Introduction to Electronic Design Automation

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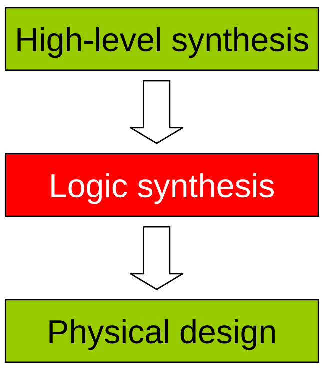
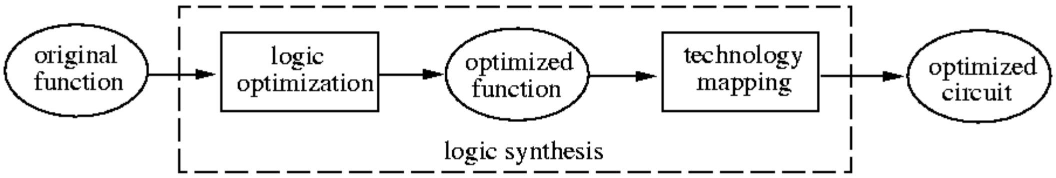
Logic Synthesis



Part of the slides are by courtesy of Prof. Andreas Kuehlmann

Logic Synthesis

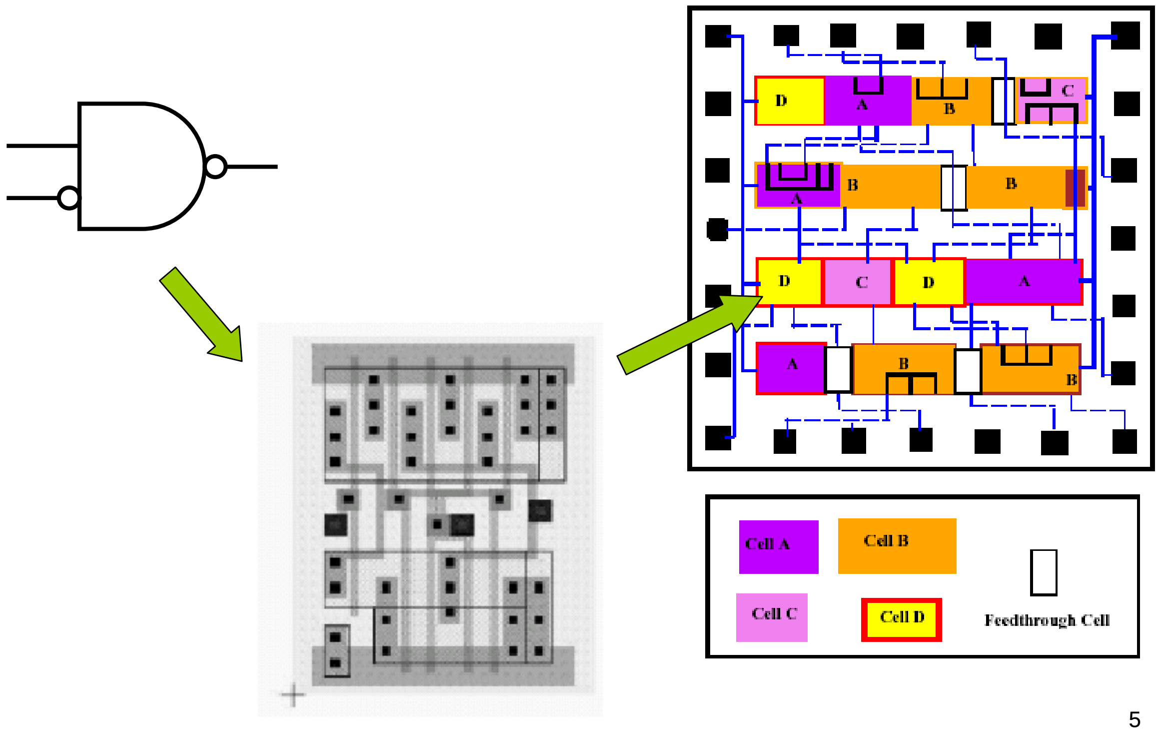
- Course contents
  - Overview
  - Boolean function representation
  - Logic optimization
  - Technology mapping
- Reading
  - Chapter 6



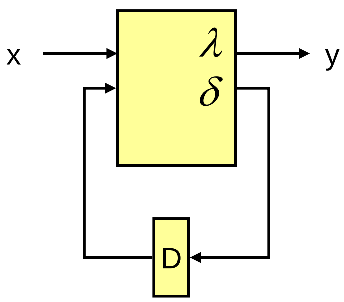
High-Level to Logic Synthesis

- Hardware is normally partitioned into two parts:
  - **Data path:** a network of functional units, registers, multiplexers and buses.
  - **Control:** the circuit that takes care of having the data present at the right place at a specific time (i.e. FSM), or of presenting the right instructions to a programmable unit (i.e. microcode).
- High-level synthesis often focuses on data-path optimization
  - The control part is then realized as an FSM
- Logic synthesis often focuses on control-logic optimization
  - Logic synthesis is widely used in application-specific IC (ASIC) design, where standard cell design style is most common

Standard-Cell Based Design

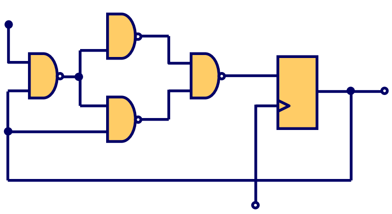


Transformation of Logic Synthesis



Given: Functional description of finite-state machine F(Q,X,Y,δ,λ) where:

Q: Set of internal states
X: Input alphabet
Y: Output alphabet
δ: X x Q → Q (next state *function*)
λ: X x Q → Y (output *function*)



Target: Circuit C(G, W) where:
G: set of circuit components g ∈ {gates, FFs, etc.}
W: set of wires connecting G

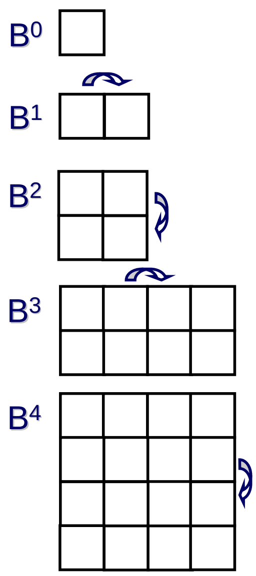
Boolean Function Representation

- Logic synthesis translates Boolean functions into circuits
- We need representations of Boolean functions for two reasons:
  - to represent and manipulate the actual circuit that we are implementing
  - to facilitate *Boolean reasoning*

