Module #9: Matrices

Rosen 5th ed., §2.7
~18 slides, ~1 lecture
2.7 Matrices

- A matrix (say MAY-trix) is a rectangular array of objects (usually numbers).
- An $m \times n$ ("$m$ by $n$") matrix has exactly $m$ horizontal rows, and $n$ vertical columns.
- Plural of matrix = matrices (say MAY-trih-sees)
- An $n \times n$ matrix is called a square matrix, whose order is $n$.

\[
\begin{pmatrix}
2 & 3 \\
5 & -1 \\
7 & 0 \\
\end{pmatrix}
\] a $3 \times 2$ matrix

Note: The singular form of "matrices" is "matrix," not "MAY-trih-see"!

Applications of Matrices

Tons of applications, including:
- Solving systems of linear equations
- Computer Graphics, Image Processing
- Models within Computational Science & Engineering
- Quantum Mechanics, Quantum Computing
- Many, many more…
Matrix Equality

Two matrices $A$ and $B$ are equal iff they have the same number of rows, the same number of columns, and all corresponding elements are equal.

\[
\begin{bmatrix}
3 & 2 \\
-1 & 6
\end{bmatrix} \neq \begin{bmatrix}
3 & 2 & 0 \\
-1 & 6 & 0
\end{bmatrix}
\]

Row and Column Order

The rows in a matrix are usually indexed 1 to $m$ from top to bottom. The columns are usually indexed 1 to $n$ from left to right. Elements are indexed by row, then column.

\[
A = [a_{i,j}] = \begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\]
Matrices as Functions

- An $m \times n$ matrix $A = [a_{ij}]$ of members of a set $S$ can be encoded as a partial function $f_A: \Box \times \Box \rightarrow S$, such that for $i < m, j < n, f_A(i, j) = a_{i,j}$.
- By extending the domain over which $f_A$ is defined, various types of infinite and/or multidimensional matrices can be obtained.

Matrix Sums

- The sum $A + B$ of two matrices $A, B$ (which must have the same number of rows, and the same number of columns) is the matrix (also with the same shape) given by adding corresponding elements.

$$A + B = [a_{ij} + b_{ij}]$$

$$\begin{bmatrix} 2 & 6 \\ 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ -11 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ -11 & -5 \end{bmatrix}$$
Matrix Products

- For an $m \times k$ matrix $A$ and a $k \times n$ matrix $B$, the product $AB$ is the $m \times n$ matrix:

$$AB = C = [c_{i,j}] = \sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j}$$

- I.e., element $(i,j)$ of $AB$ is given by the vector dot product of the $i$th row of $A$ and the $j$th column of $B$ (considered as vectors).

- Note: Matrix multiplication is not commutative!

Matrix Product Example

- An example matrix multiplication to practice in class:

\[
\begin{bmatrix}
0 & 1 & -1 \\
2 & 0 & 3
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & -1 & 1 & 0 \\
2 & 0 & -2 & 0 \\
1 & 0 & 3 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & -5 & -1 \\
3 & -2 & 11 & 3
\end{bmatrix}
\]
Identity Matrices

• The identity matrix of order $n$, $I_n$, is the order-$n$ matrix with 1’s along the upper-left to lower-right diagonal and 0’s everywhere else.

$$I_n = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}$$

Review: [2.6 Matrices, so far]

Matrix sums and products:

$$A+B = [a_{i,j}+b_{i,j}]$$

$$AB = C = [c_{i,j}] = \left[ \sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j} \right]$$

Identity matrix of order $n$:

$I_n = [\delta_{ij}]$, where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$. 
Matrix Inverses

- For some (but not all) square matrices $A$, there exists a unique multiplicative inverse $A^{-1}$ of $A$, a matrix such that $A^{-1}A = I_n$.
- If the inverse exists, it is unique, and $A^{-1}A = AA^{-1}$.
- We won’t go into the algorithms for matrix inversion...

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Matrix Multiplication Algorithm

```
procedure matmul(matrices A: m×k, B: k×n)
  for i := 1 to m do 
    for j := 1 to n do 
      c_{ij} := 0 
      for q := 1 to k do 
        c_{ij} := c_{ij} + a_{iq} b_{qj} 
      end
  end

{\text{What's the } \Theta \text{ of its time complexity?}}

\{ Answer: } \Theta(mnk) \{ \text{end} \}
```
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Powers of Matrices

If $A$ is an $n \times n$ square matrix and $p \geq 0$, then:

- $A^p \equiv AAA \cdots A$ ($A^0 \equiv I_n$)

  \[
  \begin{bmatrix}
  2 & 1 \\
  -1 & 0 
  \end{bmatrix}^3 = \begin{bmatrix}
  2 & 1 \\
  -1 & 0 
  \end{bmatrix} \cdot \begin{bmatrix}
  2 & 1 \\
  -1 & 0 
  \end{bmatrix} \cdot \begin{bmatrix}
  2 & 1 \\
  -1 & 0 
  \end{bmatrix}
  \]

- Example:

  
  \[
  \begin{bmatrix}
  2 & 1 \\
  -1 & 0 
  \end{bmatrix} \cdot \begin{bmatrix}
  3 & 2 \\
  -2 & -1 
  \end{bmatrix} = \begin{bmatrix}
  4 & 3 \\
  -3 & -2 
  \end{bmatrix}
  \]

\[\]

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Matrix Transposition

- If $A = [a_{ij}]$ is an $m \times n$ matrix, the transpose of $A$ (often written $A^t$ or $A^T$) is the $n \times m$ matrix given by $A^t = [b_{ij}] = [a_{ji}]$ ($1 \leq i \leq n, 1 \leq j \leq m$)

  \[
  \begin{bmatrix}
  2 & 3 \\
  0 & -2 
  \end{bmatrix}^t = \begin{bmatrix}
  2 & 0 \\
  1 & -1 \\
  3 & -2 
  \end{bmatrix}
  \]

  \[
  \text{Flip across diagonal}
  \]

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Symmetric Matrices

- A square matrix $A$ is symmetric iff $A = A^t$.
  \[ A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & 1 & 1 & 0 & -1 \\ 3 & -1 & 2 \\ -2 & 1 & 0 & 1 & -1 \\ -3 & 1 & 1 & -2 \end{bmatrix} \]

- Which is symmetric?

Zero-One Matrices

- Useful for representing other structures.
  - E.g., relations, directed graphs (later in course)
- All elements of a zero-one matrix are 0 or 1
  - Representing False & True respectively.
- The meet of $A, B$ (both $m \times n$ zero-one matrices):
  - $A \land B = [a_{ij} \land b_{ij}]$]
- The join of $A, B$:
  - $A \lor B = [a_{ij} \lor b_{ij}]$]
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Boolean Products

- Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix, & let $B = [b_{ij}]$ be a $k \times n$ zero-one matrix,
- The *boolean product* of $A$ and $B$ is like normal matrix $\times$, but using $\lor$ instead $+$ in the row-column "vector dot product."

$$A \boxdot B = C = [c_{ij}] = \left[ \lor_{\ell=1}^{k} a_{i\ell} \land b_{\ell j} \right]$$

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Boolean Powers

- For a square zero-one matrix $A$, and any $k \geq 0$, the $k$th *Boolean power of $A$* is simply the Boolean product of $k$ copies of $A$.

$$A^{[k]} \equiv A \boxdot A \boxdot \ldots \boxdot A$$

$k$ times