Introduction to Electronic Design Automation

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Computation & Optimization in a Nutshell

- Course contents:
  - Computational complexity
  - NP-completeness; PSPACE-completeness
  - Algorithmic paradigms
  - Mathematical optimization
- Readings
  - Chapter 4
  - Reference:
    - T. Cormen, C. Leiserson, R. Rivest, and C. Stein. *Introduction to Algorithms*. MIT Press, 2001.
    - M. Sipser. *Introduction to the Theory of Computation*. Cengage Learning, 2nd edition, 2005.

Computation Complexity

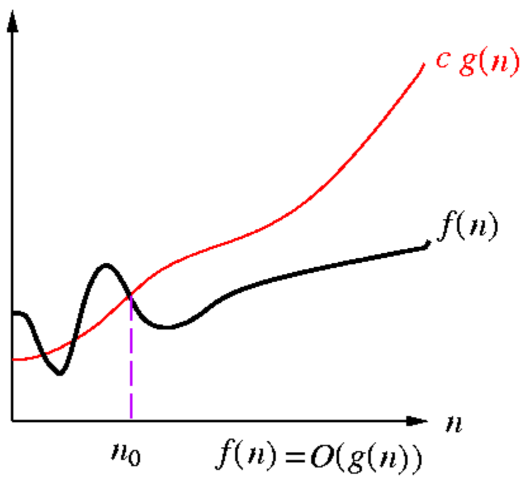
- We would like to characterize the efficiency/hardness of problem solving
- By that, we can have a better idea on how to come up with good algorithms
  - Algorithm: a well-defined procedure transforming some *input* to a desired *output* in **finite** computational resources in *time* and *space* (c.f. semi-algorithm)
- Why does complexity matter?

Time	Big-Oh	$n = 10$	$n = 100$	$n = 10^3$	$n = 10^6$
500	$O(1)$	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec
$3n$	$O(n)$	3×10^{-8} sec	3×10^{-7} sec	3×10^{-6} sec	0.003 sec
$n \log n$	$O(n \log n)$	3×10^{-8} sec	2×10^{-7} sec	3×10^{-6} sec	0.006 sec
n^2	$O(n^2)$	1×10^{-7} sec	1×10^{-5} sec	0.001 sec	16.7 min
n^3	$O(n^3)$	1×10^{-6} sec	0.001 sec	1 sec	3×10^5 cent.
2^n	$O(2^n)$	1×10^{-6} sec	3×10^{17} cent.	∞	∞
$n!$	$O(n!)$	0.003 sec	∞	∞	∞

assuming 10^9 instructions per second

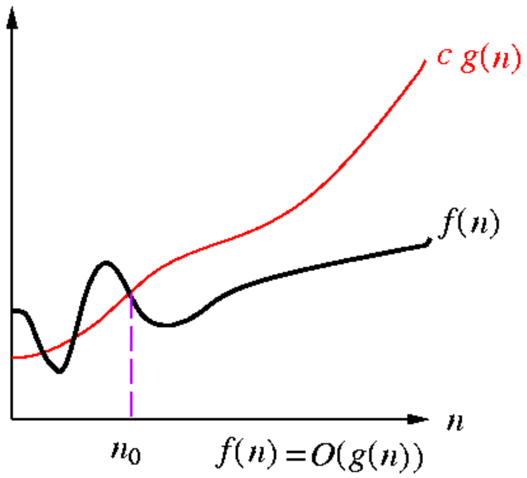
O: Upper Bounding Function

- Definition: $f(n) = O(g(n))$ if $\exists\, c>0$ and $n_0>0$ such that $0 \leq f(n) \leq c\, g(n)$ for all $n \geq n_0$
  - E.g., $2n^2 + 3n = O(n^2)$, $2n^2 = O(n^3)$, $3n \lg n = O(n^2)$
  - Intuition: $f(n)$ "$\leq$" $g(n)$ when we ignore constant multiples and small values of n



Big-O Notation

- How to show O (Big-Oh) relationships?
  - $f(n) = O(g(n))$ iff $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ for some $c \geq 0$
- "An algorithm has worst-case running time $O(g(n))$": there is a constant c s.t. for every n large enough, **every execution** on an input of size n takes **at most** $c\,g(n)$ time



Big-O Notation (cont'd)

- Only the dominating term needs to be kept while constant coefficients are immaterial

- Example

  $0.3\, n^2 = O(n^2)$

  $3\, n^2 + 152\, n + 1777 = O(n^2)$

  $n^2 \lg n + 3n^2 = O(n^2 \lg n)$

  The following are correct but not used

  $3n^2 = O(n^2 \lg n)$

  $3n^2 = O(0.1\, n^2)$

  $3n^2 = O(n^2 + n)$

Other Asymptotic Bounds

Other notations (though not important for now):

