Proofs and Types
Strong Normalisation for **F**

Bow-Yaw Wang

Academia Sinica

Spring 2012

Reducibility

- Let us try to prove **F** by reducibility.
- We first need to define what terms in a type T are reducible.
  - Reducible terms of the types without universal quantification are defined as before.
  - t of the type $\Pi X.T$ is *reducible* if for all types U, $t\ U$ (of the type $T[U/X]$) is reducible.
- Consider t of the type $\Pi X.X$.
- t is reducible if $t\ U$ is reducible for all U.
- Particularly, take $U = \Pi X.X$. We need to check if $t\ (\Pi X.X)$ of the type $\Pi X.X$ is reducible.
- But we have a circularity.
  - To check t of the type $\Pi X.X$ is reducible, we need to check if $t(\Pi X.X)$ of the type $\Pi X.X$ is reducible;
  - To check $t(\Pi X.X)$ of the type $\Pi X.X$ is reducible, we need to check if $t(\Pi X.X)(\Pi X.X)$ of the type $\Pi X.X$ is reducidble;
  - ad infinitum

Reducibility Candidates

- Instead of defining reducibility, we introduce reducibility candidates.
- Reducibility candidates do not define reducibility.
  - Hence we avoid the circularity.
- However, we intend to find the definition of reducibility among reducibility candidates.
  - Thus we can prove the strong normalisation for **F**.
- More concretely, a reducibility candidate of the type U is a reducibility predicate (set of terms of the type U) satisfying **CR 1-3**.
- A term of the type $\Pi X.T$ is reducible if for every the type U and every reducible candidate $\mathcal{R}$ of the type U, t U of the type $T[U/X]$ is reducible with respect to $\mathcal{R}$.
  - That is, using $\mathcal{R}$ as the definition of reducibility of U.
  - If $\mathcal{R}$ happens to be the "true" reducibility of the type U, it yields the "true" reducibility of the type $T[U/X]$.

Reducibility Candidates

- Consider the term $\Pi X.\lambda x^X.x$ of the type $\Pi X.X \to X$.
- $\Pi X.\lambda x^X.x$ is reducible if for every reducibility candidate $\mathcal{R}$ for U, $(\Pi X.\lambda x^X.x)\ U$ of the type $U \to U$ is reducible wrt $\mathcal{R}$.
- That is, for every $u \in \mathcal{R}$, $(\Pi X.\lambda x^X.x)\ U\ u \in \mathcal{R}$.
- Recall that $\mathcal{R}$ satisfies **CR 1-3**.
  - $\mathcal{R}$ is a reducibility candidate.
- We will use **CR 1-3** to show

$$u \in \mathcal{R} \text{ implies } (\Pi X.\lambda x^X.x)\ U\ u \in \mathcal{R}$$

- Observe how to avoid the circularity!
- Note that the properties **CR 1-3** are crucial.

Definitions

- A term t is *neutral* if it has one of the following forms:

$$x \qquad t\,u \qquad t\,U$$

- A *reducibility candidate* of the type U is a set $\mathcal{R}$ of terms of the type U that
  - **CR 1** If $t \in \mathcal{R}$, t is strongly normalisable.
  - **CR 2** If $t \in \mathcal{R}$ and $t \rightsquigarrow t'$, $t' \in \mathcal{R}$.
  - **CR 3** If t is neutral and we obtain a term $t' \in \mathcal{R}$ whenever a redex of t is converted, then $t \in \mathcal{R}$.
    - Particularly, if t is neutral and normal, $t \in \mathcal{R}$ by **CR 3**.
- For example, the set of strongly normalisable terms of the type U is a reducibility candidate of tyep U.
- If $\mathcal{R}$ and $\mathcal{S}$ are reducibility candidates of the types U and V, define a set $\mathcal{R} \to \mathcal{S}$ of terms of the type $U \to V$ by

$$t \in \mathcal{R} \to \mathcal{S} \quad \text{if} \quad t\,u \in \mathcal{S} \text{ for every } u \in \mathcal{R}.$$

