Algebric Properties: Axioms 1–3 (by Huntington, 1904)

1. **Closure Property**: A set S is **closed** wrt to a binary operator $\bullet$ <u>if and only if</u> (iff) for every $x, y \in S, x \bullet y \in S$.
   - **Axiom 1(a):** B is closed wrt the operator $+$
   - **Axiom 1(b):** B is closed wrt the operator $\cdot$
2. **Identity Element**: A set S is said to have an **identity element** wrt to a particular binary operator $\bullet$ whenever there exists an element $e \in S$ such that for every $x \in S, e \bullet x = x \bullet e = x$.
   - **Axiom 2(a):** B has an identity element wrt $+$, denoted by 0
   - **Axiom 2(b):** B has an identity element wrt $\cdot$, denoted by 1
3. **Commutativity Property**: A binary operator $\bullet$ defined on a set S is said to be **commutative** iff for every $x, y \in S, x \bullet y = y \bullet x$.
   - **Axiom 3(a):** B is commutative wrt the operator $+$
   - **Axiom 3(b):** B is commutative wrt the operator $\cdot$

Boolean Algebra

- Boolean algebra (named after G. Boole for his work in 1854) is a mathematical system for specifying and transforming logic functions.
- **Boolean (switching) algebra:** An algebra structure defined on a set of elements, B, together with two binary operators (denoted $+$ and $\cdot$) provided the following **axioms** are satisfied:
  1. Closure <u>with respect to</u> (wrt) $+$ and $\cdot$
  2. Identity elements exist for $+$ and $\cdot$
  3. Commutative wrt $+$ and $\cdot$
  4. Operator $+$ distributes over $\cdot$, and $\cdot$ distributes over $+$.
  5. Every element of B has a complement.
  6. B has at least two distinct elements.
- Purpose: Studying Boolean algebra to specify the design of digital systems!

Duality and Basic Identities

- **Duality Property:** Every algebraic expression deducible from the previous axioms remains valid after **dual** operations.
  - The **DUAL** of an algebraic expression is formed by interchanging AND ($\cdot$) $\longleftrightarrow$ OR ($+$) and 0 $\longleftrightarrow$ 1.
  - Exp: $x + y = y + x \longleftrightarrow x \cdot y = y \cdot x$; $x + x' = 1 \longleftrightarrow x \cdot x' = 0$.
- Identities of Boolean algebra:

1.	$x + 0 = x$	2.	$x \cdot 1 = x$	
3.	$x + 1 = 1$	4.	$x \cdot 0 = 0$	
5.	$x + x = x$	6.	$x \cdot x = x$	
7.	$x + \bar{x} = 1$	8.	$x \cdot \bar{x} = 0$	
9.	$\bar{\bar{x}} = x$			
10.	$x + y = y + x$	11.	$xy = yx$	Commutative
12.	$x + (y + z) = (x + y) + z$	13	$x(yz) = (xy)z$	Associative
14.	$x(y + z) = xy + xz$	15.	$x + yz = (x + y)(x + z)$	Distributive
16.	$\overline{x + y} = \bar{x} \cdot \bar{y}$	17.	$\overline{x \cdot y} = \bar{x} + \bar{y}$	DeMorgan's
18.	$x + xy = x$	19.	$x(x + y) = x$	Absorption
20.	$xy + x\bar{y} = x$	21.	$(x + y)(x + \bar{y}) = x$	
22.	$x + \bar{x}y = x + y$	23.	$x(\bar{x} + y) = xy$	

- **Operator precedence:** parenthese () $\rightarrow$ NOT $\rightarrow$ AND $\rightarrow$ OR.

Algebraic Properties: Axioms 4–6

4. **Distributivity Property**: If $\bullet$ and $\diamond$ are two binary operators on a set S, $\bullet$ is said to be **distributive** over $\diamond$ if for all $x, y, z \in S, x \bullet (y \diamond z) = (x \bullet y) \diamond (x \bullet z)$.
   - **Axiom 4(a):** The operator $\cdot$ is distributive over $+$; **Axiom 4(b):** The operator $+$ is distributive over $\cdot$.
5. **Complement Element**: For every $x \in B$, there exists an element $x' \in B$ such that $x + x' = 1$ and $x \cdot x' = 0$. x' is called the **complement** of x.
6. **Cardinality Bound**: There are at least two elements $x, y \in B$ such that $x \neq y$.

