Chapter 2  Continuous-Wave Modulation

2.1 Introduction
2.2 Consider a carrier
\[ c(t) = A_c \cos(2\pi f_c t) \]                 (2.1)

\[ X \]

The output of the modulator
\[ s(t) = A_c [1 + k_a m(t)] \cos(2\pi f_c t) \]                 (2.2)

Where \( m(t) \) is the baseband signal, \( k_a \) is the amplitude sensitivity.

1. \( |k_a m(t)| < 1 \), for all \( t \)                 (2.3)

2. \( f_c >> W \)                 (2.4)

where \( W \) is the highest frequency of \( m(t) \)
\[ s(t) = A_c \cos(2\pi f_c t) + A_c k_a m(t) \cos(2\pi f_c t) \] (2.2)

Recall
\[ \cos(2\pi f_c t) \Leftrightarrow \frac{1}{2} \left[ \delta(f - f_c) + \delta(f + f_c) \right] \]

\[ m(t) \cos(2\pi f_c t) \Leftrightarrow \frac{1}{2} \left[ M(f - f_c) + M(f + f_c) \right] \]

\[ s(f) = \frac{A_c}{2} \left[ \delta(f - f_c) + \delta(f + f_c) \right] + \frac{k_a A_c}{2} \left[ M(f - f_c) + M(f + f_c) \right] \quad (2.5) \]

where \(M(f)\) is the Fourier Transform of \(m(t)\)

1. Negative frequency component of \(m(t)\) becomes visible.
2. \(f_c - W < M(f) < f_c\) lower sideband
   \(f_c < M(f) < f_c + W\) upper sideband
3. Transmission bandwidth \(B_T = 2W\)
Virtues and Limitations of Amplitude Modulation

Transmitter

 Receiver

Major limitations
1. AM is wasteful of power.
2. AM is wasteful of bandwidth.
2.3 Linear Modulation Schemes

Linear modulation is defined by

\[ s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t) \]  \hspace{1cm} (2.7)

\[ s_I(t) = \text{In-phase component} \]

\[ s_Q(t) = \text{Quadrature component} \]

Three types of linear modulation:
1. Double sideband-suppressed carrier (DSB-SC) modulation
2. Single sideband (SSB) modulation
3. Vestigial sideband (VSB) modulation
Notes:

1. $s_I(t)$ is solely dependent on $m(t)$
2. $s_Q(t)$ is a filtered version of $m(t)$.

The spectral modification of $s(t)$ is solely due to $s_Q(t)$. 

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**Table 2.1 Different forms of linear modulation**

<table>
<thead>
<tr>
<th>Type of Modulation</th>
<th>In-Phase Component $s_I(t)$</th>
<th>Quadrature Component $s_Q(t)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSB-SC</td>
<td>$m(t)$</td>
<td>0</td>
<td>$m(t) = \text{message signal}$</td>
</tr>
<tr>
<td>SSB:a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) Upper sideband transmitted</td>
<td>$\frac{1}{2}m(t)$</td>
<td>$\frac{1}{2}\hat{m}(t)$</td>
<td>$\hat{m}(t) = \text{Hilbert transform of } m(t)$</td>
</tr>
<tr>
<td>(b) Lower sideband transmitted</td>
<td>$\frac{1}{2}m(t)$</td>
<td>$-\frac{1}{2}\hat{m}(t)$</td>
<td></td>
</tr>
<tr>
<td>VSB:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) Vestige of lower sideband transmitted</td>
<td>$\frac{1}{2}m(t)$</td>
<td>$\frac{1}{2}m'(t)$</td>
<td>$m'(t) = \text{output of the filter of frequency response } H_Q(f)$ due to $m(t)$. For the definition of $H_Q(f)$, see Eq. (2.16)</td>
</tr>
<tr>
<td>(b) Vestige of upper sideband transmitted</td>
<td>$\frac{1}{2}m(t)$</td>
<td>$-\frac{1}{2}m'(t)$</td>
<td></td>
</tr>
</tbody>
</table>

*aFor the mathematical description of single sideband modulation, see Problem 2.16.*
Double Sideband-Suppressed Carrier (DSB-SC) Modulation

\[ s(t) = A_c m(t) \cos(2\pi f_c t) \]  \hspace{1cm} (2.8)

The Fourier transform of \( S(t) \) is

\[ s(f) = \frac{1}{2} A_c \left[ M(f - f_c) + M(f + f_c) \right] \]  \hspace{1cm} (2.9)
Coherent Detection (Synchronous Detection)

The product modulator output is

\[ v(t) = A_c' \cos(2\pi f_c t + \phi) s(t) \]

\[ = A_c' A_c \cos(2\pi f_c t) \cos(2\pi f_c t + \phi) m(t) \]

\[ = \frac{1}{2} A_c A_c' \cos(4\pi f_c t + \phi) m(t) + \frac{1}{2} A_c A_c' \cos(\phi) m(t) \]  \hspace{1cm} (2.10)

Let \( V(f) \) be the Fourier transform of \( v(t) \)

\[ v_0(t) = \frac{1}{2} A_c A_c' \cos\phi m(t) \] \hspace{1cm} (Low pass filtered)  \hspace{1cm} (2.11)
Costas Receiver

I-channel and Q-channel are coupled together to form a negative feedback system to maintain synchronization $\phi \approx 0$

$$\frac{1}{4} A_c^2 \cos \phi \sin \phi \ m^2(t) = \frac{1}{8} A_c^2 m^2(t) \sin(2\phi)$$

$$\approx \frac{1}{4} A_c^2 m^2(t) \phi \quad (\sin 2\phi \approx 2\phi)$$

The phase control signal ceases with modulation.

(multiplier + very narrow band LF)
Quadrature-Carrier Multiplexing (or QAM)

Two DSB-SC signals occupy the same channel bandwidth, where pilot signal (tone) may be needed.

\[ s(t) = A_c m_1(t) \cos(2\pi f_c t) + A_c m_2(t) \sin(2\pi f_c t) \]
Single-Sideband Modulation (SSB)

The lower sideband and upper sideband of AM signal contain same information.

The frequency-discrimination method consists of a product modulator (DSB-SC) and a band-pass filter. The filter must meet the following requirements:

a. The desired sideband lies inside the passband.
b. The unwanted sideband lies inside the stopband.
c. The transition band is twice the lowest frequency of the message.

To recover the signal at the receiver, a pilot carrier or a stable oscillator is needed (Donald Duck effect).
Vestigial Sideband Modulation (VSB)

When the message contains near DC component

The transition must satisfy

a. \(|H(f - f_c)| + |H(f + f_c)| = 1\)

b. The phase response is linear :

\[ H(f - f_c) + H(f + f_c) = 1 \quad \text{for} \quad -W \leq f \leq W \]  \hspace{1cm} (2.13)

\[ B_T = W + f_v \]  \hspace{1cm} (2.14)
Consider the negative frequency response:

\[ |H(f)| \]

Here, the shift response \( |H(f-f_c)| \) is
and $|H(f+fc)|$ is $|H(f+f_c)|$
So, we get \(|H(f-fc)| + |H(f+fc)|\) is
Consider \(-W < f < W\) we get:

\[
\begin{align*}
\text{Which is equal to} \\
\left| H(f - f_c) \right| + \left| H(f + f_c) \right| = 1 \quad \text{for} \quad -W < f < W
\end{align*}
\]
\[ s(t) = \frac{1}{2} Acm(t) \cos(2\pi f_ct) \pm \frac{1}{2} Acm'(t) \sin(2\pi f_ct) \]  
\[ \pm \text{corresponds to upper or lower sideband} \]  
\[ H_Q(f) = j[H(f - f_c) - H(f + f_c)] \]  
\[ \text{for } -W \leq f \leq W \]  

\[ \frac{1}{j}H_Q(f) \]  