Reducibility with Parameters

- Let $T[\underline{X}]$ denote a type T with all free variables in $\underline{X}$.
- Let $\underline{U}$ be a sequence of types. Define $T[\underline{U}/\underline{X}]$ to be the type obtained by simultaneous substitution of $\underline{U}$ for $\underline{X}$.
- Let $\underline{\mathcal{R}}$ be a sequence of reducibility candidates of types $\underline{U}$.
- Define a set $\mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$ of terms of the type $T[\underline{U}/\underline{X}]$ as follows.
  - If $T = X_i$, $\mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathcal{R}_i$;
  - If $T = V \to W$, $\mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}] \to \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$;
  - If $T = \Pi Y.W$, $\mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$ is the set of terms t of the type $T[\underline{U}/\underline{X}]$ such that for every type V and reducibility candidate $\mathcal{S}$ of V, $t\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$.

Properties of Reducibility with Parameters

Lemma 1

$\mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$ *is a reducibility candidate of the type* $T[\underline{U}/\underline{X}]$.

Proof.

Induction on T. Recall that $\nu(t)$ bounds the length of every normalisation sequence from t. We consider $T = \Pi Y.W$.

CR 1 Let $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$, V a type, and $\mathcal{S}$ a reducibility candidate of the type V. $t\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$ by definition. By IH (**CR 1**), $t\ V$ is strongly normalisable. Observe that $\nu(t) \leq \nu(t\ V)$. Hence t is strongly normalisable.

CR 2 Let $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$ and $t \rightsquigarrow t'$. Consider any type V and a reducibility candidate $\mathcal{S}$ of the type V. $t\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$ and $t\ V \rightsquigarrow t'\ V$. By IH (**CR 2**), $t'\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$. Hence $t' \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$.

CR 3 Let t be neutral and suppose all t' one step from t are in $\mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$. Let V be a type and $\mathcal{S}$ a reducibility candidate of the type V. We have $t\ V \rightsquigarrow t'V$ for some t' one step from t since t is neutral. Since $t' \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$, $t'\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$. By IH (**CR 3**), $t\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$. Hence $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$.

Properties of Reducibility with Parameters

Proof (cont'd).

Consider $T = X_i$.

CR 1 Let $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathcal{R}_i$. t is strongly normalisable since $\mathcal{R}_i$ is a reducibility candidate.

CR 2 Let $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathcal{R}_i$ and $t \rightsquigarrow t'$. Clearly, t' is strongly normalisable. Hence $t' \in \mathcal{R}_i$ provided $\mathcal{R}_i$ contains all strongly normalisable terms of type X_i.

CR 3 Let t be neutral and $t' \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathcal{R}_i$ for every t' obtained by converting a redex in t. Since $\nu(t) = 1 + \max_{t'} \nu(t')$, t is strongly normalisable. $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathcal{R}_i$ provided $\mathcal{R}_i$ contains all strongly normalisable terms of type X_i.

Properties of Reducibility with Parameters

Proof (cont'd).

Consider $T = V \to W$.

CR 1 Let $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}] \to \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$. Let x be a variable of the type V. $x \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$ for x is neutral and normal (**CR 3**). Hence $t\ x \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$. By IH (**CR 1**), $t\ x$ is strongly normalisable. Since $\nu(t) \leq \nu(t\ x)$, t is strongly normalisable.

CR 2 Let $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}] \to \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$ and $t \rightsquigarrow t'$. For every $v \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$, $t\ v \rightsquigarrow t'\ v$. Since $t\ v \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$, $t'\ v \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$ by IH (**CR 2**). Hence $t' \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$.

CR 3 Let t be neutral and $t' \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$ for every t' obtained by converting a redex in t. Let $v \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$. By IH (**CR 1**), v is strongly normalisable. In one step, $t\ v$ converts to

(1) $t'\ v$ with t' one step from t. $t'\ v \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$ for $t' \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$ by assumption.

