin{cases} 
A^2 (1 - \frac{|\tau|}{T}), & |\tau| < T \\
0, & |\tau| \geq T 
\end{cases}, \text{ where } \tau = t_k - t_i
\]

What is the Fourier Transform of \( R_X(\tau) \)?

\[
S_x(f) = A^2 T \sin c^2(fT)
\]

Cross-correlation Function of $X(t)$ and $Y(t)$

and \[ R_{XY}(t,u) = E[X(t)Y(u)] \] \hspace{1cm} (1.19)
\[ R_{YX}(t,u) = E[Y(t)X(u)] \] \hspace{1cm} (1.20)

Note $R_{XY}(t,u)$ and $R_{YX}(t,u)$ are not general even functions.

The correlation matrix is

\[ R(t,u) = \begin{bmatrix} R_x(t,u) & R_{XY}(t,u) \\ R_{YX}(t,u) & R_y(t,u) \end{bmatrix} \]

$R_x(t,u)$, $R_y(t,u)$ are autocorrelation functions

If $X(t)$ and $Y(t)$ are jointly stationary

\[ R(\tau) = \begin{bmatrix} R_x(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_y(\tau) \end{bmatrix} \] \hspace{1cm} (1.21)

where $\tau = t - u$
Proof of $R_{XY}(\tau) = R_{YX}(-\tau)$:

$$R_{XY}(\tau) = E[ X(t)Y(t - \tau)]$$

Let $t - \tau = \mu$,

$$\Rightarrow R_{XY}(\tau) = E[ X(\mu + \tau)Y(\mu)]$$

$$= E[ Y(\mu)X(\mu + \tau)]$$

$$= E[ Y(t')X(t' - (-\tau))]$$

$$= R_{YX}(-\tau) \quad (1.22)$$
Example 1.4 Quadrature-Modulated Process

\[ X_1(t) = X(t) \cos(2\pi f_c t + \Theta) \]
\[ X_2(t) = X(t) \sin(2\pi f_c t + \Theta), \]

where \( X(t) \) is a stationary process and \( \Theta \) is uniformly distributed over \([0, 2\pi]\).

\[ R_{12}(\tau) = E \left[ X_1(t)X_2(t - \tau) \right] = \mathbb{E}_\Theta \mathbb{E}_{X(t)} \left[ X_1(t)X_2(t - \tau) \right] \]
\[ = \mathbb{E}[X(t)X(t - \tau)] \mathbb{E}[\cos(2\pi f_c t + \Theta)\sin(2\pi f_c t - 2\pi f_c \tau + \Theta)] \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \mathbb{E}[X(t)X(t - \tau)\cos(2\pi f_c t + \Theta)\sin(2\pi f_c t - 2\pi f_c \tau + \Theta)] d\Theta \]
\[ = \frac{1}{2} R_x(\tau) \left\{ E[\sin(4\pi f_c t - 2\pi f_c \tau + 2\Theta)] - E[\sin(2\pi f_c \tau)] \right\} \quad (1.23) \]
\[ = - \frac{1}{2} R_x(\tau) \sin(2\pi f_c \tau) \]

At \( \tau = 0, \sin(2\pi f_c \tau) = 0, R_{12}(\tau) = 0 \),

\( X_1(t) \) and \( X_2(t) \) are orthogonal at some fixed \( t \).
1.5 Ergodic Processes

Ensemble averages of $X(t)$ are averages “across the process”. (in sample space)

Long-term averages (time averages) are averages “along the process” (in time domain)

DC value of $X(t)$ (random variable)

$$\mu_x(T) = \frac{1}{2T} \int_{-T}^{T} x(t) \, dt$$  \hspace{1cm} (1.24)

If $X(t)$ is stationary,

$$\mathbb{E}[\mu_x(T)] = \frac{1}{2T} \int_{-T}^{T} \mathbb{E}[x(t)] \, dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \mu_X \, dt$$

$$= \mu_X$$  \hspace{1cm} (1.25)
\( \mu_x(T) \) represents an unbiased estimate of \( \mu_x \)

The process \( X(t) \) is \textbf{ergodic} in the mean, if

a. \( \lim_{T \to \infty} \mu_x(T) = \mu_x \)

b. \( \lim_{T \to \infty} \text{var}[\mu_x(T)] = 0 \)

The time-averaged autocorrelation function

\[
R_x(\tau, T) = \frac{1}{2T} \int_{-T}^{T} x(t + \tau)x(t)\,dt \tag{1.26}
\]

\( R_x(\tau, T) \) is a random variable.