Diagram showing the relationship between \( m(t) \) and \( m'(t) \) through the filter \( H_Q(f) \).
Television Signals (NTSC)

(a) Maximum radiated field strength relative to picture carrier 1.0

- 1.25 MHz
- 0.75 MHz
- 4.5 MHz
- 0.25 MHz

Picture carrier

Sound carrier

(f(MHz))

(b) Normalized response

- Channel bandwidth 6 MHz

(f(MHz))

Picture carrier

Sound carrier
2.4 Frequency Translation

**Up conversion**
\[ f_2 = f_1 + f_l, \quad f_l = f_2 - f_1 \]

**Down conversion**
\[ f_2 = f_1 - f_l, \quad f_l = f_1 - f_2 \]
2.5 Frequency-Division Multiplexing (FDM)

[Diagram showing the process of FDM with carrier frequencies and basic groups]
2.6 Angle Modulation

Basic Definitions:

Better discrimination against noise and interference (expense of bandwidth).

\[ s(t) = A_c \cos[\theta_i(t)] \]  \hspace{1cm} (2.19)

The instantaneous frequency is

\[ f_i(t) = \lim_{\Delta t \to 0} f_{\Delta t}(t) \]

\[ = \lim_{\Delta t \to 0} \left[ \frac{\theta_i(t + \Delta t) - \theta_i(t)}{2\pi \Delta t} \right] \]

\[ = \frac{1}{2\pi} \frac{d\theta_i(t)}{dt} \]  \hspace{1cm} (2.21)

For an unmodulated carrier, \( \theta_i(t) \) is

\[ \theta_i(t) = 2\pi f_c t + \phi_c \]  \hspace{1cm} (2.22)

where \( \phi_c \) is constant
1. Phase modulation (PM)
\[ \theta_i(t) = 2\pi f_c t + k_p m(t) \]

\( k_p \): phase sensitivity of the modulator

\[ s(t) = A_c \cos[2\pi f_c t + k_p m(t)] \]  \hspace{1cm} (2.23)

2. Frequency Modulation (FM)
\[ f_i(t) = f_c + k_f m(t) \]  \hspace{1cm} (2.24)

\[ \theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \]  \hspace{1cm} (2.25)

\[ s(t) = A_c \cos \left[ 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right] \]  \hspace{1cm} (2.26)

\( k_f \): frequency sensitivity of the modulator

\[ k_p^\prime(t) = 2\pi k_f \int_0^t m(\tau) d\tau \]

\( A_c \cos (2\pi f_c t) \)  \hspace{1cm} (a) generating FM signal

\( A_c \cos (2\pi f_c t) \)  \hspace{1cm} (b) generating PM signal
2.7 Frequency Modulation

FM is a nonlinear modulation process, we can not apply Fourier transform to have spectral analysis directly.

1. Consider a single-tone modulation which produces a narrowband FM \((k_f\) is small) 

2. Next consider a single-tone and wideband FM \((k_f\) is large) 

\[ \text{let } m(t) = A_m \cos(2\pi f_m t) \quad (2.27) \text{ (deterministic)} \]

\[ f_i(t) = f_c + k_f A_m \cos(2\pi f_m t) \]

\[ = f_c + \Delta f \cos(2\pi f_m t) \quad (2.28) \]

\[ \Delta f = k_f A_m : \text{frequency deviation} \]
Recall (2.25), \[ \theta_i(t) = 2\pi \int_0^t f_i(\tau) d\tau \]

\[ = 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t) \quad (2.30) \]

Modulation index \[ \beta = \frac{\Delta f}{f_m} \quad (2.31) \]

\[ \theta_i(t) = 2\pi f_c t + \beta \sin(2\pi f_m t) \quad (2.32) \]

(2.19) \Rightarrow \quad \[ s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \quad (2.33) \]

Narrowband FM, \( \beta \) is smaller than one radian.

Wideband FM, \( \beta \) is larger than one radian.
Narrowband FM

\[ s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \]

\[ = A_c \cos(2\pi f_c t) \cos(\beta \sin(2\pi f_m t)) - A_c \sin(2\pi f_c t) \sin(\beta \sin(2\pi f_m t)) \]  

(2.34)

Because \( \beta \) is small,

\[ \cos(\beta \sin(2\pi f_m t)) \approx 1 \]

\[ \sin(\beta \sin(2\pi f_m t)) \approx \beta \sin(2\pi f_m t) \]

\[ s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t) \]  

(2.35)
The output of Fig 2.21 is
\[ s'(t) = A_c \cos(2\pi f_c t) - A_c k_f \int m(\tau)d\tau \sin(2\pi f_c t) \]

\[ s(t) \] differs from ideal condition in two respects:
1. The envelope contains a residual AM.
   (FM has constant envelope)
2. \( \theta(t) \) contains odd order harmonic distortions
   \[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

For narrowband FM, \( \beta \leq 0.3 \) radians.
Recall (2.35)

\[ s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t) \]  \hspace{1cm} (2.35)

\[ \approx A_c \cos(2\pi f_c t) + \frac{1}{2} \beta A_c \{\cos[2\pi (f_c + f_m) t] - \cos[2\pi (f_c - f_m) t]\} \]  \hspace{1cm} (2.36)

For AM with sinusoidal modulating wave, \( m(t) = \cos(2\pi f_m t) \)

\[ s_{AM}(t) = A_c [1 + k_d m(t)] \cos(2\pi f_c t) \]  \hspace{1cm} (2.2)

\[ = A_c \cos(2\pi f_c t) + k_d A_c \cos(2\pi f_c t) \cos(2\pi f_m t) \]

\[ = A_c \cos(2\pi f_c t) + \frac{1}{2} \mu A_c \{\cos[2\pi (f_c + f_m) t] + \cos[2\pi (f_c - f_m) t]\} \]  \hspace{1cm} (2.37)
Wideband FM (large $\beta$)

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$  \quad (2.33)$$

$$\exp(jx) = \cos x + j \sin x$$

$$s(t) = \text{Re}[A_c \exp(j2\pi f_c t + j\beta \sin(2\pi f_m t))]$$

$$= \text{Re}[\tilde{s}(t) \exp(j2\pi f_c (t))]$$  \quad (2.38)$$

where $\text{Re}[\ ]$ denotes the real part and

$\tilde{s}(t)$ is the complex envelope defined by

$$\tilde{s}(t) = A_c \exp[j\beta \sin(2\pi f_m t)]$$  \quad (2.39)$$

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi nf_m t)$$  \quad (2.40)$$

Complex Fourier Transform
\[ c_n = f_m \int_{-\frac{1}{2} f_m}^{\frac{1}{2} f_m} \tilde{s}(t) \exp(-j2\pi nf_m t) \, dt \]

\[ = f_m A_c \int_{-\frac{1}{2} f_m}^{\frac{1}{2} f_m} \exp[j \beta \sin(2\pi f_m t) - j2\pi nf_m t] \, dt \quad (2.41) \]

Let \( x = 2\pi f_m t \)

\[ c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)] \, dx \quad (2.42) \]

**Define the \( n \)th order Bessel function of the first kind as**

\[(A3, \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0)\]

\[ J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)] \, dx \quad (2.44) \]

\[ c_n = A_c J_n(\beta) \]

\[ \tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j2\pi nf_m t) \quad (2.45) \]
\[ s(t) = A_c \text{ Re} \left[ \sum_{-\infty}^{\infty} J_n(\beta) \exp[j2\pi(f_c + nf_m)t] \right] \] (2.47)

\[ = A_c \sum_{-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + nf_m)t] \] (2.48)