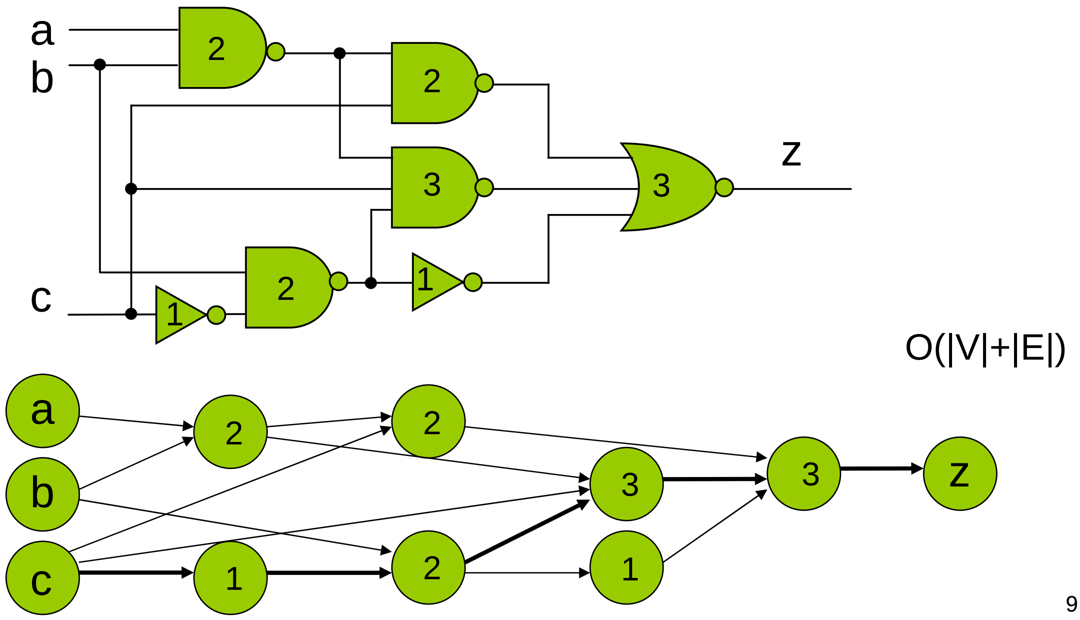
- Definition: $f(n) = \Omega(g(n))$ if $\exists\ c, n_0 > 0$ such that
  $0 \le c\ g(n) \le f(n)$ for all $n \ge n_0$.
  - Ω-notation provides an asymptotic *lower* bound on a function
- Definition: $f(n) = \Theta(g(n))$ if $\exists\ c_1, c_2, n_0 > 0$ such that
  $0 \le c_1\ g(n) \le f(n) \le c_2\ g(n)$ for all $n \ge n_0$.
  - Θ-notation provides an asymptotic *tight* bound on a function
- Showing the complexity upper bound of solving a problem (not an instance) is often much easier than showing the complexity lower bound
  - Why?

Computational Complexity

- Computational complexity: an abstract measure of the time and space necessary to execute an algorithm as function of its "input size"
- Input size examples:
  - sort n words of bounded length $\Rightarrow n$
  - the input is the integer $n \Rightarrow \lg n$
  - the input is the graph $G(V, E) \Rightarrow |V|$ and $|E|$
- Time complexity is expressed in *elementary computational steps* (e.g., an addition, multiplication, pointer indirection)
- Space complexity is expressed in *memory locations* (e.g. bits, bytes, words)

Computational Complexity

- Example
  - Computing longest delay path of a directed acyclic graph



Asymptotic Functions

- Polynomial-time complexity: $O(n^k)$, where n is the **input size** and k is a constant.
- Example polynomial functions:
  - 999: constant
  - lg n: logarithmic
  - $\sqrt{n}$: sublinear
  - n: linear
  - n lg n: loglinear
  - n^2: quadratic
  - n^3: cubic
- Example non-polynomial functions
  - 2^n, 3^n: exponential
  - $n!$: factorial

Run-time Comparison

- Assume 1000 MIPS (Yr: 200x), 1 instruction /operation

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Computation Problems

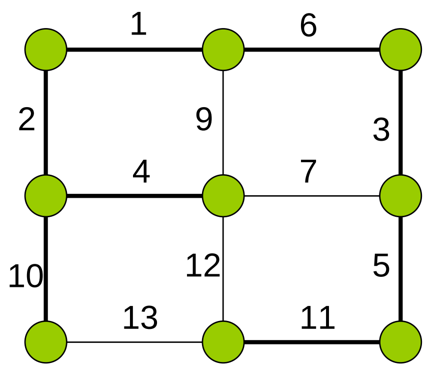
- Two common types of problems in computer science:
  - Optimization problems
    - Often discrete/combinatorial rather than continuous
    - E.g., Minimum Spanning Tree (MST), Travelling Salesman Problem (TSP), etc.
  - Decision problems
    - E.g., Fixed-weight Spanning Tree, Satisfiability (SAT), etc.

Terminology

- **Problem:** a general class, e.g., “the shortest-path problem for directed acyclic graphs”
- **Instance:** a specific case of a problem, e.g., “the shortest-path problem in a specific graph, between two given vertices”
- **Optimization problems:** those finding a legal configuration such that its cost is minimum (or maximum).
  - MST: Given a graph $G=(V, E)$, find the cost of a minimum spanning tree of G.
- An instance $I = (F, c)$ where
  - F is the set of *feasible solutions*, and
  - c is a *cost function*, assigning a cost value to each feasible solution $c : F \rightarrow R$
  - The solution of the optimization problem is the feasible solution with optimal (minimal/maximal) cost
- c.f., **optimal** solutions/costs, optimal (**exact**) algorithms (Attn: optimal $\neq$ exact in the theoretic computer science community).

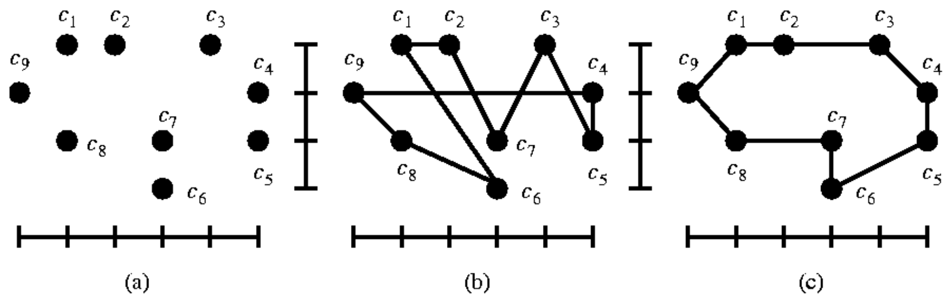
Optimization problem: Minimum Spanning Tree (MST)