If the following conditions hold, \( X(t) \) is ergodic in the autocorrelation functions

\[
\lim_{T \to \infty} R_x(\tau, T) = R_x(\tau)
\]

\[
\lim_{T \to \infty} \text{var}[R_x(\tau, T)] = 0
\]
Linear Time-Invariant Systems (stable)

a. The principle of superposition holds
b. The impulse response is defined as the response of the system with zero initial condition to a unit impulse or $\delta(t)$ applied to the input of the system
c. If the system is time invariant, then the impulse response function is the same no matter when the unit impulse is applied
d. The system can be characterized by the impulse response $h(t)$
e. The Fourier transform of $h(t)$ is denoted as $H(f)$
1.6 Transmission of a random Process Through a Linear Time-Invariant Filter (System)

\[
Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t - \tau) \, d\tau \quad \Rightarrow Y(f) = H(f)X(f)
\]

where \( h(t) \) is the impulse response of the system

\[
\mu_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau)X(t - \tau) \, d\tau\right] \quad (1.27)
\]

If \( E[X(t)] \) is finite

\[
= \int_{-\infty}^{\infty} h(\tau)E[x(t - \tau)] \, d\tau \quad (1.28)
\]

and system is stable

\[
= \int_{-\infty}^{\infty} h(\tau)\mu_X(t - \tau) \, d\tau
\]

If \( X(t) \) is stationary, \( \mu_Y = \mu_X \int_{-\infty}^{\infty} h(\tau) \, d\tau = \mu_X H(0) \),

\( H(0) \) : System DC response.
Consider autocorrelation function of \( Y(t) \):
\[
R_Y(t, \mu) = E[Y(t)Y(\mu)]
\]

\[
= E\left[ \int_{-\infty}^{\infty} h(\tau_1)X(t - \tau_1) \, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(\mu - \tau_2) \, d\tau_2 \right] \quad (1.30)
\]

If \( E[ X^2(t) ] \) is finite and the system is stable,
\[
R_Y(t, \mu) = \int_{-\infty}^{\infty} d\tau_1 h(\tau_1) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2)R_X(t - \tau_1, \mu - \tau_2) \quad (1.31)
\]

If \( R_X(t - \tau_1, \mu - \tau_2) = R_X(t - \mu - \tau_1 + \tau_2) \) (stationary)
\[
R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2) \, d\tau_1 \, d\tau_2 \quad (1.32)
\]

Function of time difference
Stationary input, Stationary output (WSS)
\[
R_Y(0) = E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2 - \tau_1) \, d\tau_1 \, d\tau_2 \quad (1.33)
\]
1.7 Power Spectral Density (PSD)

Consider the Fourier transform of \( g(t) \),

\[
G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi ft) \, dt
\]

\[g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft) \, df\]

Let \( H(f) \) denote the frequency response, \( \tau = \tau_2 - \tau_1 \)

Recall (1.30) \( E[Y(t)Y(u)] = E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1\int_{-\infty}^{\infty} h(\tau_2)X(\mu-\tau_2)d\tau_2\right] \)

\[
h(\tau_1) = \int_{-\infty}^{\infty} H(f) \exp(j2\pi f\tau_1) \, df
\]

(1.34)

\[
E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(f) \exp(j2\pi f\tau_1) \, df \, h(\tau_2)R_X(\tau_2 - \tau_1) \, d\tau_1 \, d\tau_2
\]

let \( \tau = \tau_2 - \tau_1 \)

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(f) \int_{-\infty}^{\infty} h(\tau_2)R_X(\tau_2 - \tau_1) \exp(j2\pi f\tau_1) \, d\tau_1 \, d\tau_2
\]

(1.35)

\[
\tau_1 = \tau_2 - \tau
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(f) \int_{-\infty}^{\infty} \underbrace{h(\tau_2)\exp(j2\pi f\tau_2)}_{*} \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) \, d\tau
\]

(1.36)

\( H(f) \) (complex conjugate response of the filter)
\[
E[Y^2(t)] = \int_{-\infty}^{\infty} df |H(f)|^2 \int_{-\infty}^{\infty} R_x(\tau) \exp(-j2\pi f \tau) d\tau
\]  
(1.37)

\[|H(f)| \text{: the magnitude response}\]