The Fourier transform of \( s(t) \) is

\[ S(f) = \frac{A_c}{2} \sum_{-\infty}^{\infty} J_n(\beta) \left[ \delta(f - f_c - nf_m) + \delta(f + f_c + nf_m) \right] \] (2.49)

![Figure 2.23: Plots of Bessel functions of the first kind for varying order.](image_url)
Properties of $J_n(\beta)$

1. $J_n(\beta) = (-1)^n J_{-n}(\beta)$, for all $n$ \hspace{1cm} (2.50)

2. If $\beta$ is small

   $J_0(\beta) \approx 1$

   $J_1(\beta) \approx \frac{\beta}{2}$

   $J_n(\beta) \approx 0 \hspace{1cm} n > 2$ \hspace{1cm} (2.51)

3. $\sum_{-\infty}^{\infty} J_n^2(\beta) = 1$

Observation of FM

1. An FM signal contains $f_c$, $f_m$, $2f_m$, $3f_m$, … components.

2. For small $\beta$, the FM signal is effectively composed of a carrier and a single pair of side frequencies at $f_c \pm f_m \Rightarrow$ narrowband FM

3. The amplitude of carrier depends on $\beta$

   \[ P = \frac{1}{2} A_c^2 = \frac{A_c^2}{2} \sum_{-\infty}^{\infty} J_n^2(\beta) \] \hspace{1cm} (2.54)
Example 2.2
Transmission Bandwidth of FM signals

With a specified amount of distortion, the FM signal is effectively limited to a finite number of significant side frequencies.

A. Carson’s rule

\[ B_T \approx 2\Delta f + 2f_m = 2\Delta f \left(1 + \frac{1}{\beta}\right) \]

\[ \beta = \frac{\Delta f}{f_m} \]

\[ \Delta f = \beta f_m \quad (2.55) \]
B. $B_T = 2n_{\text{max}} f_m$, \[ |J_{n_{\text{max}}} (\beta)| \gg 0.01 \] $B_T = 2n_{\text{max}} \frac{\Delta f}{\beta}$

### Table 2.2 Number of significant side frequencies of a wideband FM signal for varying modulation index

<table>
<thead>
<tr>
<th>Modulation Index $\beta$</th>
<th>Number of Significant Side Frequencies $2n_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2</td>
</tr>
<tr>
<td>0.3</td>
<td>4</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
</tr>
<tr>
<td>1.0</td>
<td>6</td>
</tr>
<tr>
<td>2.0</td>
<td>8</td>
</tr>
<tr>
<td>5.0</td>
<td>16</td>
</tr>
<tr>
<td>10.0</td>
<td>28</td>
</tr>
<tr>
<td>20.0</td>
<td>50</td>
</tr>
<tr>
<td>30.0</td>
<td>70</td>
</tr>
</tbody>
</table>

Universal curve for evaluating the 1 percent bandwidth of an FM wave
Example 2.3

In north America, the maximum value of frequency deviation $\Delta f$ is fixed at 75kHz for commercial FM broadcasting by radio. If we take the modulation frequency $W=15kHz$, which is typically the “maximum” audio frequency of interest in FM transmission, we find that corresponding value of the deviation ratio is

$$D = \frac{75}{15} = 5$$

Using Carson’s rule of Equation (2.55), replacing $\beta$ by $D$, and replacing $f_m$ by $W$, the approximate value of the transmission bandwidth of the FM signal is obtained as

$$B_T = 2(75+15) = 180kHz$$

On the other hand, use of the curve of Figure 2.26 gives the transmission bandwidth of the FM signal to be

$$B_T = 3.2 \ \Delta f = 3.2 \times 75 = 240kHz$$

In practice, a bandwidth of 200kHz is allocated to each FM transmission. On this basis, Carson’s rule underestimates the transmission bandwidth by 10 percent, whereas the universal curve of Figure 2.26 overestimates it by 20 percent.
Generation of FM signals

Baseband signal $m(t)$

Integrator

Narrowband phase modulator

Frequency multiplier

FM signal $s(t)$

Crystal-controlled oscillator

Frequency Multiplier

$\nu(t) = a_1 s(t) + a_2 s^2(t) + \cdots + a_n s^n(t)$  \hspace{1cm} (2.56)

$s(t) = A_c \cos \left[ 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]$

The frequency multiplier output

$s'(t) = A_c' \cos \left[ 2\pi n f_c t + 2\pi n k_f \int_0^t m(\tau) d\tau \right]$  \hspace{1cm} (2.58)

$f_i'(t) = n f_c + n k_f m(t)$  \hspace{1cm} (2.59)
Varactor diode VCO FM modulator
Crosby Direct FM Transmitter
Demodulation of FM signals

The frequency discrimination consists of a slope circuit followed by an envelope detector.

Consider Fig 2.29a, the frequency response of a slope circuit is

$$H_1(f) = \begin{cases} 
  j2\pi a(f - f_c + \frac{B_T}{2}), & f_c - \frac{B_T}{2} \leq f \leq f_c + \frac{B_T}{2} \\
  j2\pi a(f + f_c - \frac{B_T}{2}), & -f_c - \frac{B_T}{2} \leq f \leq -f_c + \frac{B_T}{2} \\
  0, & \text{elsewhere}
\end{cases}$$

(2.60)
\[ \tilde{H}_1(f - f_c) = 2H_1(f), \quad f > 0 \]

\[ \tilde{H}_2(f - f_c) = 2H_2(f), \quad f > 0 \]
Appendix 2.3 Hilbert Transform

Fourier Transform-frequency-selective
Hilbert Transform-phase-selective

(±90° shift)

Let \( g(t) \Longleftrightarrow G(f) \)

Denote the Hilbert transform of \( g(t) \) as

\[
\hat{g}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\tau)}{t - \tau} d\tau \tag{A2.31}
\]

The inverse Hilbert transform

\[
g(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(\tau)}{t - \tau} d\tau \tag{A2.32}
\]
\[ \frac{1}{\pi t} \Leftrightarrow -j \text{sgn}(f) \] (A2.33)

\[ \text{sgn}(f) = \begin{cases} 
1 & f > 0 \\
0 & f = 0 \\
-1 & f < 0 
\end{cases} \] (A2.34)

The Fourier transform of \( g(t) \) is

\[ \hat{G}(f) = -j \text{sgn}(f) G(f) \] (A2.35)
Properties of the Hilbert Transform
(time domain operation)

If \( g(t) \) is real

1. \( \hat{g}(t) \) and \( g(t) \) have the same magnitude spectrum

2. Hilbert transform of \( \hat{g}(t) \) is \( -g(t) \) (take H.F of \( \hat{g}(t) \) and compare with A2.32)

3. \( \int_{-\infty}^{\infty} g(t)\hat{g}(t)dt = 0 \Rightarrow g(t) \perp \hat{g}(t) \)
For a **band-pass system**, we consider

\[ x(t) \Leftrightarrow X(f) \]

\( X(f) \) is limited within \( \pm W \) Hz

\( W \ll f_c \)

\[ x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t) \tag{A2.48} \]

The complex envelope of \( x(t) \) is

\[ \tilde{x}(t) = x_I(t) + j x_Q(t) \tag{A2.49} \]

\[ h(t) = h_I(t) \cos(2\pi f_c t) - h_Q(t) \sin(2\pi f_c t) \tag{A2.50} \]
Define the complex impulse response
\[ \tilde{h}(t) = h_I(t) + j h_Q(t) \]  \hspace{1cm} (A2.51)

The complex representation of \( h(t) \)
\[ h(t) = \text{Re} \left[ \tilde{h}(t) \exp(j2\pi f_c t) \right] \]  \hspace{1cm} (A2.52)