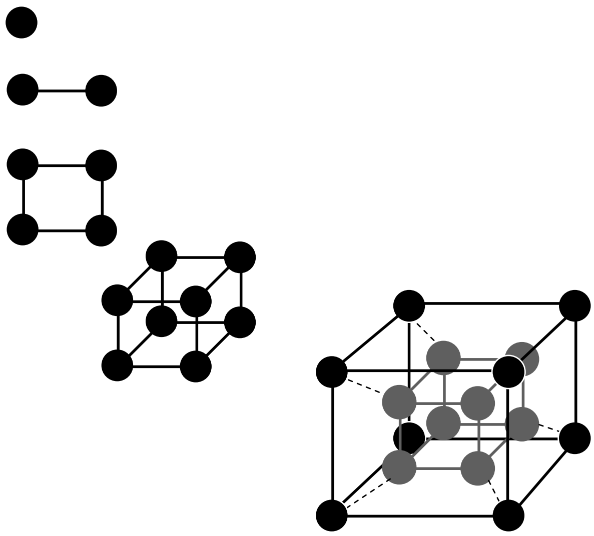
Boolean Space

- B = {0,1}
- B^2 = {0,1}×{0,1} = {00, 01, 10, 11}

Karnaugh Maps:



Boolean Lattices:



Boolean Function

- A Boolean function f over input variables: $x_1, x_2, \ldots, x_m$, is a mapping f: $\mathbf{B}^m \rightarrow Y^n$, where $\mathbf{B} = \{0,1\}$ and $Y = \{0,1,d\}$
  - E.g.
  - The output value of $f(x_1, x_2, x_3)$, say, partitions $\mathbf{B}^m$ into three sets:
    - on-set ($f = 1$)
      - E.g. {010, 011, 110, 111} (characteristic function $f^1 = x_2$)
    - off-set ($f = 0$)
      - E.g. {100, 101} (characteristic function $f^0 = x_1 \neg x_2$)
    - don't-care set ($f = d$)
      - E.g. {000, 001} (characteristic function $f^d = \neg x_1 \neg x_2$)

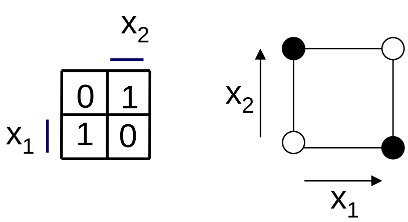
- f is an incompletely specified function if the don't-care set is nonempty. Otherwise, f is a completely specified function
  - Unless otherwise said, a Boolean function is meant to be completely specified

Boolean Function

- A Boolean function f: $\mathbf{B}^n \rightarrow \mathbf{B}$ over variables $x_1,\ldots,x_n$ maps each Boolean valuation (truth assignment) in $\mathbf{B}^n$ to 0 or 1

Example

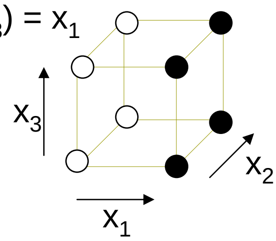
$f(x_1,x_2)$ with $f(0,0) = 0$, $f(0,1) = 1$, $f(1,0) = 1$, $f(1,1) = 0$



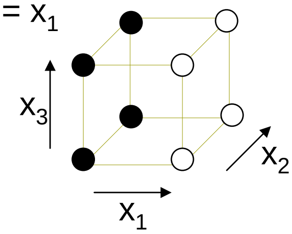
Boolean Function

- **Onset** of f, denoted as f^1, is $f^1 = \{v \in \mathbf{B}^n \mid f(v)=1\}$
  - If $f^1 = \mathbf{B}^n$, f is a tautology
- **Offset** of f, denoted as f^0, is $f^0 = \{v \in \mathbf{B}^n \mid f(v)=0\}$
  - If $f^0 = \mathbf{B}^n$, f is unsatisfiable. Otherwise, f is satisfiable.
- f^1 and f^0 are sets, not functions!
- Boolean functions f and g are equivalent if $\forall v \in \mathbf{B}^n.\ f(v) = g(v)$ where v is a truth assignment or Boolean valuation
- A literal is a Boolean variable *x* or its negation x′ (or $\overline{x}$, ¬x) in a Boolean formula

$f(x_1, x_2, x_3) = x_1$

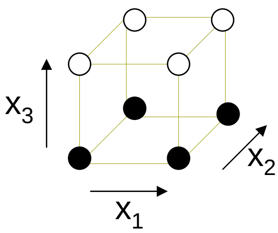


$f(x_1, x_2, x_3) = \overline{x}_1$



Boolean Function

- There are 2^n vertices in $\mathbf{B}^n$
- There are 2^{2^n} distinct Boolean functions
  - Each subset $f^1 \subseteq \mathbf{B}^n$ of vertices in $\mathbf{B}^n$ forms a distinct Boolean function f with onset f^1



$x_1x_2x_3$	f
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	0
1 0 0	⇒ 1
1 0 1	0
1 1 0	1
1 1 1	0

Boolean Operations

Given two Boolean functions:

$f : \mathbf{B}^n \rightarrow \mathbf{B}$

$g : \mathbf{B}^n \rightarrow \mathbf{B}$

- $h = f \wedge g$ from AND operation is defined as
  $h^1 = f^1 \cap g^1$; $h^0 = \mathbf{B}^n \setminus h^1$
- $h = f \vee g$ from OR operation is defined as
  $h^1 = f^1 \cup g^1$; $h^0 = \mathbf{B}^n \setminus h^1$
- $h = \neg f$ from COMPLEMENT operation is defined as
  $h^1 = f^0$; $h^0 = f^1$

Cofactor and Quantification

Given a Boolean function:

$f : \mathbf{B}^n \rightarrow \mathbf{B}$, with the input variable $(x_1, x_2, \ldots, x_i, \ldots, x_n)$