Define: Power Spectral Density (Fourier Transform of \(R(\tau)\))

\[
S_X(f) = \int_{-\infty}^{\infty} R_x(\tau) \exp(-2\pi f \tau) d\tau 
\]  
(1.38)

\[
E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df 
\]  
(1.39)

Recall \(E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_x(\tau_2 - \tau_1) d\tau_1 d\tau_2 \)  
(1.33)

Let \(|H(f)|\) be the magnitude response of an ideal narrowband filter

\[
|H(f)| = \begin{cases} 
  1, & \text{if } |f \pm f_c| < \frac{1}{2} \Delta f \\
  0, & \text{if } |f \pm f_c| > \frac{1}{2} \Delta f
\end{cases}
\]  
(1.40)

\(\Delta f\): Filter Bandwidth

If \(\Delta f \ll f_c\) and \(S_X(f)\) is continuous, 

\[
E[Y^2(t)] \approx 2\Delta f \ S_X(f_c) \quad S_X(f_c) \text{in W/Hz}
\]
Properties of The PSD

\[ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau) \, d\tau \]  \hspace{1cm} (1.42)

\[ R_X(\tau) = \int_{-\infty}^{\infty} S_X(\tau) \exp(j2\pi f \tau) \, df \]  \hspace{1cm} (1.43)

Einstein-Wiener-Khintchine relations:

\[ S_X(f) \Leftrightarrow R_X(\tau) \]

\[ S_X(f) \] is more useful than \( R_X(\tau) \)!
a. \( S_x(0) = \int_{-\infty}^{\infty} R_x(\tau) \, d\tau \)  \hspace{1cm} (1.44)

b. \( E[X^2(t)] = \int_{-\infty}^{\infty} S_x(f) \, df \)  \hspace{1cm} (1.45)

c. If \( X(t) \) is stationary,
   \[ E[Y^2(t)] \approx (2\Delta f) S_x(f) \geq 0 \]
   \[ S_x(f) \geq 0 \quad \text{for all} \quad f \]  \hspace{1cm} (1.46)

d. \( S_x(-f) = \int_{-\infty}^{\infty} R_x(\tau) \exp(j2\pi f \tau) \, d\tau \)
   \[ = \int_{-\infty}^{\infty} R_x(u) \exp(-j2\pi f u) \, du, \quad u = -\tau \]
   \[ = S_x(f) \quad \because R(-\tau) = R(\tau) \]  \hspace{1cm} (1.47)

e. The PSD can be associated with a pdf:
   \[ p_x(f) = \frac{S_x(f)}{\int_{-\infty}^{\infty} S_x(f) \, df} \]  \hspace{1cm} (1.48)
Example 1.5 Sinusoidal Wave with Random Phase

\[ X(t) = A \cos(2\pi f_c t + \Theta), \quad \Theta \sim U(-\pi, -\pi) \]

\[ R_x(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau) \quad \text{(Example 1.2)} \]

\[ S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) \exp(-j2\pi f \tau) \, d\tau \]

\[ = \frac{A^2}{4} \int_{-\infty}^{\infty} \left[ \exp(j2\pi f_c \tau) + \exp(-j2\pi f_c \tau) \right] \exp(-j2\pi f \tau) \, d\tau \]

\[ = \frac{A^2}{4} \left[ \delta(f - f_c) + \delta(f + f_c) \right] \]

\[ \Rightarrow \text{Appendix 2, } \int_{-\infty}^{\infty} \exp\left[j2\pi(f_c - f)\right] \, d\tau = \delta(f - f_c) \]

The spectral Analyzer can not detected the phase, so the phase information is lost.
Example 1.6 Random Binary Wave (Example 1.3)

Define the energy spectral density of a pulse as

\[
X(t) = \begin{cases} 
    A, & \text{if } m(t) = 1 \\
    -A, & \text{if } m(t) = 0 
\end{cases}
\]

\[
R_X(\tau) = \begin{cases} 
    A^2 (1 - \frac{|\tau|}{T}) & |\tau| < T \\
    0 & |\tau| \geq T 
\end{cases}
\]

\[
S_X(f) = \int_{-T}^{T} A^2 (1 - \frac{|\tau|}{T}) \exp(-j2\pi f \tau) \, d\tau
\]

\[= A^2 T \text{sinc}^2(fT) \quad (1.50)\]