\( h_I(t), h_Q(t) \) and \( \tilde{h}(t) \) are low-pass functions

From (A2.52) we have \( z = v + ju, 2v = z + z^* \)

\[ 2h(t) = \tilde{h}(t) \exp(j2\pi f_c t) + \tilde{h}^*(t) \exp(-j2\pi f_c t) \]  \hspace{1cm} (A2.53)

Apply Fourier transform to (A2.53)

\[ 2H(f) = \tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c) \]  \hspace{1cm} (A2.54)

Since \( h(t) \) is real
\[ H^*(f) = H(-f) \]

and \( \tilde{H}(f) \) is limited to \( |f| \leq B \) with \( B < f_c \)

\[ \Rightarrow \tilde{H}(f - f_c) = 2H(f), f > 0 \]  \hspace{1cm} (A2.55)

We can obtain \( \tilde{H}(f) \) from \( H(f) \), \( \tilde{H}(f') = 2H(f' + f_c) \)
Define the pre-envelope of \( h(t) \) as

\[
h_+(t) = h(t) + j\hat{h}(t), \quad \hat{h}(t) \text{: Hilbert T. of } h(t)
\]

\[
H_+(f) = H(f) + \text{sgn}(f)H(f)
\]

\[
H_+(f) = \begin{cases} 
2H(f) & f > 0 \\
H(0) & f = 0 \\
0 & f < 0 
\end{cases}
\]  (A2.37)

\[A2.58 \implies y(t) = \int_{-\infty}^{\infty} \text{Re}[h_+(\tau)] \text{Re}[x_+(t-\tau)] d\tau \quad (A2.59)\]
Recall $h_+ (t) = h(t) + j \hat{h}(t)$

$$h(t) = \text{Re}[h_+(t)]$$

$$x(t) = \text{Re}[x_+(t)]$$

To prove (A2.60)

$$\text{Re}\left[ \int_{-\infty}^{\infty} h_+(\tau)x_+(t-\tau)d\tau \right]$$

$$= \text{Re}\left[ \int_{-\infty}^{\infty} \left[ h(\tau) + j \hat{h}(\tau) \right] \left[ x(t-\tau) + j \hat{x}(t-\tau) \right] d\tau \right]$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau - \int_{-\infty}^{\infty} \hat{h}(\tau)\hat{x}(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau - \int_{-\infty}^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\tau-u} h(u) \hat{x}(t-\tau) d\tau du, \quad t-\tau = \nu, \quad \tau = t-\nu$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau + \int_{-\infty}^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-1}{t-u-\nu} \hat{x}(\nu)d\nu h(u) du$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau + \int_{-\infty}^{\infty} h(u)x(t-u)du$$

$$= 2 \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= 2 \int_{-\infty}^{\infty} \text{Re}[h_+(\tau)] \text{Re}[x_+(t-\tau)] d\tau$$
(A2.58) becomes

\[ y(t) = \int_{-\infty}^{\infty} \text{Re}[h_+(\tau)] \text{Re}[x_+(t-\tau)] \, d\tau \quad \text{(A2.59)} \]

\[
= \frac{1}{2} \text{Re} \left[ \int_{-\infty}^{\infty} h_+(\tau)x_+(t-\tau) \, d\tau \right]
\]

\[
= \frac{1}{2} \text{Re} \left[ \int_{-\infty}^{\infty} \tilde{h}(\tau) \exp(j2\pi f_c \tau) \tilde{x}(t-\tau) \exp(j2\pi f_c (t-\tau)) \, d\tau \right]
\]

\[
= \frac{1}{2} \text{Re} \left[ \exp(j2\pi f_c t) \int_{-\infty}^{\infty} \tilde{h}(\tau) \tilde{x}(t-\tau) \, d\tau \right]
\]
Comparing (A2.57) and (A2.61) we have

\[ 2\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{h}(t)\tilde{x}(t - \tau)\,d\tau \quad (A2.62) \]

or \[ 2\tilde{y}(t) = \tilde{h}(t) \ast \tilde{x}(t) \quad (A2.63) \]

We can represent bandpass signals and systems by the equivalent lowpass functions \( \tilde{x}(t), \tilde{y}(t) \) and \( \tilde{h}(t) \) (without the factor \( \exp(j2\pi f_ct) \))
\[ 2\tilde{y}(t) = \left[ h_I(t) + jh_Q(t) \right] * \left[ x_I(t) + jx_Q(t) \right] \quad (A2.64) \]
\[ = \left[ h_I(t) * x_I(t) - h_Q(t) * x_Q(t) \right] \]
\[ + j\left[ h_Q(t) * x_I(t) + h_I(t) * x_Q(t) \right] \quad (A2.65) \]

let \( \tilde{y}(t) = \tilde{y}_I(t) + j\tilde{y}_Q(t) \quad (A2.66) \)

\[ 2y_I(t) = h_I(t) * x_I(t) - h_Q(t) * x_Q(t) \quad (A2.67) \]

\[ 2y_Q(t) = h_Q(t) * x_I(t) + h_I(t) * x_Q(t) \quad (A2.68) \]
Procedure for evaluating the response of a band-pass system

1. Replace $x(t)$ by $\tilde{x}(t)$
   
   $$x(t) = \text{Re} \left[ \tilde{x}(t) \exp(j2\pi f_c t) \right]$$

2. $h(t) = \text{Re} \left[ \tilde{h}(t) \exp(j2\pi f_c t) \right]$  

3. Obtain $2\tilde{y}(t) = \tilde{h}(t) * \tilde{x}(t)$

4. $y(t) = \text{Re} \left[ \tilde{y}(t) \exp(j2\pi f_c t) \right]$
To simplify the analysis

1. shift $\tilde{H}_1(f)$ to the right by $f_c$ to align to the band-pass frequency

2. set $\tilde{H}_1(f - f_c) = 2H_1(f)$, for $f > 0$ \hspace{1cm} (2.61)

Recall

$$H_1(f) = \begin{cases} 
  j2\pi a(f - f_c + \frac{B_T}{2}) & f_c - \frac{B_T}{2} \leq f \leq f_c + \frac{B_T}{2} \\
  j2\pi a(f + f_c - \frac{B_T}{2}) & -f_c - \frac{B_T}{2} \leq f \leq -f_c + \frac{B_T}{2} \\
  0 & \text{elsewhere}
\end{cases} \hspace{1cm} (2.60)$$

From (2.60) and (2.61), we get

$$\tilde{H}_1(f) = \begin{cases} 
  j4\pi a(f + \frac{B_T}{2}) & -\frac{B_T}{2} \leq f \leq \frac{B_T}{2} \\
  0 & \text{elsewhere}
\end{cases} \hspace{1cm} (2.62)$$
Recall FM signal \( s(t) \)

\[
s(t) = A_c \cos \left[ 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]
\]

The complex envelope is

\[
\tilde{s}(t) = A_c \exp \left[ j2\pi k_f \int_0^t m(\tau) d\tau \right] \tag{2.63}
\]

Let \( \tilde{s}_1(t) \) denote the complex envelope of the slope ckt. response output.