- Positive cofactor on variable x_i
  $h = f_{x_i}$ is defined as $h = f(x_1, x_2, \ldots, 1, \ldots, x_n)$
- Negative cofactor on variable x_i
  $h = f_{\neg x_i}$ is defined as $h = f(x_1, x_2, \ldots, 0, \ldots, x_n)$
- Existential quantification over variable x_i
  $h = \exists x_i. f$ is defined as $h = f(x_1, x_2, \ldots, 0, \ldots, x_n) \vee f(x_1, x_2, \ldots, 1, \ldots, x_n)$
- Universal quantification over variable x_i
  $h = \forall x_i. f$ is defined as $h = f(x_1, x_2, \ldots, 0, \ldots, x_n) \wedge f(x_1, x_2, \ldots, 1, \ldots, x_n)$
- Boolean difference over variable x_i
  $h = \partial f / \partial x_i$ is defined as $h = f(x_1, x_2, \ldots, 0, \ldots, x_n) \oplus f(x_1, x_2, \ldots, 1, \ldots, x_n)$

Boolean Function Representation

- Some common representations:
  - Truth table
  - Boolean formula
    - SOP (sum-of-products, or called disjunctive normal form, DNF)
    - POS (product-of-sums, or called conjunctive normal form, CNF)
  - BDD (binary decision diagram)
  - Boolean network (consists of nodes and wires)
    - Generic Boolean network
      - Network of nodes with generic functional representations or even subcircuits
    - Specialized Boolean network
      - Network of nodes with SOPs (PLAs)
      - And-Inv Graph (AIG)
- Why different representations?
  - Different representations have their own strengths and weaknesses (no single data structure is best for all applications)

Boolean Function Representation Truth Table

- Truth table (function table for multi-valued functions):

  The truth table of a function f : $\mathbf{B}^n \rightarrow \mathbf{B}$ is a tabulation of its value at each of the 2^n vertices of $\mathbf{B}^n$.

  In other words the truth table lists all mintems

  Example: f = a′b′c′d + a′b′cd + a′bc′d + ab′c′d + ab′cd + abc′d + abcd′ + abcd

  The truth table representation is
  - impractical for large *n*
  - canonical

  If two functions are the equal, then their canonical representations are isomorphic.

	abcd	f		abcd	f
0	0000	0	8	1000	0
1	0001	1	9	1001	1
2	0010	0	10	1010	0
3	0011	1	11	1011	1
4	0100	0	12	1100	0
5	0101	1	13	1101	1
6	0110	0	14	1110	1
7	0111	0	15	1111	1

Boolean Function Representation
Boolean Formula

- A Boolean formula is defined inductively as an expression with the following formation rules (syntax):

formula ::=		'(' formula ')'	
	|	Boolean constant	(**true** or **false**)
	|	<Boolean variable>	
	|	formula "+" formula	(OR operator)
	|	formula "·" formula	(AND operator)
	|	¬ formula	(complement)

Example

$f = (x_1 \cdot x_2) + (x_3) + \neg(\neg(x_4 \cdot (\neg x_1)))$

typically "·" is omitted and '(', ')' are omitted when the operator priority is clear, e.g., $f = x_1\ x_2 + x_3 + x_4 \neg x_1$

Boolean Function Representation
Boolean Formula in SOP

- Any function can be represented as a sum-of-products (SOP), also called sum-of-cubes (a cube is a product term), or disjunctive normal form (DNF)

Example

$\varphi = ab + a'c + bc$

Boolean Function Representation
Boolean Formula in POS

- Any function can be represented as a product-of-sums (POS), also called conjunctive normal form (CNF)
  - Dual of the SOP representation

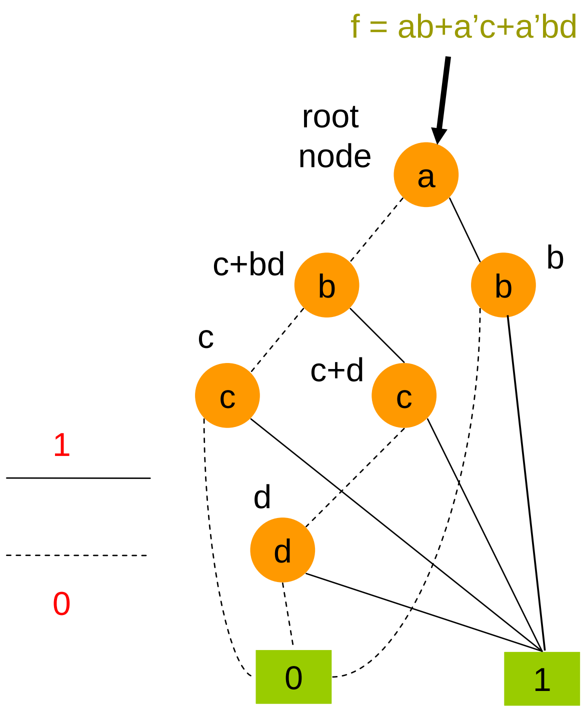
Example

$\varphi = (a+b'+c)\,(a'+b+c)\,(a+b'+c')\,(a+b+c)$

- Exercise: Any Boolean function in POS can be converted to SOP using De Morgan's law and the distributive law, and vice versa

Boolean Function Representation
Binary Decision Diagram

- BDD – a graph representation of Boolean functions
  - A leaf node represents constant 0 or 1
  - A non-leaf node represents a decision node (multiplexer) controlled by some variable
  - Can make a BDD representation **canonical** by imposing the variable ordering and reduction criteria (ROBDD)



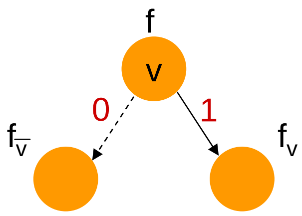
Boolean Function Representation
Binary Decision Diagram

- Any Boolean function *f* can be written in term of **Shannon expansion**

  $$f = v\, f_v + \neg v\, f_{\neg v}$$

  - Positive cofactor: $f_{xi} = f(x_1, \dots, x_i = 1, \dots, x_n)$
  - Negative cofactor: $f_{\neg xi} = f(x_1, \dots, x_i = 0, \dots, x_n)$

- BDD is a compressed Shannon cofactor tree:
  - The two children of a node with function *f* controlled by variable *v* represent two sub-functions f_v and $f_{\neg v}$

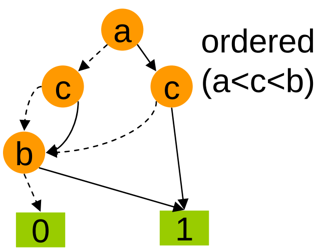


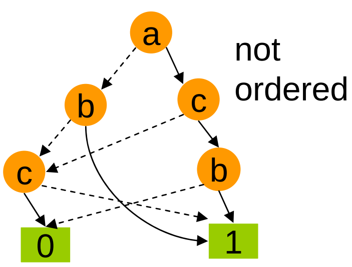
Boolean Function Representation
Binary Decision Diagram