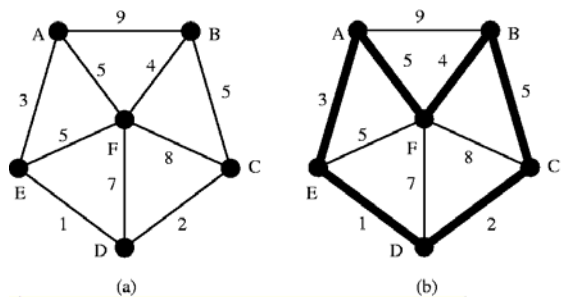
- MST: Given an undirected graph $G = (V, E)$ with weights on the edges, a minimum spanning tree of G is a subgraph $T \subseteq G$ such that
  - T has no cycles (i.e. a tree)
  - T contains all vertices in V
  - Sum of the weights of all edges in T is minimum



Optimization Problem: Traveling Salesman Problem (TSP)

- TSP: Given a set of cities and that distance between each pair of cities, find the distance of a minimum tour starts and ends at a given city and visits every city exactly once





Terminology

- **Decision problems:** problem that can only be answered with "yes" or "no"
  - MST: Given a graph $G=(V, E)$ and a bound K, is there a spanning tree with a cost at most K?
  - TSP: Given a set of cities, distance between each pair of cities, and a bound B, is there a route that starts and ends at a given city, visits every city exactly once, and has total distance at most B?
- A decision problem Π, has instances: $I = (F, c, k)$
  - The set of of instances for which the answer is "yes" is given by Y_Π
  - A subtask of a decision problem is *solution checking*: given $f \in F$, checking whether the cost is less than k
- Can apply binary search on decision problems to obtain solutions to optimization problems
- NP-completeness is associated with decision problems

Decision Problem: Fixed-weight Spanning Tree

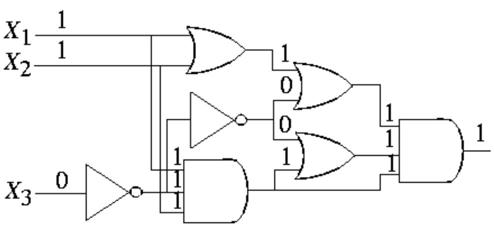
- Given an undirected graph $G = (V, E)$, is there a spanning tree of G with weight c?
- Can solve MST by posing it as a sequence of decision problems (with binary search)

Decision Problem: Satisfiability Problem (SAT)

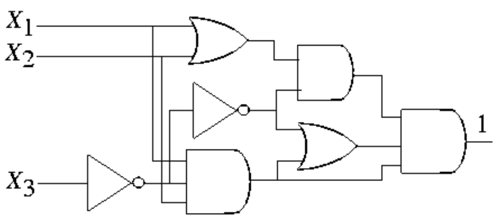
- **Satisfiability Problem (SAT):**
  - **Instance:** A Boolean formula φ in conjunctive normal form (CNF), a.k.a. product-of-sums (POS)
  - **Question:** Is there an assignment of Boolean values to the variables that makes φ true ?
- A Boolean formula φ is *satisfiable* if there exists a a set of Boolean input values that makes φ valuate to true. Otherwise, φ is *unsatisfiable*.
  - (a+b)(¬a+c)(¬b+¬c) is satisfiable since <a, b, c> = <0, 1, 0> makes the formula true.
  - (a+b)(¬a+c)(¬b)(¬c) is unsatisfiable

Decision Problem: Circuit Satisfiability Problem (CSAT)

- **Circuit-Satisfiability Problem (CSAT):**
  - **Instance:** A combinational circuit *C* composed of AND, OR, and NOT gates
  - **Question:** Is there an assignment of Boolean values to the inputs that makes the output of *C* to be 1?
- A circuit is satisfiable if there exists a a set of Boolean input values that makes the output of the circuit to be 1
  - Circuit (a) is satisfiable since <x_1, x_2, x_3> = <1, 1, 0> makes the output to be 1



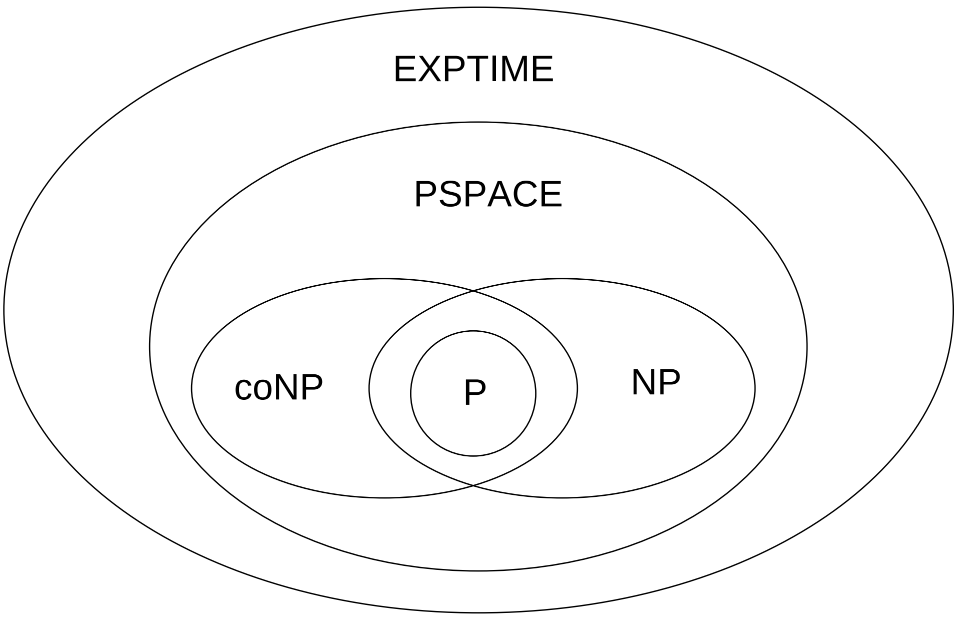
(a) satisfiable circuit



(b) unsatisfiable circuit

Complexity Hierarchy

- Tractable: solvable in deterministic polynomial time (P)
- Intractable: unsolvable in deterministic polynomial time (P)