Recall (A2.63) \( 2\tilde{y}(t) = \tilde{h}(t) * \tilde{x}(t) \), we have

\[
\tilde{S}_1(f) = \frac{1}{2} \tilde{H}_1(f) \tilde{S}(f) \text{ (upper arm of Fig 2.30 in text)}
\]

\[
= \begin{cases} 
  j2\pi a(f + \frac{B_T}{2}) \tilde{S}(f) & -\frac{B_T}{2} \leq f \leq \frac{B_T}{2} \\
  0 & \text{elsewhere}
\end{cases} \tag{2.64}
\]

\[
\Rightarrow \tilde{s}_1(t) = a \left[ \frac{d \tilde{s}(t)}{dt} + j\pi B_T \tilde{s}(t) \right] \tag{2.65}
\]

From (2.63) and (2.65), we have

\[
\tilde{s}_1(t) = j\pi B_T a A_c \left[ 1 + \frac{2k_f}{B_T} m(t) \right] \exp \left[ j2\pi k_f \int_0^t m(\tau) d\tau \right] \tag{2.66}
\]
\[ s_1(t) = \text{Re} \left[ \tilde{s}_1(t) \exp(j2\pi f_ct) \right] \]
\[ = \pi B_T aA_c \left[ 1 + \frac{2k_f}{B_T} m(t) \right] \cos \left[ 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau + \frac{\pi}{2} \right] - \sin \left[ 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right] \]

\[ s_1(t) \text{ is a hybrid-modulated signal (amplitude, frequency)} \]

However, provided that we choose \( \left| \frac{2k_f}{B_T} m(t) \right| < 1 \), for all \( t \)

using an envelope detector, we have

\[ |\tilde{s}_1(t)| = \pi B_T aA_c \left[ 1 + \frac{2k_f}{B_T} m(t) \right] \]

(2.68)

The bias term \( \pi B_T aA_c \) can be removed by a second frequency discriminator with \( H_2(f) \), where \( \tilde{H}_2(f) = \tilde{H}_1(-f) \).
Balanced Frequency Discriminator

Let the transfer function of the second branch of Fig 2.30 be (complementary slope circuit)

\[ \tilde{H}_2(f) = \tilde{H}_1(-f) \]  \hspace{1cm} (2.69)

\[ |\tilde{s}_2(t)| = \pi B_T a A_c \left[ 1 - \frac{2k_f}{B_T} m(t) \right] \]  \hspace{1cm} (2.70)

\[ s_0(t) = |\tilde{s}_1(t)| - |\tilde{s}_2(t)| \]

\[ = 4\pi k_f a A_c m(t) \]  \hspace{1cm} (2.71)
FM Stereo Multiplexing

Two factors which influence FM stereo standards

1. Operation within the allocated FM channels.
2. Compatible with monophonic radio receiver.

\[
m(t) = [m_l(t) + m_r(t)] + [m_l(t) - m_r(t)] \cos(4\pi f_c t) + K \cos(2\pi f_c t) \tag{2.72}
\]
Stereo FM

Figure 9-40. FM stereo generation block diagram.
In Figure 9-40, audio signals from both left and right microphones are combined in an linear matrixing network to produce an \( L+R \) signal and an \( L-R \) signal.

Both \( L+R \) and \( L-R \) are signals in the audio band and must be separated before modulating the carrier for transmission. This is accomplished by translating the \( L-R \) audio signal up in the spectrum.

As seen in Figure 9-40, the frequency translation is achieved by amplitude-modulating a 38-kHz subsidiary carrier in a balanced modulator to produce DSB-SC.
Stereo FM transmitter using frequency-division multiplexing.
Stereo FM transmitter: (a) block diagram; (b) resulting spectrum.
SAC: Subsidiary Communication Authorization
Stereo FM

• The stereo receiver will need a frequency-coherent 38-kHz reference signal to demodulate the DSB-SC.

• To simplify the receiver, a frequency- and phase-coherent signal is derived from the subcarrier oscillator by frequency division (÷2) to produce a pilot.

• The 19-kHz pilot fits nicely between the \( L+R \) and DSB-SC \( L-R \) signals in the baseband frequency spectrum.
As indicated by its relative amplitude in the baseband composite signal, the pilot is made small enough so that its FM deviation of the carrier is only about 10% of the total 75-kHz maximum deviation.

After the FM stereo signal is received and demodulated to baseband, the 19-kHz pilot is used to phase-lock an oscillator, which provides the 38-kHz subcarrier for demodulation of the $L-R$ signal.

A simple example using equal frequency but unequal amplitude audio toned in the $L$ and $R$ microphones is used to illustrate the formation of the composite stereo (without pilot) in Figure 9-41.
Figure 9-41. Development of composite stereo signal. The 38 kHz alternately multiplies \(L-R\) signal by +1 and −1 to produce the DSB-SC in the balanced AM modulator (part d). The adder output (shown in e without pilot) will be filtered to reduce higher harmonics before FM modulation.
Stereo FM

Spectrum of stereo FM signal.

**SCA:** Subsidiary communication authorization (commercial-free program)
2.8 Nonlinear Effects in FM Systems

1. Strong nonlinearity, e.g., square-law modulators, hard limiter, frequency multipliers.
2. Weak nonlinearity, e.g., imperfections

Nonlinear input-output relation

\[ v_0(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t) \]  \hspace{1cm} (2.73)
For FM signal
\[ v_i(t) = A_c \cos[2\pi f_c t + \phi(t)] \]
\[ \phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau \]
\[ v_0(t) = a_1 A_c \cos[2\pi f_c t + \phi(t)] + a_2 A_c^2 \cos^2[2\pi f_c t + \phi(t)] + a_3 A_c^3 \cos^3[2\pi f_c t + \phi(t)] \]
\[ = \frac{1}{2} a_2 A_c^2 + (a_1 A_c + \frac{3}{4} a_3 A_c^3) \cos[2\pi f_c t + \phi(t)] \]
\[ + \frac{1}{2} a_2 A_c^2 \cos[4\pi f_c t + 2\phi(t)] \]
\[ + \frac{1}{4} a_3 A_c^3 \cos[6\pi f_c t + 3\phi(t)] \]  
(2.75)
Carson's rule, \( B_T = 2\Delta f + 2f_m = 2\Delta f + 2W \)

\[
\begin{array}{cccccccc}
W & \quad & 2\Delta f & \quad & W & \quad & W & \quad & 4\Delta f & \quad & W \\
\end{array}
\]

In order to separate the desired FM signal from the second harmonic, we have

\[2f_c - (2\Delta f + W) > f_c + \Delta f + W \]

\[f_c > 3\Delta f + 2W \quad (2.76)\]

The output of the band-pass filter is

\[v_0'(t) = (a_1A_c + \frac{3}{4} a_3A_c^3) \cos[2\pi f_c t + \phi(t)] \quad \text{(no effect to } m(t))\]

An FM system is extremely sensitive to phase nonlinearities. Common type of source: AM-to-PM conversion.
2.9 Super Heterodyne Receiver

(Carrier-frequency tuning, filtering, amplification, and demodulation)

\[ f_{IF} = f_{LO} - f_{RF} \quad (2.78) \]

A FM system may use a limiter to remove amplitude variations.

<table>
<thead>
<tr>
<th>Table 2.3 Typical frequency parameters of AM and FM radio receivers</th>
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</thead>
<tbody>
<tr>
<td>AM Radio</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>RF carrier range</td>
</tr>
<tr>
<td>Midband frequency of IF section</td>
</tr>
<tr>
<td>IF bandwidth</td>
</tr>
</tbody>
</table>
Commercial FM Broadcast Allocations and Sidebands

- 150 kHz
- Guard bands

88 MHz  88.1  88.3  107.9 MHz  108 MHz
2.10 Noise in CW modulation System

1. **Channel model**: additive white Gaussian noise (AWGN)

2. **Receiver model**: a band-pass filter followed by an ideal demodulator

The PSD of $w(t)$ is denoted by $\frac{N_0}{2}$. 

