Module #12 - Summations

University of Florida
Dept. of Computer & Information Science & Engineering

COT 3100
Applications of Discrete Structures
Dr. Michael P. Frank

Slides for a Course Based on the Text
Discrete Mathematics & Its Applications
(5th Edition)
by Kenneth H. Rosen

Module #12: Summations

Rosen 5th ed., §3.2
~19 slides, ~1 lecture
Module #12 - Summations

Summation Notation

- Given a series \( \{a_n\} \), an integer lower bound (or limit) \( j \geq 0 \), and an integer upper bound \( k \geq j \), then the summation of \( \{a_n\} \) from \( j \) to \( k \) is written and defined as follows:

\[
\sum_{i=j}^{k} a_i \equiv a_j + a_{j+1} + \ldots + a_k
\]

- Here, \( i \) is called the index of summation.

Generalized Summations

- For an infinite series, we may write:

\[
\sum_{i=j}^{\infty} a_i \equiv a_j + a_{j+1} + \ldots
\]

- To sum a function over all members of a set \( X = \{x_1, x_2, \ldots\} \):

\[
\sum_{x \in X} f(x) \equiv f(x_1) + f(x_2) + \ldots
\]

- Or, if \( X = \{x \mid P(x)\} \), we may just write:

\[
\sum_{P(x)} f(x) \equiv f(x_1) + f(x_2) + \ldots
\]
Module #12 - Summations

Simple Summation Example

\[ \sum_{i=2}^{4} i^2 + 1 = (2^2 + 1) + (3^2 + 1) + (4^2 + 1) \]
\[ = (4 + 1) + (9 + 1) + (16 + 1) \]
\[ = 5 + 10 + 17 \]
\[ = 32 \]

More Summation Examples

• An infinite series with a finite sum:
  \[ \sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \ldots = 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2 \]

• Using a predicate to define a set of elements to sum over:
  \[ \sum x^2 = 2^2 + 3^2 + 5^2 + 7^2 = 4 + 9 + 25 + 49 = 87 \]

  \( x \text{ is prime} \land x < 10 \)
Module #12 - Summations

### Summation Manipulations

- Some handy identities for summations:

  \[ \sum_x cf(x) = c \sum_x f(x) \]  
  (Distributive law.)

  \[ \sum_x f(x) + g(x) = \left( \sum_x f(x) \right) + \sum_x g(x) \]  
  (Application of commutativity.)

  \[ \sum_{i=j}^k f(i) = \sum_{i=j+n}^{k+n} f(i-n) \]  
  (Index shifting.)

### More Summation Manipulations

- Other identities that are sometimes useful:

  \[ \sum_{i=j}^k f(i) = \left( \sum_{i=j}^m f(i) \right) + \sum_{i=m+1}^k f(i) \]  
  if \( j \leq m < k \)  
  (Series splitting.)

  \[ \sum_{i=j}^k f(i) = \sum_{i=0}^{k-j} f(k-i) \]  
  (Order reversal.)

  \[ \sum_{i=0}^{2k} f(i) = \sum_{i=0}^k f(2i) + f(2i+1) \]  
  (Grouping.)
Module #12 - Summations

Example: Impress Your Friends

- Boast, “I’m so smart; give me any 2-digit number \( n \), and I’ll add all the numbers from 1 to \( n \) in my head in just a few seconds.”
- \( I.e. \), Evaluate the summation: \[ \sum_{i=1}^{n} i \]
- There is a simple closed-form formula for the result, discovered by Euler at age 12!

Leonhard Euler (1707-1783)

Module #12 - Summations

Euler’s Trick, Illustrated

- Consider the sum: \[ 1 + 2 + \ldots + \frac{n}{2} + \frac{(n/2)+1}{2} + \ldots + \frac{n-1}{2} + n \]
- \( n/2 \) pairs of elements, each pair summing to \( n+1 \), for a total of \( (n/2)(n+1) \).
Symbolic Derivation of Trick

\[ \sum_{i=1}^{k} \left( \sum_{r=0}^{i-1} r \right) = \sum_{i=1}^{k} i + \sum_{r=0}^{n-(k+1)} \left((n+(k+1)-i)\right) = \sum_{i=1}^{k} i + \sum_{r=0}^{n-k-1} \left((n-i)\right) = \sum_{i=1}^{k} i + \sum_{r=1}^{n+1-i} i + \sum_{r=1}^{n} i = \sum_{i=1}^{k} i + \sum_{i=1}^{n+1} i = \left( \sum_{i=1}^{k} i \right) + \left( \sum_{i=1}^{n+1} i \right) = \frac{k(k+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{k(k+1)}{2} + \frac{n(n+1)}{2} = \frac{n(n+1)}{2} \]