Complexity Class P

- **Complexity class *P*** contains those problems that can be **solved** in polynomial time in the **size of input**
  - **Input size:** size of encoded “binary” strings
  - Edmonds: Problems in P are considered **tractable**
- The computer concerned is a *deterministic Turing machine*
  - *Deterministic* means that each step in a computation is predictable
  - A *Turing machine* is a mathematical model of a universal computer (any computation that needs polynomial time on a Turing machine can also be performed in polynomial time on any other machine)
- MST and shortest path problems are in *P*

Complexity Class NP

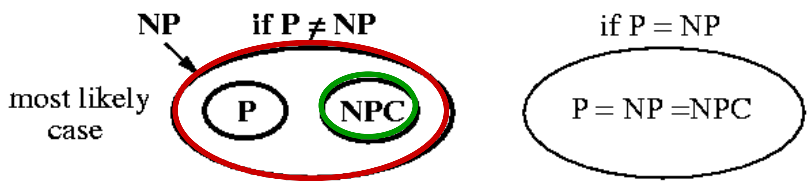
- Suppose that *solution checking* for some problem can be done in polynomial time on a deterministic machine ⇒ the problem can be solved in polynomial time on a *nondeterministic Turing machine*
  - *Nondeterministic*: the machine makes a guess, e.g., the right one (or the machine evaluates all possibilities in parallel)
- **The class NP (Nondeterministic Polynomial)**: class of problems that can be **verified** in polynomial time in the size of input
  - NP: class of problems that can be solved in polynomial time on a nondeterministic machine
- Is TSP ∈ NP?
  - Need to **check** a solution in polynomial time
    - Guess a tour
    - Check if the tour visits every city exactly once
    - Check if the tour returns to the start
    - Check if total distance ≤ *B*
  - All can be done in $O(n)$ time, so TSP ∈ NP

P vs. NP

- An issue which is still unsettled:

  $P \subset NP$ or $P = NP$?

  - There is a strong belief that $P \neq NP$, due to the existence of *NP*-complete problems.
  - One of the 7 *Clay Millennium Prize Problems*



NP-Completeness

- **The NP-complete (NPC) class:**
  - Developed by S. Cook and R. Karp in early 1970
    - Cook showed the first NP-complete problem (SAT) in 1971
    - Karp showed many other problems are NP-complete (by polynomial reduction) in 1972
  - Thousands of combinatorial problems are known to be NP-complete
    - NP-complete problems: SAT, 3SAT, CSAT, TSP, Bin Packing, Hamiltonian Cycles, …
  - All problems in NPC have the same degree of difficulty:

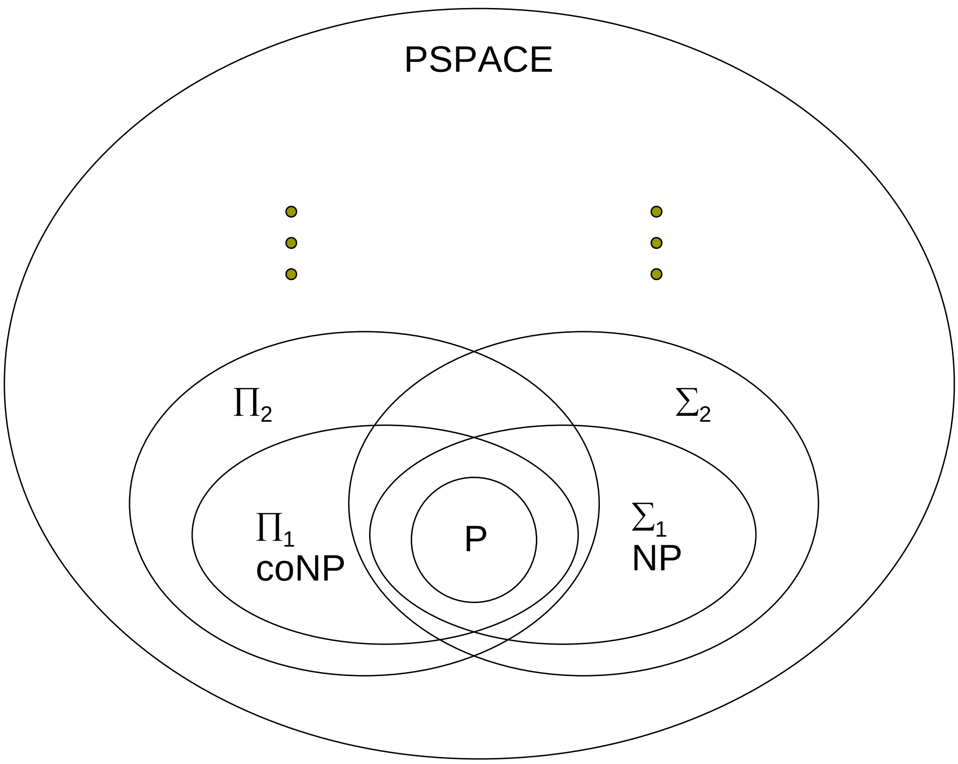
    Any NPC problem can be solved in polynomial time ⇒

    All problems in NP can be solved in polynomial time

Beyond NP

- A quantified Boolean formula (QBF) is
  $Q_1\ x_1,\ Q_2\ x_2,\ \ldots,\ Q_n\ x_n.\ \varphi$
  where Qi is either a existential ($\exists$) or universal quantifier ($\forall$), x_i is a Boolean variable, and φ is a Boolean formula.
  - Σ_i: $\exists x_1, \forall x_2, \exists x_3, \ldots, Q_n x_i.\ \varphi$
  - Π_i: $\forall x_1, \exists x_2, \forall x_3, \ldots, Q_n x_i.\ \varphi$
- The polynomial-time hierarchy
  - Σ_1 (= NP) $\subseteq \Sigma_2 \subseteq \ldots \subseteq \Sigma_i \subseteq \ldots$.
  - Π_1 (= coNP) $\subseteq \Pi_2 \subseteq \ldots \subseteq \Pi_i \subseteq \ldots$.