![Diagram showing the CW modulation system with channel and receiver models, including the PSD of the noise $w(t)$ with $\frac{N_0}{2}$ and frequency $f_c$.](image)
The filtered noise in narrowband noise representation:

\[ n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \]  \hspace{1cm} (2.79)

The filtered signal for demodulation is

\[ x(t) = s(t) + n(t) \]  \hspace{1cm} (2.80)

The channel signal-to-noise ratio

\[ (\text{SNR})_c = \frac{\text{average power of } s(t)}{\text{average power of } n(t)} \]

The output signal-to-noise ratio

\[ (\text{SNR})_o = \frac{\text{average power of the demodulated signal}}{\text{average power of noise at the output}} \]

Figure of merit = \[ \frac{(\text{SNR})_o}{(\text{SNR})_c} \]  \hspace{1cm} (2.81)
2.11 Noise in Linear Receiver Using Coherent Detection

The DSD-SC system

\[ s(t) = C A_c \cos(2\pi f_c t) m(t) \quad m(t) \leftrightarrow S_M(f) \]

\[ P = \int_{-W}^{W} S_M(f) df \]  

\[ (SNR)^{C, DSB} = \frac{C^2 A_c^2 P}{2 WN_0} \]

\[ = \frac{C^2 A_c^2 P}{2 WN_0} \quad (baseband) \quad (2.84) \]

\( C: \text{system dependent scaling factor} \)
\[ x(t) = s(t) + n(t) \]
\[ = CA_c \cos(2\pi f_c t)m(t) + n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \quad (2.85) \]

\[ v(t) = x(t) \cos(2\pi f_c t) \]
\[ = \frac{1}{2} CA_c m(t) + \frac{1}{2} n_I(t) \]
\[ + \frac{1}{2} [CA_c m(t) + n_I(t)] \cos(4\pi f_c t) - \frac{1}{2} n_Q(t) \sin(4\pi f_c t) \]

high frequency components

Low-pass filter \(\Rightarrow y(t) = \frac{1}{2} CA_c m(t) + \frac{1}{2} n_I(t) \quad (2.86)\)

(2.86) indicates:

1. \(m(t)\) and \(n_I(t)\) are additive at the receiver output.
2. \(n_Q(t)\) is completely rejected by the coherent detector.
The average output signal \( \frac{1}{2} CA_c m(t) \) power = \( C^2 A_c^2 \frac{P}{4} \)

Let \( B_T = 2W \)

The average noise \( \frac{1}{2} n_I(t) \) power = \( \frac{1}{2} 2WN_0 = \frac{1}{2} WN_0 \)

\[
(SNR)_{O,DSB-SC} = \frac{C^2 A_c^2 P}{W N_0 / 2} = \frac{C^2 A_c^2 P}{2WN_0}
\]

(2.87)

\[
\frac{(SNR)_O}{(SNR)_C}_{DSB-SC} = 1
\]

(2.88)

1. Coherent SSB has the same figure of merit of DSB - SC
2. No trade-off between performance and bandwidth.

Serious problem!
2.12 Noise in AM Receivers Using Envelope Detection

\[ s(t) = A_c \left[ 1 + k_a m(t) \right] \cos(2\pi f_c t) \]
\[ = A_c \cos(2\pi f_c t) + A_c k_a m(t) \cos(2\pi f_c t) \]  \hspace{1cm} (2.89)

\[ (\text{SNR})_{C, \text{AM}} = \frac{A_c^2 (1 + k_a^2 P)}{2WN_0} \]  \hspace{1cm} (2.90)

At the output of the filter:

\[ x(t) = s(t) + n(t) \]
\[ = \left[ A_c + A_c k_a m(t) + n_1(t) \right] \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \]  \hspace{1cm} (2.91)
$y(t) = \text{envelope of } x(t)$

$$= \left\{ \left[ A_c + A_c k_a m(t) + n_I(t) \right]^2 + n_Q^2(t) \right\}^{1/2} \quad (2.92)$$

Assume $|A_c + A_c k_a m(t) + n_I(t)| >> |n_Q(t)|$

$y(t) = A_c + A_c k_a m(t) + n_I(t)$

1. $\frac{A_c^2}{2} >> WN_0 \quad \text{(carrier power > noise power)}$

2. $|k_a| < 1$

$$\left(\text{SNR}\right)_{O,AM} \approx \frac{A_c^2 k_a^2 P}{2WN_0} \quad (2.94)$$

$$\left(\frac{\text{SNR}}{\text{SNR}}\right)_O \bigg|_{AM} \approx \frac{k_a^2 P}{1 + k_a^2 P} \quad (2.95)$$
Supplements

Define the pre-demodulation SNR as

\[
\text{SNR pre-de} = \frac{\text{The average power of the modulated signal}}{\text{The average noise power at the input of the demodulator}}
\]

The Bandwidth of the bandpass filter is \( B_T \) then the average noise power at the input of the demodulator is

\[
N_oB_T
\]

For an AM system

\[
\text{SNR}^{\text{AM}}_{\text{pre-de}} = \frac{A_c^2 (1 + k_a^2 p)}{2N_oB_T} = \frac{A_c^2 (1 + k_a^2 p)}{2N_oB_T}
\]

If \( B_T = 2W \)

\[
\text{SNR}^{\text{AM}}_{\text{pre-de}} = \frac{A_c^2 (1 + k_a^2 p)}{4N_oW}
\]
Supplements

For a DSB-SC system,

\[
\text{SNR}_{\text{pre-de}}^{\text{DSB-SC}} = \frac{C^2 A_c^2 P / 2}{N_o B_T} = \frac{C^2 A_c^2 P}{4N_o W}
\]

For an FM system

\[
\text{SNR}_{\text{pre-de}}^{\text{FM}} = \frac{A_c^2 / 2}{N_o B_T} = \frac{A_c^2}{2N_o B_T}
\]

If using Carson’s rule, we have

\[B_T = 2\Delta f + 2f_m >> f_m = w\]

For the purpose of comparing different CW modulation systems, we define

The average power of the modulated signal

\[(\text{SNR})_c = \text{The average power of channel noise in the message band}\]

Message signal with the same power as modulated wave

LP filter with bandwidth w

output

The equivalent baseband transmission model.
Supplements

More precisely, we may express the DSB-SC as

\[ m(t) \rightarrow \mathbf{X} \rightarrow S'(t) \]

\[ \cos(2\pi f_c t + \theta) \]

\( \theta \) is uniformly distributed over \([0, 2\pi]\)

\[ S'(t) = A_c m(t) \cos(2\pi f_c t + \theta) \]

At the receiver we may write

\[ S(t) = C A_c m(t) \cos(2\pi f_c t + \theta) \]

\[ P_s = E[S^2(t)] = R_s(0) \]

\[ = \int_{-\infty}^{\infty} S_x(f)df \]

\[ = E[(CA_c m(t) \cos(2\pi f_c t + \theta))^2] \]

\[ = C^2 A_c^2 E[\cos^2(2\pi f_c t + \theta)]E[m^2(t)] \]

\[ = C^2 A_c^2 R_m(0)/2 = C^2 A_c^2 P/2 \]

\[ R_m(0) = P = \int_{-w}^{w} S_m(f)df \]

The average noise power in \(-w < f < w\)

\[ P_n = \int_{-w}^{w} \frac{N_0}{2} df = N_o W \]
Supplements

**SNR_c** = \frac{\text{The average power of } S(t)}{\text{The average power of channel noise in the message band}}

= \frac{\text{The average power of the modulated signal}}{\text{The average power of channel noise in the message band}}

= \frac{P_s}{P_n} = \frac{C^2 A_c^2 P}{2 N_0 W}

For convenience we write the modulated signal as

\[ S(t) = CA_c m(t) \cos(2\pi f_c t) \quad \theta \text{不出現} \]

Since \( \cos(2\pi f_c t + \theta) \) is ergodic and we take \( \cos(2\pi f_c t) \) as a sample function

\[ P_s = C^2 A_c^2 R_m(0) \quad \text{[time average of } [\cos^2(2\pi f_c t)] \text{]} \]