Define the energy spectral density of a pulse as

\[
\varepsilon_s(f) = A^2 T^2 \text{sinc}^2(fT) \quad (1.51)
\]

\[
S_X(f) = \frac{\varepsilon_s(f)}{T} \quad (1.52)
\]
Example 1.7 Mixing of a Random Process with a Sinusoidal Process

\[ Y(t) = X(t) \cos(2\pi f_c t + \Theta), \quad \Theta \sim U(0, 2\pi) \quad (1.53) \]

\[ R_Y(\tau) = E[Y(t + \tau)Y(t)] \]
\[ = E[X(t + \tau)X(t)]E[\cos(2\pi f_c t + 2\pi f_c \tau + \Theta)\cos(2\pi f_c t + \Theta)] \]
\[ = \frac{1}{2} R_X(\tau)E[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)] \]
\[ = \frac{1}{2} R_X(\tau)\cos(2\pi f_c \tau) \quad (1.54) \]

\[ S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau)\exp(-j2\pi f \tau) d\tau \]
\[ = \frac{1}{4} \int_{-\infty}^{\infty} R_X(\tau)[\exp(-j2\pi(f - f_c))\tau + \exp(-j2\pi(f + f_c))\tau] d\tau \]
\[ = \frac{1}{4}[S_X(f - f_c) + S_X(f + f_c)] \quad (1.55) \]

We shift the \( S_X(f) \) to the right by \( f_c \), shift it to the left by \( f_c \), add them and divide by 4.
Relation Among The PSD of The Input and Output Random Processes

Recall (1.32)

\[ R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2) \, d\tau_1 \, d\tau_2 \]  \hspace{1cm} (1.32)

\[ S_Y(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)\exp(-j2\pi f \tau) \, d\tau_1 \, d\tau_2 \, d\tau \]

Let \( \tau - \tau_1 + \tau_2 = \tau_0 \), or \( \tau = \tau_0 + \tau_1 - \tau_2 \)

\[ S_Y(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_x(\tau_0)\exp(-j2\pi f \tau_0)\exp(j2\pi f \tau_2)\exp(-j2\pi f \tau_1) \, d\tau_1 \, d\tau_2 \, d\tau_0 \\
= S_X(f)H(f)H^*(f) \\
= |H(f)|^2 S_X(f) \]  \hspace{1cm} (1.58)
Relation Among The PSD and The Magnitude Spectrum of a Sample Function

Let \( x(t) \) be a sample function of a stationary and ergodic Process \( X(t) \).

In general, the condition for Fourier transformable is

\[
\int_{-\infty}^{\infty} |x(t)| \, dt < \infty
\]  

(1.59)

This condition can never be satisfied by any stationary \( x(t) \) of infinite duration.

We may write

\[
X(f, T) = \int_{-T}^{T} x(t) \exp(-j2\pi ft) \, dt
\]  

(1.60)

Ergodic \( \Rightarrow \) Take time average

\[
R_X(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau)x(t) \, dt
\]  

(1.61)

If \( x(t) \) is a power signal (finite average power)

\[
\frac{1}{2T} \int_{-T}^{T} x(t+\tau)x(t)\, dt \Leftrightarrow \frac{1}{2T} |X(f, T)|^2
\]  

(1.62)

Time-averaged autocorrelation periodogram function

For fixed \( f \), it is a r.v. (from one sample function to another)
Take inverse Fourier Transform of right side of (1.62)

\[
\frac{1}{2\pi} \int_{-T}^{T} x(t + \tau)x(t)dt = \int_{-\infty}^{\infty} \frac{1}{2T} |X(f,T)|^2 \exp(j2\pi fT)df
\]  

(1.63)

From (1.61),(1.63), we have

\[
R_X(\tau) = \lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{1}{2T} |X(f,T)|^2 \exp(j2\pi f\tau)df
\]  

(1.64)

Note that for any given \( x(t) \) periodogram does not converge as \( T \to \infty \).