(2) $t\ v'$ with v' one step from v. By IH (**CR 2**), $v' \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$ and $\nu(v') < \nu(v)$. Hence $t\ v' \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$ by IH ($\nu(v)$).

By IH (**CR 3**), $t\ v \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$. Hence $t \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$.

Substitution

Lemma 2

$\mathsf{RED}_{T[V/Y]}[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_{T}[\underline{\mathcal{R}}/\underline{X}, \mathsf{RED}_{V}[\underline{\mathcal{R}}/\underline{X}]/Y]$.

Proof.

By induction on T. □

- On the left hand side, we have a reducibility candidate of the type $T[V/Y]$ with parameters $\underline{\mathcal{R}}$ for $\underline{X}$.
- On the right hand side, we have a reducibility candidate of the type T with parameters $\underline{\mathcal{R}}, \mathsf{RED}_{V}[\underline{\mathcal{R}}/\underline{X}]$ for $\underline{X}, Y$.
- The reducibility candidate for the type Y is the reducibility candidate
- This lemma says that reducibility with parameters respects substitutions.

Universal Abstraction

Lemma 3

If $w[V/Y] \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$ for every type V and reducibility candidate $\mathcal{S}$, then $\Lambda Y.w \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$.

Proof.

We want to show $(\Lambda Y.w)\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$ for every type V and reducibility candidate $\mathcal{S}$ of V. By induction on $\nu(w)$, converting a redex in $(\Lambda Y.w)\ V$ gives:

- $(\Lambda Y.w')\ V$ with $\nu(w') < \nu(w)$. By IH, $(\Lambda Y.w')\ V \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$;
- $w[V/Y]$. By assumption, $w[V/Y] \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$.

$(\Lambda Y.w)\ V$ is neutral and every one-step conversion gives a term in $\mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$. The result follows by **CR 3**. □

Universal Application

Lemma 4

If $t \in \mathsf{RED}_{\Pi Y.W}[\underline{\mathcal{R}}/\underline{X}]$*,* $t\ V \in \mathsf{RED}_{W[V/Y]}[\underline{\mathcal{R}}/\underline{X}]$ *for every type* V*.*

Proof.

Since $t \in \mathsf{RED}_{\Pi Y.W}[\underline{\mathcal{R}}/\underline{X}]$, $t\ V \in \mathsf{RED}_{W}[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$ for every reducibility candidate $\mathcal{S}$ of V. Take $\mathcal{S} = \mathsf{RED}_{V}[\underline{\mathcal{R}}/\underline{X}]$. The result follows by Lemma 2. □

Abstraction

Lemma 5

If $v[u/x] \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$ *for all* $u \in \mathsf{RED}_U[\underline{\mathcal{R}}/\underline{X}]$*, then* $\lambda x^U.v \in \mathsf{RED}_{U \to V}[\underline{\mathcal{R}}/\underline{X}]$*.*

Proof.

Recall $x \in \mathsf{RED}_U[\underline{\mathcal{R}}/\underline{X}]$ and $v[x/x] = v \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$. Let $u \in \mathsf{RED}_U[\underline{\mathcal{R}}/\underline{X}]$. By induction on $\nu(v) + \nu(u)$, converting a redex in $(\lambda x^U.v)u$ gives:

- $v[u/x]$. $v[u/x] \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$ by assumption.
- $(\lambda x^U.v)u'$ with u' one step from u. By **CR 2**, $u' \in \mathsf{RED}_U[\underline{\mathcal{R}}/\underline{X}]$ and $\nu(u') < \nu(u)$. By IH, $(\lambda x^U.v)u' \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$.
- $(\lambda x^U.v')u$ with v' one step from v. By **CR 2**, $v' \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$ and $\nu(v') < \nu(v)$. By IH, $(\lambda x^U.v')u \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$.