= \frac{C^2 A_c^2 P}{2}

**SNR_c** = \frac{\frac{C^2 A_c^2 P}{2}}{N_0 W} = \frac{C^2 A_c^2 P}{2N_0 W}
Example 2.4  Single-Tone Modulation

Consider the special case of a sinusoidal wave of frequency $f_m$ and amplitude $A_m$ as the modulating wave, as shown by

$$m(t) = A_m \cos(2\pi f_m t)$$

The corresponding AM wave is

$$s(t) = A_c[1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$

where $\mu = k_a A_m$ is the modulation factor. The average power of the modulating wave $m(t)$ is (assuming a load resistor of 1 ohm)

$$P = \frac{1}{2} A_m^2$$

Therefore, using Equation (2.95), we get

$$\frac{(\text{SNR})_O}{(\text{SNR})_C} \bigg|_{AM} = \frac{\frac{1}{2} k_a^2 A_m^2}{1 + \frac{1}{2} k_a^2 A_m^2} = \frac{\mu^2}{2 + \mu^2} \quad (2.96)$$

When $\mu = 1$, which corresponds to 100 percent modulation, we get a figure of merit equal to 1/3. This means that, other factors being equal, an AM system (using envelope detection) must transmit three times as much average power as a suppressed-carrier system (using coherent detection) to achieve the same quality of noise performance.
Threshold Effect

\[(\text{SNR})_O = \rho\]

\[(\text{SNR})_O \approx 0.91\rho^2\]

noise power > carrier power
2.13 Noise in FM Receivers

The discriminator consists of a slope network and an envelope detector.

Let \( n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \)

\[
\begin{align*}
n(t) &= r(t) \cos \left[ (2\pi f_c t) + \psi(t) \right] \\
\text{The envelope is } r(t) &= \left[ n_I^2(t) + n_Q^2(t) \right]^{1/2} \\
\text{The phase is } \psi(t) &= \tan^{-1} \left[ \frac{n_Q(t)}{n_I(t)} \right]
\end{align*}
\]

where \( r(t) \) is Rayleigh distributed, and \( \psi(t) \) is uniform distributed over \( 2\pi \).

\[
\begin{align*}
f_R(r) &= \frac{r}{\sigma^2} \exp \left( -\frac{r^2}{2\sigma^2} \right), \quad r \geq 0 \\
f_\psi(\psi) &= \frac{1}{2\pi}, \quad 0 < \psi < 2\pi
\end{align*}
\]
The incoming FM signal $s(t)$ is defined by

$$
s(t) = A_c \cos \left[ 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]
$$

(2.133)

$$
= A_c \cos[2\pi f_c t + \phi(t)]
$$

(2.135)

where $\phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau$

(2.134)

At the bandpass filter output

$$
x(t) = s(t) + n(t)
$$

$$
= A_c \cos[2\pi f_c t + \phi(t)] + r(t) \cos[2\pi f_c t + \psi(t)]
$$

(2.136)

where $A_c >> |r(t)|$

$$
\theta(t) = \phi(t) + \tan^{-1} \left\{ \frac{r(t) \sin[\psi(t) - \phi(t)]}{A_c + r(t) \cos[\psi(t) - \phi(t)]} \right\}
$$

(2.137)
Note that the envelope of \( x(t) \) is of no interest to us (limiter)

Because \( A_c >> |r(t)| \)

\[
\theta(t) \approx \phi(t) + \frac{r(t)}{A_c} \sin[\psi(t) - \phi(t)] \tag{2.138}
\]

\[
= 2\pi k_f \int_0^t m(\tau)d\tau + \frac{r(t)}{A_c} \sin[\psi(t) - \phi(t)] \tag{2.139}
\]

The discriminator output is (Fig 2.40)

\[
v(t) = \frac{1}{2\pi} \frac{d\theta(t)}{dt}
\]

\[
= k_f m(t) + n_d(t) \tag{2.140}
\]

where

\[
n_d(t) = \frac{1}{2\pi A_c} \frac{d}{dt} \left\{ r(t) \sin[\psi(t) - \phi(t)] \right\} \tag{2.141}
\]
Assume $\psi(t) - \phi(t)$ is uniformly distributed over $(0, 2\pi)$, then $n_d(t)$ is independent of message signal.

We may simplify $n_d(t)$ as

$$n_d(t) \approx \frac{1}{2\pi A_c} \frac{d}{dt} \{r(t) \sin[\psi(t)]\} \quad (2.142)$$

From definition of $r(t)$ and $\psi(t)$, we have

$$n_Q(t) = r(t) \sin[\psi(t)] \quad (2.143)$$

$$n_d(t) \approx \frac{1}{2\pi A_c} \frac{dn_Q(t)}{dt} \quad (2.144)$$

The quadrature component appears
From (2.140)
The average output signal power = $k_f^2P$

Recall
\[ \frac{d}{dt} \iff j2\pi f \]

\[
\begin{array}{c}
n_Q(t) \\
S_{N_Q}(f) \\
\end{array}
\xrightarrow{\frac{1}{2\pi A_c} \frac{d}{dt}}
\begin{array}{c}
n_d(t) \\
S_{N_d}(f) \\
\end{array}
\]

\[ S_{N_d}(f) = \frac{f^2}{A_c^2} S_{N_Q}(f) \quad (2.145) \]

noise is enhanced at high frequency
Assume that $n_Q(t)$ has ideal low-pass characteristic with bandwidth $B_T$

$$S_{N_d}(f) = \frac{N_0 f^2}{A_c^2}, \quad |f| \leq \frac{B_T}{2} \quad (2.146)$$

If $\frac{B_T}{2} > W$

At the receiver output

$$S_{N_0}(f) = \frac{N_0 f^2}{A_c^2}, \quad |f| \leq W \quad (2.147)$$
Average power of \( n_0(t) \) is:

\[
\frac{N_0}{A_c^2} \int_{-W}^{W} f^2 df
\]

\[
= \frac{2N_0 W^3}{3A_c^2} \quad (2.148)
\]

\[
\propto \frac{1}{A_c^2} \text{ noise quieting effect}
\]

when increasing carrier power

\[
(SNR)_{o,FM} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}
\]

The average power of \( s(t) \) is \( \frac{A_c^2}{2} \),

the average noise power in message bandwidth is \( WN_0 \)

\[
\Rightarrow (SNR)_{C,FM} = \frac{A_c^2}{2WN_0} \quad (2.150)
\]

\[
\Rightarrow \frac{(SNR)_o}{(SNR)_C}^{FM} = \frac{3k_f^2 P}{W^2} \quad (2.151)
\]

\[
\Delta f = k_f A_m \quad (2.29)
\]

\[(SNR)_{o,FM} \propto (\Delta f)^2\]
Example 2.5 Single-Tone Modulation

\[ s(t) = A_c \cos \left[ 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t) \right] \]

We may write, \( 2\pi k_f \int_0^t m(\tau) d\tau = \frac{\Delta f}{f_m} \sin(2\pi f_m t) \)

\[ \frac{d}{dt} \text{ both side } \Rightarrow m(t) = \frac{\Delta f}{k_f} \cos(2\pi f_m t) \]

The average power of \( m(t) \) (across the load) is \( P = \frac{(\Delta f)^2}{2k_f} \)

From (2.149), \( (SNR)^{FM}_{o} = \frac{3A_c^2(\Delta f)^2}{4N_0W^3} = \frac{3A_c^2\beta^2}{4N_0W} \), \( \beta = \frac{\Delta f}{W} \)

\[ \Rightarrow \left. \frac{(SNR)^{o}}{(SNR)^{c}} \right|_{FM} = \frac{3}{2} \left( \frac{\Delta f}{W} \right)^2 = \frac{3}{2} \beta^2 \quad (2.152) \]

compare to AM, \( \left. \frac{(SNR)^{o}}{(SNR)^{c}} \right|_{AM} = \frac{1}{3} \) (from Example 2.4)