Polynomial Hierarchy

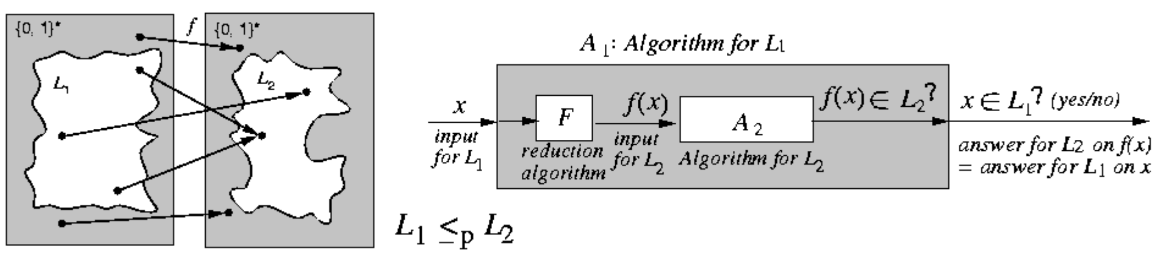


PSPACE-completeness

- The satisfiability problem for quantified Boolean formulae (QSAT) is PSPACE-complete
  - GO is PSPACE-complete!
  - Many sequential verification problems are PSPACE-complete

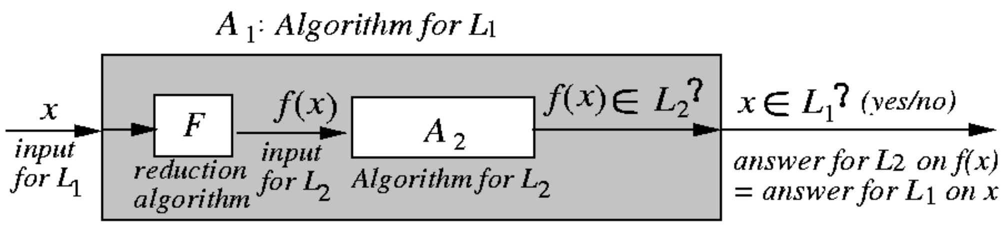
Polynomial-time Reduction

- **Motivation:** Let L_1 and L_2 be two decision problems. Suppose algorithm A_2 can solve L_2. Can we use A_2 to solve L_1?
- **Polynomial-time reduction** ***f*** **from** L_1 **to** L_2: $L_1 \leq_P L_2$
  - f reduces input for L_1 into an input for L_2 s.t. the reduced input is a "yes" input for L_2 iff the original input is a "yes" input for L_1
    - $L_1 \leq_P L_2$ if $\exists$ polynomial-time computable function f: $\{0, 1\}^* \rightarrow \{0, 1\}^*$ s.t. $x \in L_1$ iff $f(x) \in L_2$, $\forall x \in \{0, 1\}^*$
    - **L_2 is at least as hard as L_1**
- f is computable in polynomial time



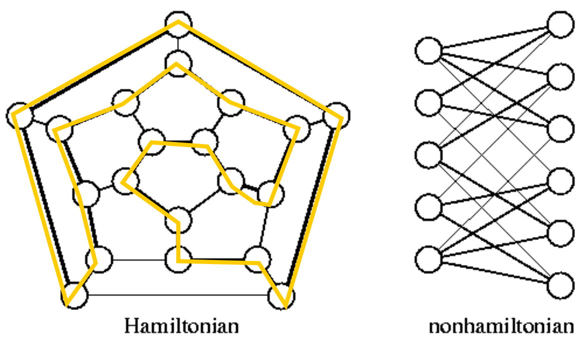
Significance of Reduction

- Significance of $L_1 \le_P L_2$:
  - $\exists$ polynomial-time algorithm for $L_2 \Rightarrow \exists$ polynomial-time algorithm for L_1 ($L_2 \in$ P $\Rightarrow L_1 \in$ P)
  - $\nexists$ polynomial-time algorithm for $L_1 \Rightarrow \nexists$ polynomial-time algorithm for L_2 ($L_1 \notin$ P $\Rightarrow L_2 \notin$ P)
- $\le_P$ is transitive, i.e., $L_1 \le_P L_2$ and $L_2 \le_P L_3 \Rightarrow L_1 \le_P L_3$



Polynomial-time Reduction

- The Hamiltonian Circuit, a.k.a. Hamiltonian Cycle, Problem (HC)
  - **Instance:** an undirected graph $G = (V, E)$
  - **Question:** is there a cycle in G that includes every vertex exactly once?
- TSP (The Traveling Salesman Problem)
- How to show HC $\le_P$ TSP?
  1. Define a function f mapping any HC instance into a TSP instance, and show that f can be computed in polynomial time
  2. Prove that G has an HC iff the reduced instance has a TSP tour with distance $\le B$ ($x \in$ HC $\Leftrightarrow f(x) \in$ TSP)

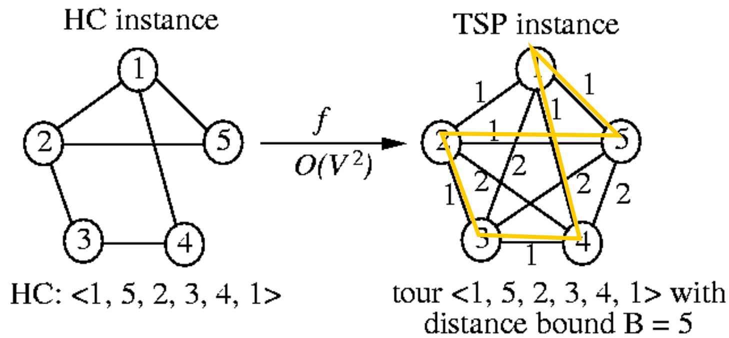


HC $\leq_P$ TSP: Step 1

- Define a reduction function *f* for HC $\leq_P$ TSP
  - Given an arbitrary HC instance $G = (V, E)$ with *n* vertices
    - Create a set of *n* cities labeled with names in *V*
    - Assign distance between *u* and *v*