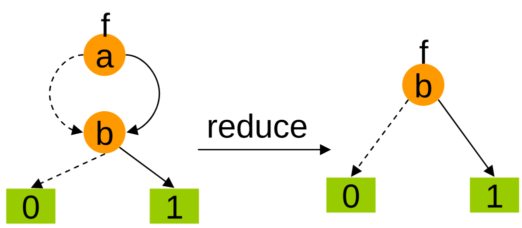
- Reduced and ordered BDD (ROBDD) is a **canonical** Boolean function representation
  - Ordered:
    - cofactor variables are in the same order along all paths

      $x_{i_1} < x_{i_2} < x_{i_3} < \dots < x_{i_n}$
  - Reduced:
    - any node with two identical children is removed
    - two nodes with isomorphic BDD's are merged

    These two rules make any node in an ROBDD represent a distinct logic function

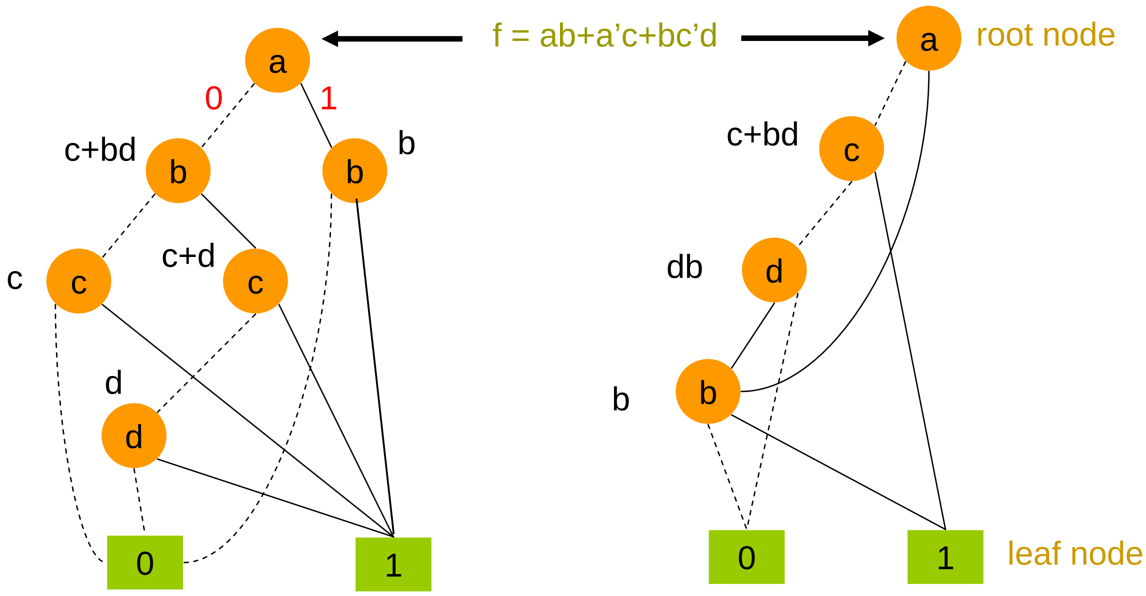






Boolean Function Representation
Binary Decision Diagram

- For a Boolean function,
  - ROBDD is unique with respect to a given variable ordering
  - Different orderings may result in different ROBDD structures



Boolean Function Representation
Boolean Network

- A Boolean network is a directed graph C(G,N) where G are the gates and N $\subseteq$ (G×G) are the directed edges (nets) connecting the gates.

  Some of the vertices are designated:
  Inputs: $I \subseteq G$
  Outputs: $O \subseteq G$
  $I \cap O = \varnothing$

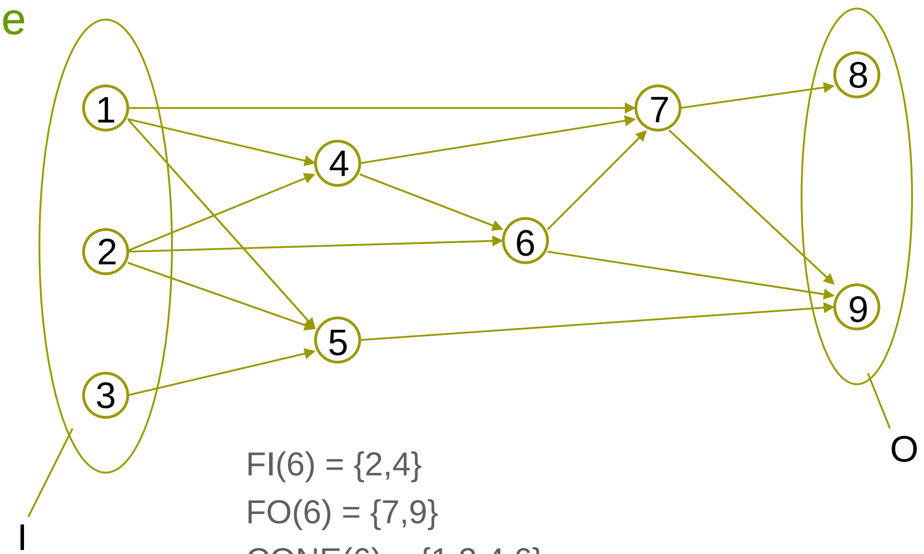
  Each gate g is assigned a Boolean function f_g which computes the output of the gate in terms of its inputs.

Boolean Function Representation
Boolean Network

- The fanin FI(g) of a gate g are the predecessor gates of g:
  FI(g) = {g' | (g',g) $\in$ N} (N: the set of nets)
- The fanout FO(g) of a gate g are the successor gates of g:
  FO(g) = {g' | (g,g') $\in$ N}
- The cone CONE(g) of a gate g is the transitive fanin (TFI) of g and g itself
- The support SUPPORT(g) of a gate g are all inputs in its cone:
  SUPPORT(g) = CONE(g) $\cap$ I