Concluding Euler’s Derivation

\[ \sum_{i=1}^{n} i = \left( \sum_{i=1}^{k} i \right) + \sum_{i=k+1}^{n} \left( \frac{n+1-i}{2} \right) = \left( \sum_{i=1}^{k} i \right) + \sum_{i=1}^{n-k} \left( i \right) = \frac{k(k+1)}{2} + \frac{n(n+1)}{2} = \frac{n(n+1)}{2} \]

- So, you only have to do 1 easy multiplication in your head, then cut in half.
- Also works for odd \( n \) (prove this at home).
Example: Geometric Progression

- A geometric progression is a series of the form \( a, ar, ar^2, ar^3, \ldots, ar^k \), where \( a, r \in \mathbb{R} \).
- The sum of such a series is given by:
  \[
  S = \sum_{i=0}^{k} ar^i
  \]
- We can reduce this to closed form via clever manipulation of summations...

Geometric Sum Derivation

- Here we go...

\[
S = \sum_{i=0}^{n} ar^i
\]

\[
S = \sum_{i=0}^{n+1} ar^i - \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} ar^{i+1} + ar^{n+1} = \ldots
\]
Module #12 - Summations

Derivation example cont...

\[ rS = \left( \sum_{i=1}^{n} ar^i \right) + ar^{n+1} = (ar^1 - ar^0) + \left( \sum_{i=1}^{n} ar^i \right) + ar^{n+1} \]

\[ = ar^1 + \sum_{i=1}^{n} ar^i + ar^{n+1} - ar^0 \]

\[ = \left( \sum_{i=0}^{n} ar^i \right) + \left( \sum_{i=1}^{n} ar^i \right) + ar^{n+1} - a \]

\[ = \sum_{i=0}^{n} ar^i + a(r^{n+1} - 1) = S + a(r^{n+1} - 1) \]

Module #12 - Summations

Concluding long derivation...

\[ rS = S + a(r^{n+1} - 1) \]

\[ rS - S = a(r^{n+1} - 1) \]

\[ S(r-1) = a(r^{n+1} - 1) \]

\[ S = a \left( \frac{r^{n+1} - 1}{r-1} \right) \quad \text{when } r \neq 1 \]

When \( r = 1 \), \( S = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} a^i = \sum_{i=0}^{n} a \cdot 1 = (n+1)a \)
Module #12 - Summations

### Nested Summations

- These have the meaning you’d expect.
  \[
  \sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} \left( \sum_{j=1}^{3} ij \right) = \sum_{i=1}^{4} i \left( \sum_{j=1}^{3} j \right) = \sum_{i=1}^{4} i(1 + 2 + 3)
  \]
  \[
  = \sum_{i=1}^{4} 6i = 6 \sum_{i=1}^{4} i = 6(1 + 2 + 3 + 4)
  \]
  \[
  = 6 \cdot 10 = 60
  \]
- Note issues of free vs. bound variables, just like in quantified expressions, integrals, etc.

---

Module #12 - Summations

### Some Shortcut Expressions

- Geometric series:
  \[
  \sum_{k=0}^{n} ar^k = a(r^{n+1} - 1)/(r - 1), r \neq 1
  \]
- Euler’s trick:
  \[
  \sum_{k=1}^{n} k = n(n+1)/2
  \]
- Quadratic series:
  \[
  \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6
  \]
- Cubic series:
  \[
  \sum_{k=1}^{n} k^3 = n^2(n+1)^2/4
  \]
Module #12 - Summations

Using the Shortcuts

- Example: Evaluate \( \sum_{k=50}^{100} k^2 \).
  - Use series splitting.
  - Solve for desired summation.
  - Apply quadratic series rule.
  - Evaluate.

\[
\sum_{k=50}^{100} k^2 = \left( \sum_{k=1}^{49} k^2 \right) + \sum_{k=50}^{100} k^2
\]

\[
= \frac{100\cdot101\cdot201}{6} - \frac{49\cdot50\cdot99}{6}
\]

\[
= 338,350 - 40,425
\]

\[
= 297,925.
\]

Module #12 - Summations

Summations: Conclusion

- You need to know:
  - How to read, write & evaluate summation expressions like:
    \[
    \sum_{i=j}^{k} a_i \quad \sum_{i=j}^{\infty} a_i \quad \sum_{x \in X} f(x) \quad \sum_{P(x)} f(x)
    \]
  - Summation manipulation laws we covered.
  - Shortcut closed-form formulas, & how to use them.