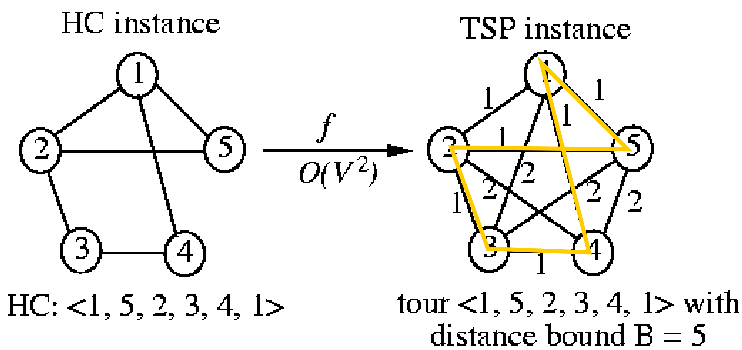
$$d(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E, \\ 2, & \text{if } (u,v) \notin E. \end{cases}$$

    - Set bound $B = n$
  - *f* can be computed in $O(V^2)$ time



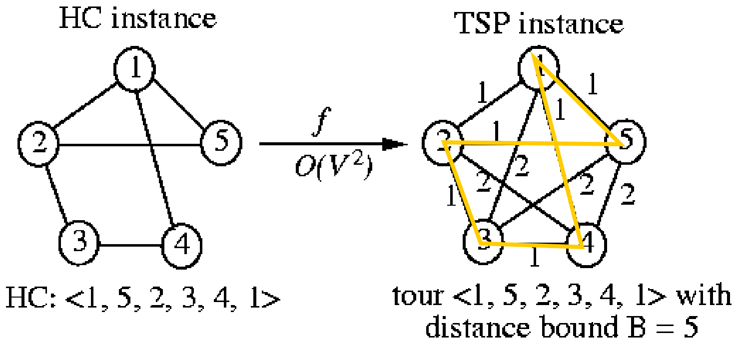
HC $\leq_P$ TSP: Step 2

- *G* has an HC iff the reduced instance has a TSP with distance $\leq B$
  - $x \in \text{HC} \Rightarrow f(x) \in \text{TSP}$
    - Suppose the HC is $h = <v_1, v_2, \ldots, v_n, v_1>$. Then, *h* is also a tour in the transformed TSP instance
    - The distance of the tour *h* is $n = B$ since there are *n* consecutive edges in *E*, and so has distance 1 in $f(x)$
    - Thus, $f(x) \in$ TSP ($f(x)$ has a TSP tour with distance $\leq B$)



HC $\leq_P$ TSP: Step 2 (cont'd)

- *G* has an HC iff the reduced instance has a TSP with distance $\leq B$
  - $f(x) \in$ TSP $\Rightarrow x \in$ HC
    - Suppose there is a TSP tour with distance $\leq n = B$. Let it be $<v_1, v_2, \ldots, v_n, v_1>$.
    - Since distance of the tour $\leq n$ and there are *n* edges in the TSP tour, the tour contains only edges in *E*
    - Thus, $<v_1, v_2, \ldots, v_n, v_1>$ is a Hamiltonian cycle ($x \in$ HC)



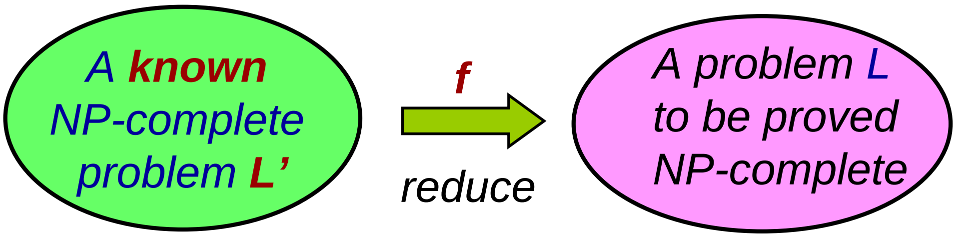
NP-Completeness and NP-Hardness

- **NP-completeness: worst-case** analyses for **decision** problems
- L is **NP-complete** if
  - **L $\in$ NP**
  - **NP-Hard: L' $\leq_P$ L for every L' $\in$ NP**
- **NP-hard:** If *L* satisfies the 2nd property, but not necessarily the 1st property, we say that *L* is **NP-hard**
- Significance of NPC class:

  Suppose $L \in$ NPC
  - If $L \in P$, then there exists a polynomial-time algorithm for every $L' \in$ NP (i.e., P = NP)
  - If $L \notin P$, then there exists no polynomial-time algorithm for any $L' \in$ NPC (i.e., P $\neq$ NP)

Proving NP-Completeness

- **Five steps for proving that *L* is NP-complete:**
  1. Prove $L \in$ NP
  2. Select a known NP-complete problem ***L'***
  3. Construct a reduction *f* transforming **every** instance of ***L'*** to an instance of *L*
  4. Prove that $x \in L'$ iff $f(x) \in L$ for all $x \in \{0, 1\}^*$
  5. Prove that *f* is a polynomial-time transformation
  - E.g., we showed that TSP is NP-complete