Since \( x(t) \) is ergodic

\[
E[R_X(\tau)] = R_X(\tau) = \lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{1}{2T} E\left[ |X(f-T)|^2 \right] \exp(j2\pi f \tau)df
\]

(1.66)

Recall (1.43) \( R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f \tau)df \)

\[
S_X(f) = \lim_{T \to \infty} \frac{1}{2T} E\left[ |X(f,T)|^2 \right]
\]

(1.67)

(1.67) is used to estimate the PSD of \( x(t) \).
Cross-Spectral Densities

\[ S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) \exp(-j2\pi f \tau) d\tau \]  
(1.68)

\[ S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau) \exp(-j2\pi f \tau) d\tau \]  
(1.69)

\[ S_{XY}(f) \text{ and } S_{YX}(f) \text{ may not be real.} \]

\[ R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(f) \exp(j2\pi f \tau) df \]

\[ R_{YX}(\tau) = \int_{-\infty}^{\infty} S_{YX}(f) \exp(j2\pi f \tau) df \]

\[ \therefore R_{XY}(\tau) = R_{YX}(-\tau) \]  
(1.22)

\[ S_{XY}(f) = S_{YX}(-f) = S_{YX}^{*}(f) \]  
(1.72)
Example 1.8  \( X(t) \) and \( Y(t) \) are uncorrelated and zero mean stationary processes.

Consider \( Z(t) = X(t) + Y(t) \)

\[
S_Z(f) = S_X(f) + S_Y(f) \tag{1.75}
\]

Example 1.9  \( X(t) \) and \( Y(t) \) are jointly stationary.

\[
\begin{align*}
X(t) & \rightarrow h_1(t) \rightarrow V(t) \\
Y(t) & \rightarrow h_2(t) \rightarrow Z(t)
\end{align*}
\]

\[
R_{VZ}(t,u) = E[V(t)Z(u)]
\]

\[
= E\left[ \int_{-\infty}^{\infty} h_1(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2)Y(u-\tau_2)d\tau_2 \right]
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)R_{XY}(t-\tau_1,u-\tau_2)d\tau_1d\tau_2
\]

\[
\begin{align*}
\text{Let } \tau &= t-u \\
&= t-\tau_1-u+\tau_2 = \tau - \tau_1 + \tau_2
\end{align*}
\]

\[
R_{VZ}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)R_{XY}(\tau - \tau_1 + \tau_2)d\tau_1d\tau_2 \tag{1.77}
\]

\[
F \rightarrow S_{VZ}(f) = H_1(f)H_2^*(f)S_{XY}(f)
\]
1.8 Gaussian Process
Define: \( Y \) as a linear functional of \( X(t) \) (泛函數)

\[
Y = \int_0^T g(t) X(t) \, dt \quad (g(t): \text{some function and the integral exists}) \quad (1.79)
\]

(e.g. \( g(t): \mathcal{C} \))

The process \( X(t) \) is a Gaussian process if every linear functional of \( X(t) \) is a Gaussian random variable

\[
f_Y(y) = \frac{1}{\sqrt{2\pi} \sigma_Y} \exp \left[ - \frac{(y - \mu_Y)^2}{2\sigma_Y^2} \right] \quad (1.80)
\]

Normalized \( f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp \left(- \frac{y^2}{2} \right)\), as \( N(0,1) \) \quad (1.81)

Fig. 1.13 Normalized Gaussian distribution
Central Limit Theorem
Let $X_i$ , $i=1,2,3,\ldots,N$ be (a) statistically independent R.V.
and (b) have mean $\mu_x$ and variance $\sigma_x^2$.
Since they are independently and identically distributed (i.i.d.)
Normalized $X_i$

$$\Rightarrow Y_i = \frac{1}{\sigma_x}(X_i - \mu_x) \quad i = 1,2,\ldots,N$$

Hence, $\mathbb{E}[Y_i] = 0,$
$$\text{Var}[Y_i] = 1.$$ Define $V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i$

The Central Limit Theorem
The probability distribution of $V_N$ approaches $N(0,1)$
as $N$ approaches infinity.
Note: For some random variables, the approximation is poor even $N$ is quite large.
Properties of A Gaussian Process

1. \( \frac{X(t)}{h(t)} \rightarrow Y(t) \)

Gaussian in \( h(t) \) Gaussian out

\[
Y(t) = \int_{0}^{T} h(t - \tau)X(\tau)d\tau
\]

Define \( Z = \int_{0}^{\infty} g_Y(t)\int_{0}^{T} h(t - \tau)X(\tau)\,d\tau\,dt \)

\[
= \int_{0}^{\infty} \int_{0}^{T} g_Y(t)h(t - \tau)\,dt\,X(\tau)\,d\tau
\]

\[
= \int_{0}^{T} g(\tau)X(\tau)\,d\tau
\]

where \( g(\tau) = \int_{0}^{\infty} g_Y(t)h(t - \tau)dt \)

By definition \( Z \) is a Gaussian random variable (1.81)

\[
\Rightarrow Y(t) = \int_{0}^{T} h(t - \tau)X(\tau)d\tau, \quad 0 \leq t < \infty \text{ is Gaussian}
\]
2. If \( X(t) \) is Gaussian

Then \( X(t_1) , X(t_2) , X(t_3) , \ldots , X(t_n) \) are jointly Gaussian.