Boolean Function Representation
Boolean Network

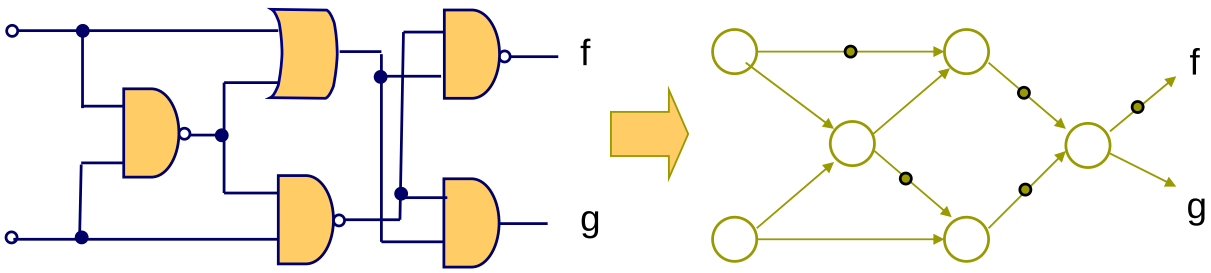
Example



FI(6) = {2,4}
FO(6) = {7,9}
CONE(6) = {1,2,4,6}
SUPPORT(6) = {1,2}
Every node may have its own function

Boolean Function Representation
And-Inverter Graph

- AND-INVERTER graphs (AIGs)
  vertices: 2-input AND gates
  edges: interconnects with (optional) dots representing INVs
- Hash table to identify and reuse structurally isomorphic circuits



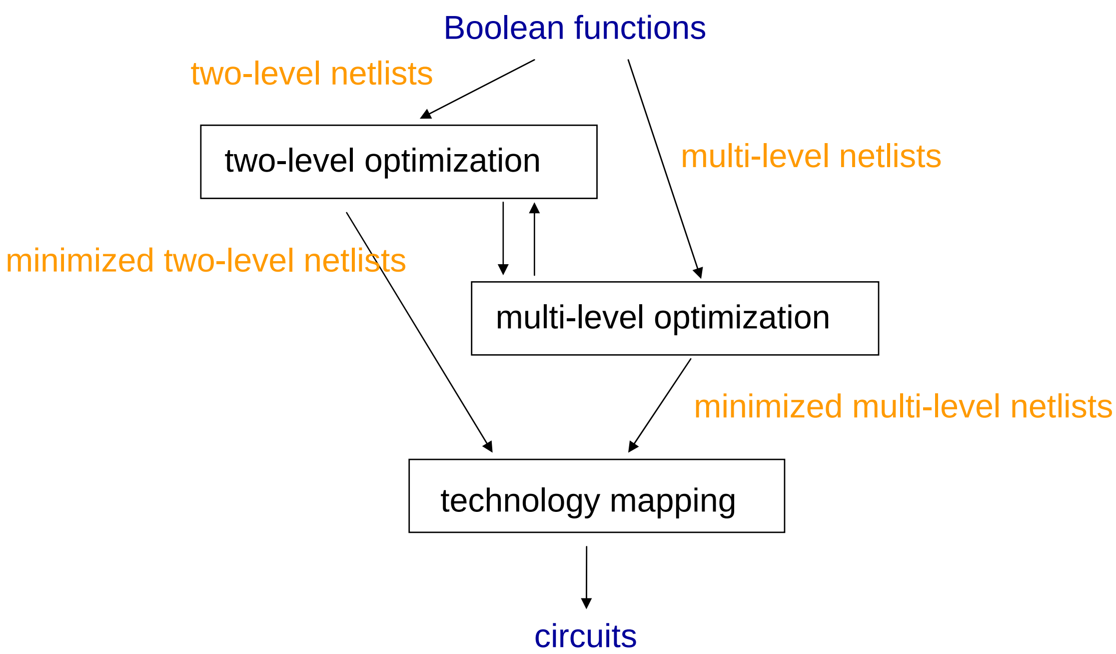
Boolean Function Representation

- A canonical form of a Boolean function is a unique representation of the function
  - It can be used for verification purposes
- Example
  - Truth table is canonical
    - It grows exponentially with the number of input variables
  - ROBDD is canonical
    - It is of practical interests because it may represent many Boolean functions compactly
  - SOP, POS, Boolean networks are NOT canonical

Boolean Function Representation

- Truth table
  - Canonical
  - Useful in representing small functions
- SOP
  - Useful in two-level logic optimization, and in representing local node functions in a Boolean network
- POS
  - Useful in SAT solving and Boolean reasoning
  - Rarely used in circuit synthesis (due to the asymmetric characteristics of NMOS and PMOS)
- ROBDD
  - Canonical
  - Useful in Boolean reasoning
- Boolean network
  - Useful in multi-level logic optimization
- AIG
  - Useful in multi-level logic optimization and Boolean reasoning

Logic Optimization



Two-Level Logic Minimization

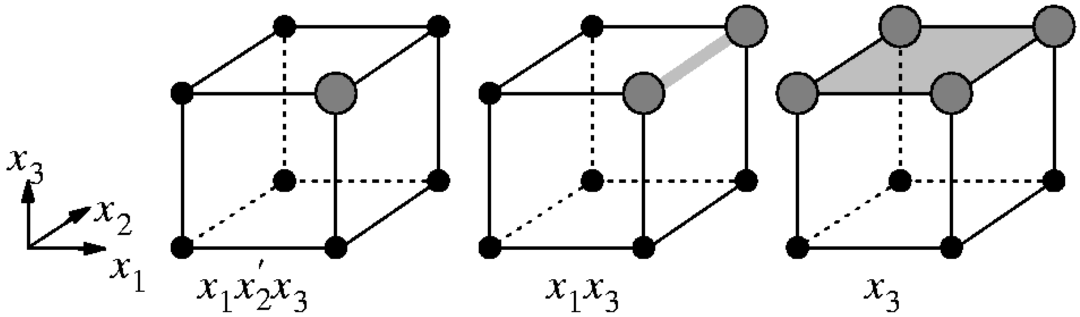
- Any Boolean function can be realized using PLA in two levels: AND-OR (sum of products), NAND-NAND, etc.
  - Direct implementation of two-level logic using PLAs (programmable logic arrays) is not as popular as in the nMOS days
- Classic problem solved by the *Quine-McCluskey* algorithm
  - Popular cost function: #cubes and #literals in an SOP expression
    - #cubes – #rows in a PLA
    - #literals – #transistors in a PLA
  - The goal is to find a minimal irredundant prime cover

Two-Level Logic Minimization

- Exact algorithm
  - Quine-McCluskey's procedure
- Heuristic algorithm
  - Espresso

Two-Level Logic Minimization
Minterms and Cubes

- A minterm is a product of every input variable or its negation
  - A minterm corresponds to a single point in $\mathbf{B}^n$
- A cube is a product of literals
  - The fewer the number of literals is in the product, the bigger the space is covered by the cube