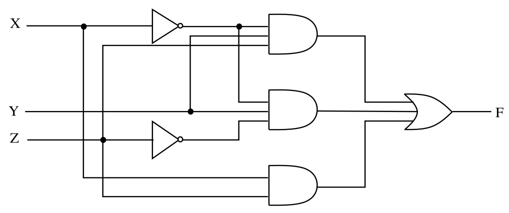
- Other notes:
  - The **Associative Law** $x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ hold and can be derived from the above axioms.
  - Axioms 4(b) and 5 are not available in ordinary algebra.
  - Boolean algebra applies to a finite set of elements; ordinary algebra applies to the infinite set of real numbers.

Algebraic Manipulation to Minimize Literals

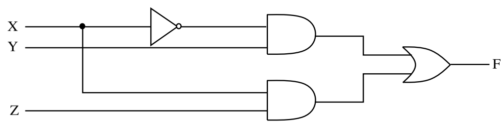
- **Literal:** A single variable within a **term** that may or may not be complemented.
- Minimize # of terms and literals: $\bar{x}yz + \bar{x}y\bar{z} + xz$

$$
\begin{aligned}
F &= \bar{x}yz + \bar{x}y\bar{z} + xz & \\
&= \bar{x}y(z + \bar{z}) + xz & \text{Identity 14} \\
&= \bar{x}y \cdot 1 + xz & \text{Identity 7} \\
&= \bar{x}y + xz & \text{Identity 2}
\end{aligned}
$$

- # of terms reduced from 3 ($\bar{x}yz, \bar{x}y\bar{z}, xz$) to 2 ($\bar{x}y, xz$).
- # of literals reduced from 8 ($\bar{x}, y, z, \bar{x}, y, \bar{z}, x, z$) to 4 ($\bar{x}, y, x, z$).



(a) $F=\bar{X}YZ+\bar{X}Y\bar{Z}+XZ$



(b) $F=\bar{X}Y+XZ$

DeMorgan's Law

- Truth tables to verify DeMorgan's Law: $\overline{X+Y} = \bar{X} \cdot \bar{Y}$

A)	X	Y	X+Y	$\overline{\text{X+Y}}$	B)	X	Y	$\overline{\text{X}}$	$\overline{\text{Y}}$	$\overline{\text{X}}\bullet\overline{\text{Y}}$
	0	0	0	1		0	0	1	1	1
	0	1	1	0		0	1	1	0	0
	1	0	1	0		1	0	0	1	0
	1	1	1	0		1	1	0	0	0

- DeMorgan's Law can be extended to multiple variables.
  - $\overline{X_1 + X_2 + \ldots + X_n} = \bar{X}_1\bar{X}_2\ldots\bar{X}_n$.
  - $\overline{X_1X_2\ldots X_n} = \bar{X}_1 + \bar{X}_2 + \ldots + \bar{X}_n$.

Function Complement

- Use DeMorgan's Law to complement a function
  1. Interchange AND and OR operators (**duality operation**).
  2. Complement each literal.

- Applying DeMorgan's Law: $F = \bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z$

$$\begin{aligned} \bar{F} &= \overline{\bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z} = \overline{\bar{X}Y\bar{Z}} \cdot \overline{\bar{X}\bar{Y}Z} \\ &= (X + \bar{Y} + Z)(X + Y + \bar{Z}) \end{aligned}$$

- Taking duals and complementing literals: $F = \bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z$

$$\begin{aligned} F &= \bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z \\ \text{Taking dual:} \quad & (\bar{X} + Y + \bar{Z})(\bar{X} + \bar{Y} + Z) \\ \text{Complementing literals:} \quad & (X + \bar{Y} + Z)(X + Y + \bar{Z}) = \bar{F} \end{aligned}$$

Consensus Theorem

- $xy + \bar{x}z + yz = xy + \bar{x}z$

$$\begin{aligned} xy + \bar{x}z + yz &= xy + \bar{x}z + yz(x + \bar{x}) \\ &= xy + \bar{x}z + xyz + \bar{x}yz \\ &= xy + xyz + \bar{x}z + \bar{x}yz \\ &= xy(1 + z) + \bar{x}z(1 + y) \\ &= xy + \bar{x}z \end{aligned}$$

- Dual: $(x + y)(\bar{x} + z)(y + z) = (x + y)(\bar{x} + z)$

- Exp: $(A + B)(\bar{A} + C) = AC + \bar{A}B$

$$\begin{aligned} (A + B)(\bar{A} + C) &= A\bar{A} + AC + \bar{A}B + BC \\ &= AC + \bar{A}B + BC \\ &= AC + \bar{A}B \end{aligned}$$