Let \( \mu_{X(t_i)} = E[X(t_i)] \) \( i = 1,2,\ldots,n \)

and the set of covariance functions be

\[
C_{X}(t_k,t_i) = E\left[\left( X(t_k) - \mu_{X(t_k)} \right)\left( X(t_i) - \mu_{X(t_i)} \right) \right], \quad \text{for } k,i = 1,2,\ldots,n
\]

where \( X = [X(t_1), X(t_2), \ldots, X(t_n)]^T \)

Then \( f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n/2}\Delta^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^T\Sigma^{-1}(x - \mu)\right) \) \hspace{1cm} (1.85)

where \( \mu = \text{mean vector} = [\mu_1, \mu_2, \ldots, \mu_n]^T \)

\( \Sigma = \text{covariance matrix} = \{C_{X}(t_k,t_i)\}_{k,i=1}^n \)

\( \Delta = \text{determinant of covariance matrix} \Sigma \)
3. If a Gaussian process is stationary then it is strictly stationary.  
   (This follows from Property 2)  
4. If $X(t_1), X(t_2), \ldots, X(t_n)$ are uncorrelated as  
   
   $$E = [(X(t_k) - \mu_{X(t_k)})(X(t_i) - \mu_{X(t_i)})] = 0$$  
   
   Then they are independent  
   Proof: uncorrelated
   $$
   \Sigma = \begin{bmatrix}
   \sigma_1^2 & 0 \\
   \vdots & \ddots \\
   0 & \sigma_n^2
   \end{bmatrix}, \text{ where } \sigma_i^2 = E[(X(t_i) - E(X(t_i))^2], i = 1,2,\ldots,n.
   $$
   
   $\Sigma^{-1}$ is also a diagonal matrix, $\Delta = \text{determinant of } \Sigma$

   $$(1.85) \quad f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n/2}\Delta^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)$$

   $$
   \Rightarrow f_X(x) = \prod_{i=1}^n f_{X_i}(x_i) \Leftarrow \text{Independent}
   $$

   where $X_i = X(t_i)$ and $f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi \sigma_i}} \exp\left(-\frac{(x_i - \mu_{X_i})^2}{2\sigma_i^2}\right)$
1.9 Noise

- Shot noise
- Thermal noise

\[ E[V_{TN}^2] = 4kTR\Delta f \] \text{volts}^2

\[ E[I_{TN}^2] = \frac{1}{R^2} E[V_{TN}^2] = 4kT \frac{1}{R} \Delta f = 4kTG\Delta f \] \text{amps}^2

\( k \): Boltzmann’s constant = 1.38 x 10^{-23} \text{ joules/K}, \( T \) is the absolute temperature in degree Kelvin.
White noise

\[ S_W(f) = \frac{N_0}{2} \quad (1.93) \]

\[ N_0 = kT_e \quad (1.94) \]

\( T_e \): equivalent noise temperature of the receiver

\[ R_W(\tau) = \frac{N_0}{2} \delta(\tau) \quad (1.95) \]

\[ S_W(f) = \int_{-\infty}^{\infty} R_w(\tau) \exp(-j2\pi f \tau) d\tau = \frac{N_0}{2} \]

\( \delta(t) \leftrightarrow 1, \quad 1 \leftrightarrow \delta(f) \quad \text{Table A6.3} \)
Example 1.10 Ideal Low-Pass Filtered White Noise

\[ S_N(f) = \begin{cases} \frac{N_0}{2} & -B < f < B \\ 0 & |f| > B \end{cases} \]  

\( R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f \tau) \, df \)  

\[ = N_0 B \text{sinc}(2B \tau) \]  

(1.96)  

(1.97)
Example 1.11 Correlation of White Noise with a Sinusoidal Wave

White noise \( w(t) \) is correlated with a sinusoidal wave \( \cos(2\pi f_c t) \), where \( f_c = \frac{k}{T} \), \( k \) is integer.