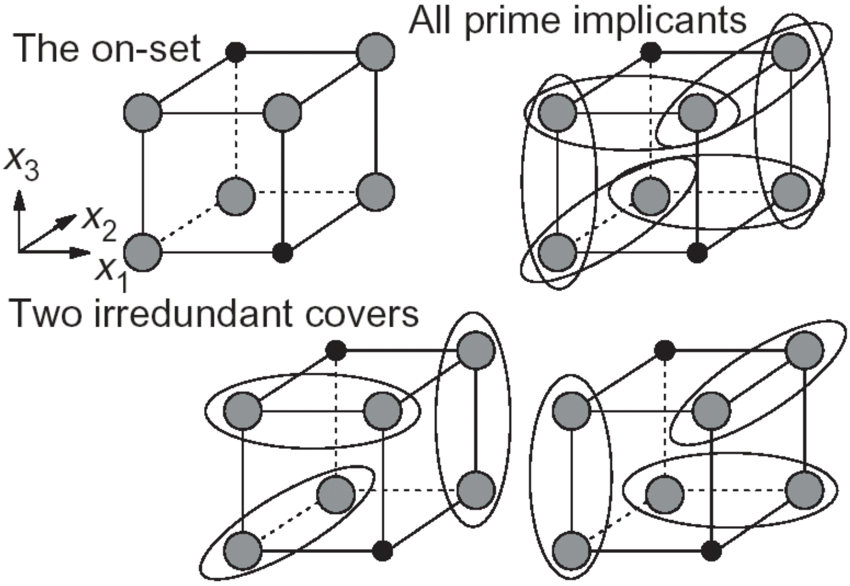


Two-Level Logic Minimization
Implicant and Cover

- An implicant is a cube whose points are either in the on-set or the dc-set.
- A prime implicant is an implicant that is not included in any other implicant.
- A set of prime implicants that together cover all points in the on-set (and some or all points of the dc-set) is called a prime cover.
  - A prime cover is irredundant when none of its prime implicants can be removed from the cover.
  - An irredundant prime cover is minimal when the cover has the minimal number of prime implicants.
    (c.f. minimum vs. minimal)

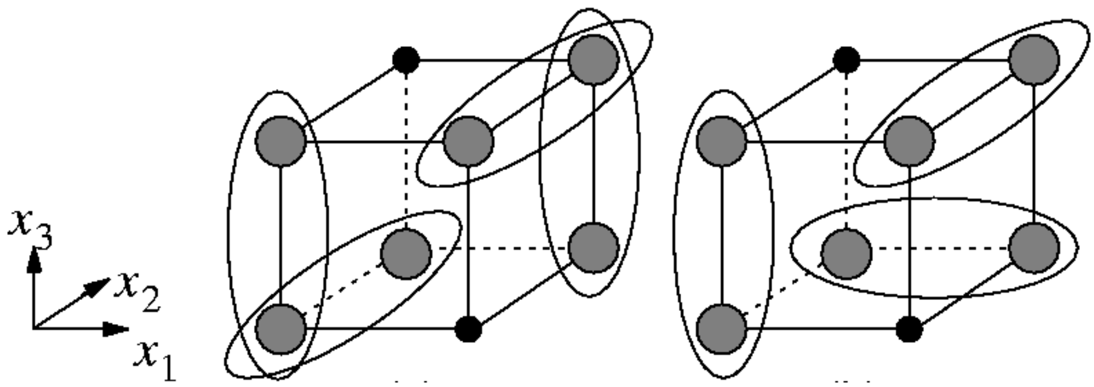
Two-Level Logic Minimization
Cover

- Example
  - $f = \neg x_1 \neg x_3 + \neg x_2 x_3 + x_1 x_2$
  - $f = \neg x_1 \neg x_2 + x_2 \neg x_3 + x_1 x_3$



Two-Level Logic Minimization
Cover

- Example



local minimal global minimal

Two-Level Logic Minimization
Quine-McCluskey Procedure

- Given G and D (covers for $\Im$ = (f,d,r) and d, respectively), find a minimum cover G* of primes where:

  $f \subseteq G^* \subseteq f+d$ (G* is a prime cover of $\Im$)
  - f is the onset, d don’t-care set, and r offset

- Q-M Procedure:
  1. Generate all primes of $\Im$, $\{P_j\}$ (i.e. primes of (f+d) = G+D)
  2. Generate all minterms $\{m_i\}$ of f = $G \wedge \neg D$
  3. Build Boolean matrix B where

     B_{ij} = 1 if $m_i \in P_j$
     = 0 otherwise
  4. Solve the minimum column covering problem for B (unate covering problem)

Two-Level Logic Minimization
Quine-McCluskey Procedure

Generating Primes

Tabular method
(based on *consensus* operation):

- Start with all minterm canonical form of *F*
- Group *pairs* of adjacent minterms into cubes
- Repeat merging cubes until no more merging possible; mark (√) + remove all covered cubes.
- Result: set of *primes* of *f.*

Example

F = x’ y’ + w x y + x’ y z’ + w y’ z

F = x’ y’ + w x y + x’ y z’ + w y’ z

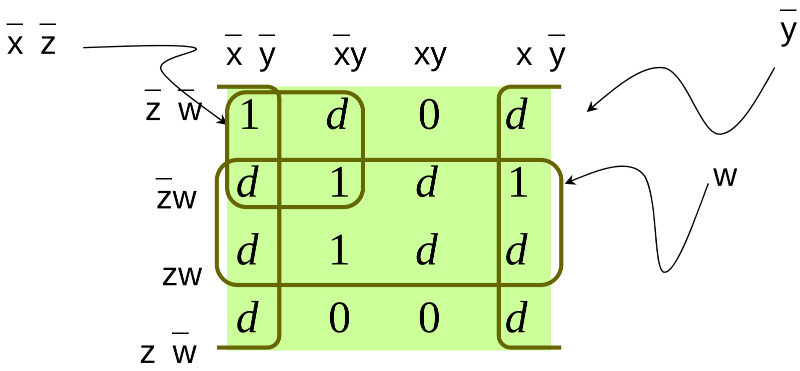
w’ x’ y’ z’ √	w’ x’ y’ √ w’ x’ z’ √ x’ y’ z’ √	x’ y’ x’ z’
w’ x’ y’ z √ w’ x’ y z’ √ w x’ y’ z’ √	x’ y’ z √ x’ y z’ √ w x’ y’ √ w x’ z’ √	
w x’ y’ z √ w x’ y z’ √	w y’ z w y z’	
w x y z’ √ w x y’ z √	w x y w x z	
w x y z √		