Reducibility Theorem

Lemma 6

Let t be a term of type T. Suppose all free variables of t are among x_i of types U_i $(i = 1, \ldots, n)$, and all free type variables of $T, U_1, \ldots, U_n$ are among X_j $(j = 1, \ldots, m)$. If $\mathcal{R}_j$ are reducibility candidates of types V_j $(j = 1, \ldots, m)$, and $u_i \in \mathsf{RED}_{U_i}[\underline{\mathcal{R}}/\underline{X}]$, is a term of the type $U_i[\underline{V}/\underline{X}]$ $(i = 1, \ldots, n)$, then $t[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in \mathsf{RED}_T[\underline{\mathcal{R}}/\underline{X}]$.

Proof.

Induction on t.

- t is x_i. Trivial since $u_i \in \mathsf{RED}_{U_i}[\underline{\mathcal{R}}/\underline{X}]$.
- t is vw. By IH, $v[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in \mathsf{RED}_{W\to V}[\underline{\mathcal{R}}/\underline{X}]$ and $w[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]$. By the definition of $\mathsf{RED}_{W\to V}[\underline{\mathcal{R}}/\underline{X}]$, $(v[\underline{V}/\underline{X}][\underline{u}/\underline{x}])(w[\underline{V}/\underline{X}][\underline{u}/\underline{x}]) \in \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$.
- t is $\lambda y^V.w$. By IH, $w[\underline{V}/\underline{X}][\underline{u}/\underline{x}, v/y] \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_{W[\underline{V}/\underline{X}]}[\underline{\mathcal{R}}/\underline{X}]$ for all $v \in \mathsf{RED}_{V[\underline{V}/\underline{X}]}[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}]$. By Lemma 5, $\lambda y^{V[\underline{V}/\underline{X}]}.(w[\underline{V}/\underline{X}][\underline{u}/\underline{x}]) \in \mathsf{RED}_{V[\underline{V}/\underline{X}]\to W[\underline{V}/\underline{X}]}[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_{(V\to W)[\underline{V}/\underline{X}]}[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_{V\to W}[\underline{\mathcal{R}}/\underline{X}]$.
- t is $\Lambda Y.w$. By IH, $w[\underline{V}/\underline{X}, V/Y][\underline{u}/\underline{x}] \in \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}, \mathcal{S}/Y]$ for every type V and reducibility candidate $\mathcal{S}$ of V. By Lemma 3, $\Lambda Y.w[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in \mathsf{RED}_{\Pi Y.W}[\underline{\mathcal{R}}/\underline{X}]$.
- t is $v\ W$. By IH, $v[\underline{V}/\underline{X}][\underline{u}/\underline{x}] \in \mathsf{RED}_{\Pi Y.V}[\underline{\mathcal{R}}/\underline{X}]$. By Lemma 4, $(v[\underline{V}/\underline{X}][\underline{u}/\underline{x}])\ (W[\underline{V}/\underline{X}]) \in \mathsf{RED}_{V[W[\underline{V}/\underline{X}]/Y]}[\underline{\mathcal{R}}/\underline{X}] = \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}, \mathsf{RED}_{W[\underline{V}/\underline{X}]}[\underline{\mathcal{R}}/\underline{X}]/Y] = \mathsf{RED}_V[\underline{\mathcal{R}}/\underline{X}, \mathsf{RED}_W[\underline{\mathcal{R}}/\underline{X}]/Y] =$

Strong Normalisation Theorem

- A term of type T is *reducible* if it is in $\mathsf{RED}_T[\underline{\mathcal{SN}}/\underline{X}]$ where X_i are free type variables of T ($i = 1, \ldots, n$) and $\mathcal{SN}_i$ is the set of strongly normalisable terms of type X_i ($i = 1, \ldots, n$).

Theorem 7

All terms of **F** *are reducible.*

Proof.

Take $\underline{u} = \underline{x}$ and $\underline{\mathcal{R}} = \underline{\mathcal{SN}}$ in Lemma 6. □

Corollary 8

All terms of **F** *are strongly normalisable.*

Proof.

By **CR 1**. □