\[
E[w'(T)] = \sqrt{\frac{2}{T}} \int_0^T w(t) \cos(2\pi f_c t) dt
\]
(1.98)

The variance of \( w'(T) \) is

\[
\sigma^2 = E\left[ \frac{2}{T} \int_0^T \int_0^T w(t_1) \cos(2\pi f_c t_1) w(t_2) \cos(2\pi f_c t_2) dt_1 dt_2 \right]
\]
\[
= \frac{2}{T} \int_0^T \int_0^T E[w(t_1)w(t_2)] \cos(2\pi f_c t_1) \cos(2\pi f_c t_2) dt_1 dt_2
\]
\[
= \frac{2}{T} \int_0^T \int_0^T R_w(t_1, t_2) \cos(2\pi f_c t_1) \cos(2\pi f_c t_2) dt_1 dt_2
\]

From (1.95)

\[
\sigma^2 = \frac{2}{T} \int_0^T \int_0^T \frac{N_0}{2} \delta(t_1 - t_2) \cos(2\pi f_c t_1) \cos(2\pi f_c t_2) dt_1 dt_2
\]
\[
= \frac{N_0}{T} \int_0^T \cos^2(2\pi f_c t) dt = \frac{N_0}{2}
\]
(1.99)
1.10 Narrowband Noise (NBN)

Two representations

a. in-phase and quadrature components \( (\cos(2\pi f_c t), \sin(2\pi f_c t)) \)

b. envelope and phase

1.11 In-phase and quadrature representation

\[
n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \tag{1.100}
\]

\( n_I(t) \) and \( n_Q(t) \) are low-pass signals
Important Properties

1. $n_f(t)$ and $n_Q(t)$ have zero mean.

2. If $n(t)$ is Gaussian then $n_f(t)$ and $n_Q(t)$ are jointly Gaussian.

3. If $n(t)$ is stationary then $n_f(t)$ and $n_Q(t)$ are jointly stationary.

4. $S_{N_f}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \leq f \leq B \\ 0, & \text{otherwise} \end{cases}$ \hspace{1cm} (1.101)

5. $n_f(t)$ and $n_Q(t)$ have the same variance $\frac{N_0}{2}$.

6. Cross-spectral density is purely imaginary. (problem 1.28) $S_{N_fN_Q}(f) = -S_{N_QN_f}(f)$

$$= \begin{cases} j[S_N(f + f_c) - S_N(f - f_c)], & -B \leq f \leq B \\ 0, & \text{otherwise} \end{cases} \hspace{1cm} (1.102)$$

7. If $n(t)$ is Gaussian, its PSD is symmetric about $f_c$, then $n_f(t)$ and $n_Q(t)$ are statistically independent. (problem 1.29)
Example 1.12 Ideal Band-Pass Filtered White Noise

\[ R_N(\tau) = \int_{-f_c-B}^{-f_c-B} \frac{N_0}{2} \exp(j2\pi f \tau) df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} \exp(j2\pi f \tau) df \]
\[ = N_0 B \text{sinc}(2B\tau)[\exp(-j2\pi f_c\tau) + \exp(j2\pi f_c\tau)] \]
\[ = 2N_0 B \text{sinc}(2B\tau) \cos(2\pi f_c\tau) \]

(1.103)

Compare with (1.97) (a factor of 2),

\[ R_{N_t}(\tau) = R_{N_Q}(\tau) = 2N_0 B \text{sinc}(2B\tau). \]

(Low-Pass filtered \( R_N(\tau) = N_0 B \text{sinc}(2B\tau) \))
1.12 Representation in Terms of Envelope and Phase Components

\[ n(t) = r(t) \cos[2\pi f_c t + \psi(t)] \]  \hspace{1cm} (1.105)

Envelope

\[ r(t) = \left[ n_I^2(t) + n_Q^2(t) \right]^{1/2} \]  \hspace{1cm} (1.106)

Phase

\[ \psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right] \]  \hspace{1cm} (1.107)