Courtesy: Maciej Ciesielski, UMASS

Two-Level Logic Minimization
Quine-McCluskey Procedure

- Example

Karnaugh map



	$\bar{y}$	w	$\bar{x}\bar{z}$
$\bar{x}\bar{y}\bar{z}\bar{w}$	1	0	1
$\bar{x}y\bar{z}w$	0	1	1
$x\bar{y}\bar{z}w$	1	1	0
$\bar{x}yzw$	0	1	0

$$F = \bar{x}\bar{y}\bar{z}\bar{w} + \bar{x}y\bar{z}w + x\bar{y}\bar{z}w + \bar{x}yzw \quad \text{(cover of } \Im\text{)}$$

$$D = \bar{y}z + xyw + \bar{x}\bar{y}\bar{z}w + x\bar{y}\bar{w} + \bar{x}y\bar{z}\bar{w} \quad \text{(cover of d)}$$

Primes: $\bar{y}$ + w + $\bar{x}\bar{z}$

Covering Table

Solution: {1,2} ⇒ $\bar{y}$ + w is a minimum prime cover (also w + $\bar{x}\bar{z}$)

Two-Level Logic Minimization
Quine-McCluskey Procedure

Column covering of Boolean matrix

Primes of f+d

Minterms of f

	$\bar{y}$	w	$\bar{x}\bar{z}$
$\bar{x}\bar{y}\bar{z}\bar{w}$	1	0	1
$\bar{x}y\bar{z}w$	0	1	1
$x\bar{y}\bar{z}w$	1	1	0
$\bar{x}yzw$	0	1	0

Row singleton (essential minterm)

Essential prime

- Definition. An essential prime is a prime that covers an onset minterm of f not covered by any other primes.

Two-Level Logic Minimization
Quine-McCluskey Procedure

- Row equality in Boolean matrix:
  - In practice, many rows in a covering table are identical. That is, there exist minterms that are contained in the same set of primes.
  - Example

    m_1 0101101

    m_2 0101101

Two-Level Logic Minimization
Quine-McCluskey Procedure

- Row dominance in Boolean matrix:
  - A *row* i_1 whose set of primes is contained in the set of primes of *row* i_2 is said to dominate i_2.
  - Example

    i_1 011010

    i_2 011110

    - i_1 dominates i_2
    - Can remove row i_2 because have to choose a prime to cover i_1, and any such prime also covers i_2. So i_2 is automatically covered.

Two-Level Logic Minimization Quine-McCluskey Procedure

- Column dominance in Boolean matrix:
  - A *column* j_1 whose rows are a superset of another *column* j_2 is said to dominate j_2.
  - Example

j_1	j_2
1	0
0	0
1	1
0	0
1	1

    - j_1 dominates j_2
    - We can remove column j_2 since j_1 covers all those rows and more. We would never choose j_2 in a minimum cover since it can always be replaced by j_1.

Two-Level Logic Minimization Quine-McCluskey Procedure

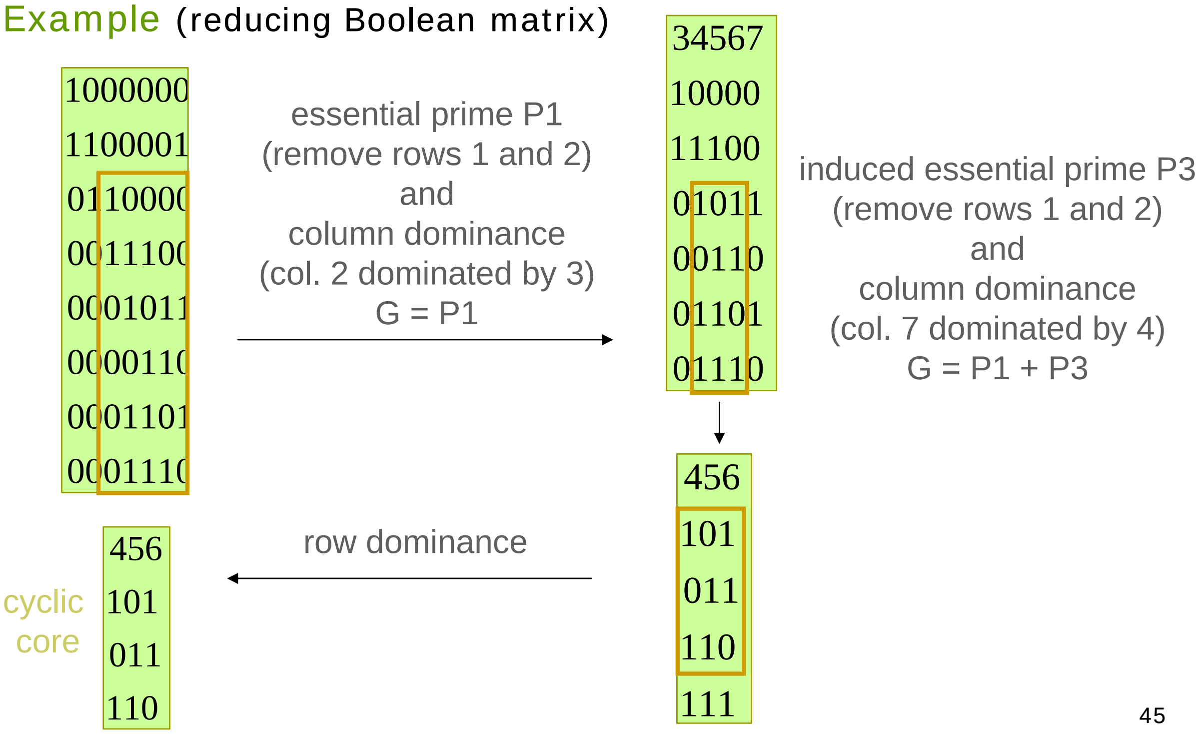
Reducing Boolean matrix

1. Remove all rows covered by essential primes (columns in row singletons). Put these primes in the cover G.
2. Group identical rows together and remove dominated rows.
3. Remove dominated columns. For equal columns, keep one prime to represent them.
4. Newly formed row singletons define induced essential primes.
5. Go to 1 if covering table decreased.

- The resulting reduced covering table is called the cyclic core. This has to be solved (unate covering problem). A minimum solution is added to G. The resulting G is a minimum cover.