When \( \frac{3}{2} \beta^2 > \frac{1}{3} \), FM has better performance.

\[ \Rightarrow \beta > \frac{\sqrt{2}}{3} = 0.471 \]

Define \( \beta = 0.5 \) as the transition between narrowband FM and wideband FM.
FM Threshold Effect (When CNR is low)

The composite signal at the frequency discriminator input

\[ x(t) = [A_c + n_I(t)] \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \]  

(2.153)

\[ \theta(t) = \tan^{-1} \frac{n_Q(t)}{A_c + n_I(t)} \]

Occasionally, \( P_1 \) may sweep around the origin, \( (r(t) > A_c) \)
\( \theta(t) \) increases or decreases \( 2\pi \)

The discriminator output is equal to

\[ \frac{\theta'(t)}{2\pi} \]
Figure 2.44 Illustrating impulselike components in $\theta'(t) = d\theta(t)/dt$ produced by changes of $2\pi$ in $\theta(t)$; (a) and (b) are graphs of $\theta(t)$ and $\theta'(t)$, respectively.
A positive-going click occurs, when
\[ r(t) > A_c, \quad \psi(t) < \pi \leq \psi(t) + d\psi(t), \quad \frac{d\psi(t)}{dt} > 0 \]

A negative-going click occurs when
\[ r(t) > A_c, \quad \psi(t) > -\pi > \psi(t) + d\psi(t), \quad \frac{d\psi(t)}{dt} < 0 \]

The carrier-to-noise ratio is defined by
\[ \rho = \frac{A_c}{2B_TN_0} \quad (2.154) \]

The output signal-to-noise ratio is calculated as
1. The average output signal power is calculated assuming
   a sinusoidal modulation which produces \( \Delta f = \frac{B_T}{2} \). (noise free)
2. The average output noise power is calculated when no signal is present (The carrier is unmodulated).
When \( \rho = \frac{A_c^2}{2B_T N_0} \geq 20 \) or \( \frac{A_c^2}{2} \geq 20B_T N_0 \) (2.155), threshold effects may be avoided.

**Figure 2.45** Dependence of output signal-to-noise ratio on input carrier-to-noise ratio for FM receiver. In curve I, the average output noise power is calculated assuming an unmodulated carrier. In curve II, the average output noise power is calculated assuming a sinusoidally modulated carrier. Both curves I and II are calculated from theory.
The procedure to calculate minimum $A_c$ ($\rho \geq 20$)

1. Given $\beta$ and $W$, determine $B_T$
   (using Figure 2.26 or Carson's rule)

2. Given $N_0$, we have $\frac{A_c^2}{2} \geq 20B_TN_0$

**Capture Effect:**

The receiver locks onto the stronger signal and suppresses the weaker one.
FM Threshold Reduction (tracking filter)

- FM demodulator with negative feedback (FMFB)
- Phase locked loop

**Figure 2.46**
FM threshold extension.

**Figure 2.47**
FM demodulator with negative feedback.
Pre-emphasis and De-emphasis on FM

**Figure 2.48** (a) Power spectral density of noise at FM receiver output.
(b) Power spectral density of a typical message signal.

**Figure 2.49** Use of pre-emphasis and de-emphasis in an FM system.
\[ H_{de}(f) = \frac{1}{H_{pe}(f)} , \quad -W \leq f \leq W \quad (2.156) \]

The PSD at the discriminator output is

\[ S_{N_d}(f) = \frac{N_0 f^2}{A_c^2} , \quad |f| \leq \frac{B_T}{2} \quad (2.146) \]

\[ |H_{de}(f)|^2 S_{N_d}(f) = \frac{N_0 f^2}{A_c^2} |H_{de}(f)|^2 , \quad |f| \leq \frac{B_T}{2} \quad (2.157) \]

\[ \left( \text{Average output noise power with de-emphasis} \right) = \frac{N_0}{A_c^2} \int_{-W}^{W} f^2 |H_{de}(f)|^2 df \quad (2.158) \]

The improvement factor \( I \) is

\[ I = \frac{2W^3}{3 \int_{-W}^{W} f^2 |H_{de}(f)|^2 df} \quad (2.162) \]
Example 2.6

Figure 2.50 (a) Pre-emphasis filter.
(b) De-emphasis filter.

A simple pre-emphasis filter response is

\[ H_{pe}(f) = 1 + \frac{jf}{f_0} \]

A de-emphasis filter response is

\[ H_{de}(f) = \frac{1}{1 + \frac{jf}{f_0}} \]

\[
I = \frac{2W^3}{3\int_{-w}^{w} \frac{f^2 df}{1 + \left(\frac{f}{f_0}\right)^2}} = \frac{(W/f_0)^3}{3 \left[ (W/f_0) - \arctan \left(\frac{W}{f_0}\right) \right]} \tag{2.161}
\]
Preemphasis for FM

- The main difference between FM and PM is in the relationship between frequency and phase.
  \[ f = \frac{1}{2\pi} \frac{d\theta}{dt}. \]
- A PM detector has a flat noise power (and voltage) output versus frequency (power spectral density). This is illustrated in Figure 9-38a.
- However, an FM detector has a parabolic noise power spectrum, as shown in Figure 9-38b. The output noise voltage increases linearly with frequency.
- If no compensation is used for FM, the higher audio signals would suffer a greater S/N degradation than the lower frequencies. For this reason compensation, called emphasis, is used for broadcast FM.
Figure 9-38. Detector noise output spectra for (a). PM and (b). FM.
Preemphasis for FM

- A preemphasis network at the modulator input provides a constant increase of modulation index $m_f$ for high-frequency audio signals.

- Such a network and its frequency response are illustrated in Figure 9-39.

Fig. 9-39. (a) Preemphasis network, and (b) Frequency response.
With the RC network chosen to give $\tau = R_1C = 75\mu s$ in North America (150$\mu s$ in Europe), a constant input audio signal will result in a nearly constant rise in the VCO input voltage for frequencies above 2.12 kHz. The larger-than-normal carrier deviations and $m_f$ will preemphasize high-audio frequencies.

At the receiver demodulator output, a low-pass RC network with $\tau = RC = 75\mu s$ will not only decrease noise at higher audio frequencies but also deemphasize the high-frequency information signals and return them to normal amplitudes relative to the low frequencies.

The overall result will be nearly constant $S/N$ across the 15-kHz audio baseband and a noise performance improvement of about 12dB over no preemphasis. Phase modulation systems do not require emphasis.
Pre-emphasis and De-emphasis on FM

Pre-emphasis and deemphasis: (a) schematic diagrams; (b) attenuation curves
Pre-emphasis and De-emphasis on FM

Example of S/N without preemphasis and deemphasis.
Example of S/N with preemphasis and deemphasis.
Dolby dynamic preemphasis
Figure 2.55 Comparison of the noise performance of various CW modulation systems. Curve I: Full AM, $\mu = 1$. Curve II: DSB-SC, SSB. Curve III: FM, $\beta = 2$. Curve IV: FM, $\beta = 5$. (Curves III and IV include 13-dB pre-emphasis, de-emphasis improvement.)
In making the comparison, it is informative to keep in mind the transmission bandwidth requirement of the modulation systems in question. Therefore, we define normalized transmission bandwidth as

$$B_n = \frac{B_T}{W}$$

Table 2.4 Values of $B_n$ for various CW modulation schemes

<table>
<thead>
<tr>
<th>Modulation Scheme</th>
<th>AM, DSB-SC</th>
<th>SSB</th>
<th>$\beta = 2$</th>
<th>$\beta = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>2</td>
<td>1</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>

**FM**