Let \( N_I \) and \( N_Q \) be R.V.s obtained (at some fixed time) from \( n_I(t) \) and \( n_Q(t) \). \( N_I \) and \( N_Q \) are independent Gaussian with zero mean and variance \( \sigma^2 \).
\[ f_{N_1,N_Q}(n_I,n_Q) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_I^2 + n_Q^2}{2\sigma^2}\right) \]  \hspace{1cm} (1.108)

\[ f_{N_1,N_Q}(n_I,n_Q)dn_I dn_Q = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_I^2 + n_Q^2}{2\sigma^2}\right) \, dn_I \, dn_Q \]  \hspace{1cm} (1.109)

Let \( n_I = r \cos\psi \) \hspace{1cm} (1.110)

\( n_Q = r \sin\psi \) \hspace{1cm} (1.111)

\( \Rightarrow dn_I \, dn_Q = r \, dr \, d\psi \) \hspace{1cm} (1.112)
Substituting (1.110) - (1.112) into (1.109)\
\[ f_{N_1,N_Q}(n_1,n_Q)dn_1,dn_Q = f_{R,\psi}(r,\psi)rdrd\psi \]
\[ = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)drd\psi \]
\[ f_{R,\psi}(r,\psi) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad (1.113) \]
\[ 0 \leq \psi \leq 2\pi, \quad f_{\psi}(\psi) = \begin{cases} 1 & 0 \leq \psi \leq 2\pi \\ \frac{1}{2\pi} & \text{elsewhere} \\ 0 & \text{elsewhere} \end{cases} \quad (1.114) \]
\[ f_R(r) = \begin{cases} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), & r \geq 0 \\ 0 & \text{elsewhere} \end{cases} \quad (1.115) \]

\( f_R(r) \) is Rayleigh distribution.

For convenience, let \( \nu = \frac{r}{\sigma} \Rightarrow f_\nu(\nu) = \sigma f_R(r) \) (Normalized)

\[ f_\nu(\nu) = \begin{cases} \nu \exp\left(-\frac{\nu^2}{2}\right), & \nu \geq 0 \\ 0 & \text{elsewhere} \end{cases} \quad (1.118) \]
Figure 1.22 Normalized Rayleigh distribution.
1.13 Sine Wave Plus Narrowband Noise

\[ x(t) = A \cos(2\pi f_c t) + n(t) \quad (1.119) \]

\[ x(t) = n'_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \]

\[ n'_I(t) = A + n_I(t) \]

If \( n(t) \) is Gaussian with zero mean and variance \( \sigma^2 \)

1. \( n'_I(t) \) and \( n_Q(t) \) are Gaussian and statistically independent.

2. The mean of \( n'_I(t) \) is \( A \) and that of \( n_Q(t) \) is zero.

3. The variance of \( n'_I(t) \) and \( n_Q(t) \) is \( \sigma^2 \).

\[
 f_{N_I',N_Q}(n'_I, n_Q) = \frac{1}{2\pi\sigma^2} \exp\left[ -\frac{(n'_I - A)^2 + n_Q^2}{2\sigma^2} \right]
\]

Let \( r(t) = \left\{ \left[ n'_I(t) \right]^2 + n_Q^2(t) \right\}^{1/2} \quad (1.123) \)

\[
 \psi(t) = \tan^{-1} \left[ \frac{n_Q(t)}{n'_I(t)} \right] \quad (1.124)
\]

Follow a similar procedure, we have

\[
 f_{R,\psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \exp\left( -\frac{r^2 + A^2 - 2A r \cos \psi}{2\sigma^2} \right)
\]

\( \Rightarrow R \) and \( \psi \) are dependent.
The modified Bessel function of the first kind of zero order is defined as (Appendix 3)

\[ f_R(r) = \frac{1}{2\pi} \int_0^{2\pi} f_{R,\psi}(r,\psi) d\psi \]

\[ = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) \int_0^{2\pi} \exp\left(\frac{Ar}{\sigma^2} \cos\psi\right) d\psi \]  (1.126)

Let \( x = \frac{Ar}{\sigma^2} \), \( f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) I_0\left(\frac{Ar}{\sigma^2}\right) \)  (1.128)

It is called Rician distribution.

If \( A = 0 \), \( I_0(0) = \frac{1}{2\pi} \int_0^{2\pi} d\psi = 1 \),

it is Rayleigh distribution.
Normalized \( \nu = \frac{r}{\sigma}, \quad a = \frac{A}{\sigma} \)

\[
f_V(\nu) = \sigma f_R(r) = \nu \exp(-\frac{\nu^2 + a^2}{2}) I_0(\nu a) \quad (1.132)
\]