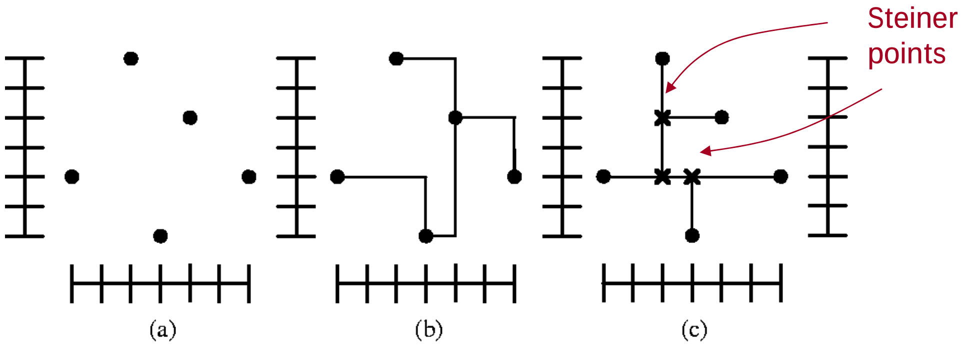


Easy vs. Hard Problems

- Many seemly similar problems may have substantial difference in their inherent hardness
  - Shortest path $\in$ P; longest path $\in$ NPC
  - Spanning tree $\in$ P; Steiner tree $\in$ NPC
  - Linear programming (LP) $\in$ P; integer linear programming (ILP) $\in$ NPC
  - …

Spanning Tree vs. Steiner Tree

- **Manhattan distance:** If two points (nodes) are located at coordinates (x_1, y_1) and (x_2, y_2), the Manhattan distance between them is given by $d_{12} = |x_1 - x_2| + |y_1 - y_2|$
- **Rectilinear spanning tree:** a spanning tree that connects its nodes using Manhattan paths (Fig. (b) below)
- **Steiner tree:** a tree that connects its nodes, and additional points (**Steiner points**) are permitted to used for the connections
- The minimum rectilinear spanning tree problem is in P, while the minimum rectilinear Steiner tree (Fig. (c)) problem is NP-complete
  - The spanning tree algorithm can be an *approximation* for the Steiner tree problem (at most 50% away from the optimum)



Hardness of Problem Solving

- Most optimization problems are intractable
  - Cannot afford to search the exact optimal solution
  - Global optimal (optimum) vs. local optimal (optimal)
- Search a reasonable solution within a reasonable bound on computational resources

Coping with NP-hard Problems

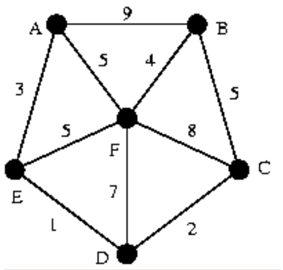
- **Approximation algorithms**
  - Guarantee to be a fixed percentage away from the optimum
  - E.g., MST for the minimum Steiner tree problem
- **Randomized algorithms**
  - Trade determinism for efficiency
- **Pseudo-polynomial time algorithms**
  - Has the form of a polynomial function for the complexity, but is not to the problem size
  - E.g., $O(nW)$ for the 0-1 knapsack problem
- **Restriction**
  - Work on some subset of the original problem
  - E.g., longest path problem restricted to directed acyclic graphs
- **Exhaustive search / Branch and bound**
  - Is feasible only when the problem size is small
- **Local search:**
  - Simulated annealing (hill climbing), genetic algorithms, etc.
- **Heuristics:** No guarantee of performance

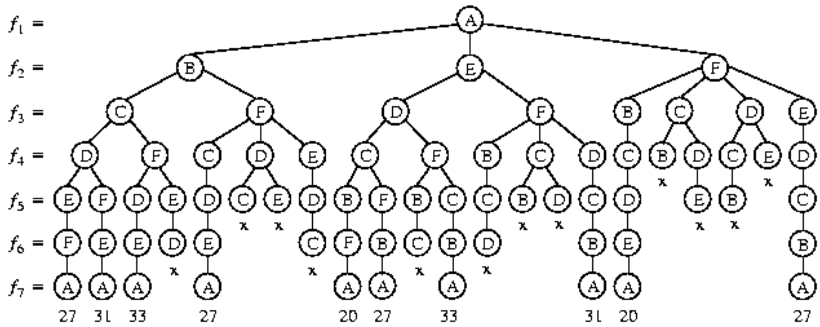
Algorithmic Paradigms

- **Exhaustive search:** Search the entire solution space
- **Branch and bound:** A search technique with pruning
- **Greedy method:** Pick a locally optimal solution at each step
- **Dynamic programming:** Partition a problem into a collection of sub-problems, the sub-problems are solved, and then the original problem is solved by combining the solutions (applicable when the sub-problems are **NOT independent**)
- **Hierarchical approach:** Divide-and-conquer
- **Mathematical programming:** A system of solving an objective function under constraints
- **Simulated annealing:** An adaptive, iterative, non-deterministic algorithm that allows "uphill" moves to escape from local optima
- **Tabu search:** Similar to simulated annealing, but does not decrease the chance of "uphill" moves throughout the search
- **Genetic algorithm:** A population of solutions is stored and allowed to evolve through successive generations via mutation, crossover, etc.