Two-Level Logic Minimization
Quine-McCluskey Procedure

Example (reducing Boolean matrix)



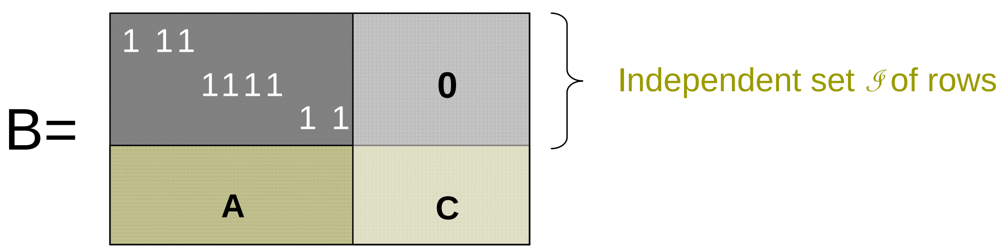
Two-Level Logic Minimization
Quine-McCluskey Procedure

Solving cyclic core

- Best known method (for unate covering) is branch and bound with some clever bounding heuristics
- Independent Set Heuristic:
  - Find a maximum set I of "independent" rows. Two rows B_{i1}, B_{i2} are independent if **not** $\exists j$ such that $B_{i_1 j} = B_{i_2 j} = 1$. (They have no column in common.)

Example

A covering matrix B rearranged with independent sets first

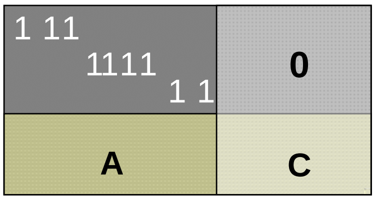


Two-Level Logic Minimization
Quine-McCluskey Procedure

Solving cyclic core

- Heuristic algorithm:
  - Let $\mathscr{I} = \{I_1, I_2, \ldots, I_k\}$ be the independent set of rows

1. choose $j \in I_i$ such that column j covers the most rows of A. Put Pj in G
2. eliminate all rows covered by column j
3. $\mathscr{I} \leftarrow \mathscr{I} \setminus \{I_i\}$
4. go to 1 if $|\mathscr{I}| > 0$
5. If B is empty, then done (in this case achieve minimum solution)
6. If B is not empty, choose an independent set of B and go to 1



Two-Level Logic Minimization
Quine-McCluskey Procedure

- Summary
  - Calculate all prime implicants (of the union of the onset and don't care set)
  - Find the minimal cover of all minterms in the onset by prime implicants
    - Construct the covering matrix
    - Simplify the covering matrix by detecting essential columns, row and column dominance
    - What is left is the cyclic core of the covering matrix.
      - The covering problem can then be solved by a branch-and-bound algorithm.

Two-Level Logic Minimization
Exact vs. Heuristic Algorithms

- Quine-McCluskey Method:

1. Generate cover of all primes $G = p_1 + p_2 + \cdots + p_{3^n/n}$
2. Make G irredundant (in optimum way)
   - Q-M is exact, i.e., it gives an exact minimum

- Heuristic Methods:

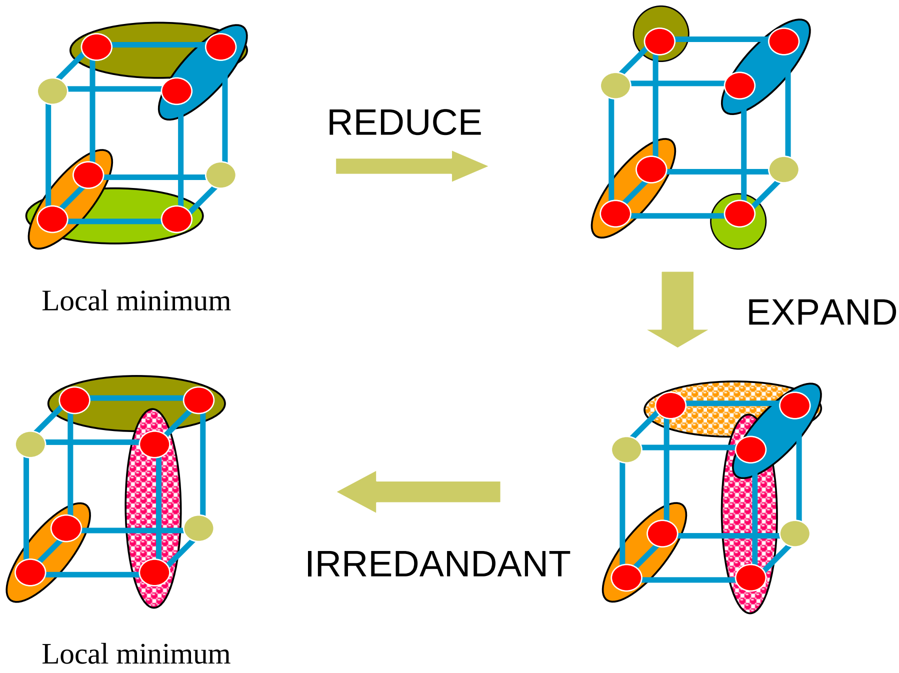
1. Generate (somehow) a cover of $\Im$ using some of the primes $G = p_{i_1} + p_{i_2} + \cdots + p_{i_k}$
2. Make G irredundant (maybe not optimally)
3. Keep best result - try again (i.e. go to 1)

Two-Level Logic Minimization
ESPRESSO

- Heuristic two-level logic minimization

```
ESPRESSO(ℑ)
{
  (F,D,R) ← DECODE(ℑ)
  F ← EXPAND(F,R)
  F ← IRREDUNDANT(F,D)
  E ← ESSENTIAL_PRIMES(F,D)
  F ← F-E;  D ← D+E
  do{
    do{
      F ← REDUCE(F,D)
      F ← EXPAND(F,R)
      F ← IRREDUNDANT(F,D)
    }while fewer terms in F
    //LASTGASP
    G ← REDUCE_GASP(F,D)
    G ← EXPAND(G,R)
    F ← IRREDUNDANT(F+G,D)
    //LASTGASP
  }while fewer terms in F
  F ← F+E;  D ← D-E
  LOWER_OUTPUT(F,D)
  RAISE_INPUTS(F,R)
  error ← (F_old ⊄ F) or (F ⊄ F_old + D)
  return (F,error)
}
```

Two-Level Logic Minimization
ESPRESSO



Logic Minimization