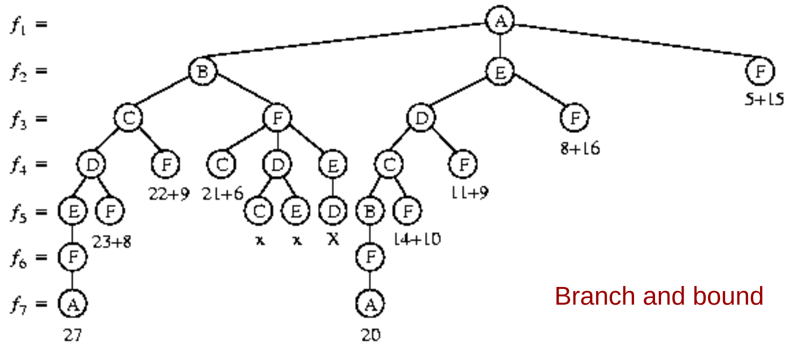
Exhaustive Search v.s. Branch and Bound

- TSP example





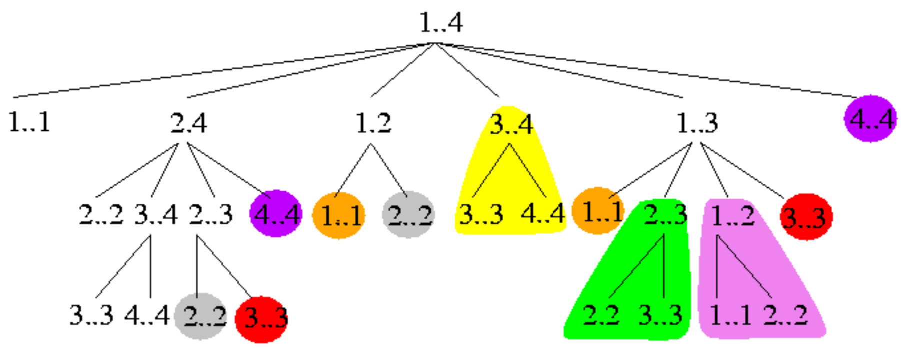
Backtracking/exhaustive search



Branch and bound

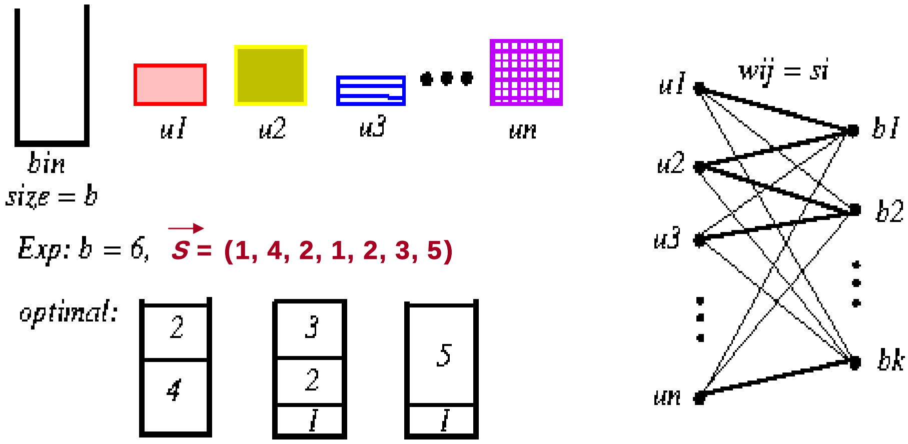
Dynamic Programming (DP) v.s. Divide-and-Conquer

- Both solve problems by combining the solutions to sub-problems
- Divide-and-conquer algorithms
  - Partition a problem into **independent** sub-problems, solve the sub-problems recursively, and then combine their solutions to solve the original problem
  - Inefficient if they solve the same sub-problem more than once
- Dynamic programming (DP)
  - Applicable when the sub-problems are **not independent**
  - DP solves each sub-problem just once



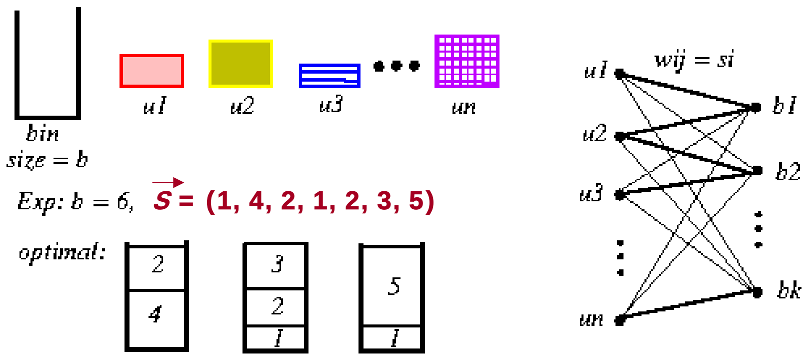
Example: Bin Packing

- **The Bin-Packing Problem Π :**
  Items $U = \{u_1, u_2, \ldots, u_n\}$, where u_i is of an integer size s_i; set B of bins, each with capacity b
- **Goal:**
  Pack all items, minimizing # of bins used (**NP-hard!**)



Algorithms for Bin Packing

- Greedy approximation algorithm:
  First-Fit Decreasing (FFD)
  - *FFD*(Π) ≤ 11OPT(Π)/9 + 4)
- Dynamic Programming? Hierarchical Approach? Genetic Algorithm? ...
- Mathematical Programming:
  Use **integer linear programming (ILP)** to find a solution using $|B|$ bins, then search for the smallest feasible $|B|$



ILP Formulation for Bin Packing

- 0-1 variable: x_{ij}=1 if item u_i is placed in bin b_j, 0 otherwise

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij}x_{ij} \\
\textit{subject to} \quad & \\
& \sum_{\forall i \in U} w_{ij}x_{ij} \le b_j, \forall j \in B \quad /* \textit{ capacity constraint } */ \quad (1) \\
& \sum_{\forall j \in B} x_{ij} = 1, \forall i \in U \quad /* \textit{ assignment constraint } */ \quad (2) \\
& \sum_{ij} x_{ij} = n \quad /* \textit{ completeness constraint } */ \quad (3) \\
& x_{ij} \in \{0, 1\} \quad /* \; 0,\ 1 \textit{ constraint } */ \quad (4)
\end{aligned}
$$

  - **Step 1:** Set $|B|$ to the lower bound of the # of bins
  - **Step 2:** Use the ILP to find a **feasible solution**
  - **Step 3:** If the solution exists, the # of bins required is $|B|$. Then exit.
  - **Step 4:** Otherwise, set $|B| \leftarrow |B| + 1$. Goto Step 2.

Mathematical Programming

- Many optimization problems can be formulated as

  minimize (or maximize) $f_0(\mathbf{x})$ objective function

  subject to $f_i(\mathbf{x}) \le c_i$, i = 1, …, m.   constraints

- Some special common mathematical programming
  - Linear programming (LP)
  - Integer linear programming (ILP)
  - Nonlinear programming
    - Convex optimization
      - Semi-definite programming, geometric programming, …